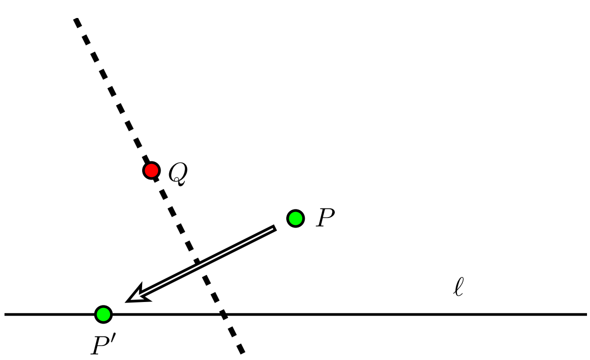
If Euclid Had Been Japanese

Bill Casselman

Starting with the two points $(0, 0)$ and $(1, 0)$, applying the standard operations with straight edge and compass, one can obtain any point with coordinates in a tower of quadratic extensions of $\mathbb{Q}$. There is an analogous result about origami constructions.

In origami construction, one starts with the configuration of the three lines $x = 0$, $y = 0$, $x + y = 1$ and applies certain basic origami operations I'll describe in a moment. In this case, the points that one obtains are those with coordinates in a tower of quadratic and cubic extensions. One half of the proof follows closely the argument for the classical case. The interesting part is the explicit construction of roots.

The basic object in origami is a line, and one constructs it by folding along it. Mathematically, folding amounts to an orthogonal reflection through the line. The simplest principles of origami construction are that (1) **points are constructed by intersecting two lines**, and, conversely, (2) **any two points determine a fold line through them**. But there are more interesting ways to "construct" lines. A more complicated but still practical origami operation is that (3) **given two points P and Q and a line ℓ, one folds along a line through Q, taking P to a point P' on ℓ**. In this way P, Q, and ℓ give rise to a new line.



Folding P to ℓ along a line through Q.

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One has to be careful. If P lies on ℓ then the fold line constructed is the line through Q perpendicular to ℓ. But if P doesn't lie on ℓ then this operation is not always possible, and when it is in fact possible it will usually not be unique. Why is this? In terms of algebra, we are looking for a line $y = mx + b$ such that

$$y_Q = mx_Q + b$$

$$(x_{P'}, 0) = (x_P, y_P) - 2\left(\frac{y_P - mx_P - b}{1 + m^2}\right)[-m, 1]$$

leading to a quadratic equation for m, which may have two, one, or zero solutions. If $P = (0, 1)$, for example, then we get the equations

$$m^2 + m(2x_Q) + (2y_Q - 1) = 0, \; b = y_Q - mx_Q$$

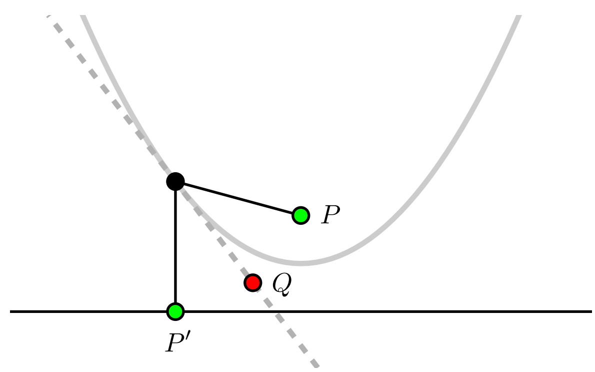
from which we see that

$$m = -x_Q \pm \sqrt{x_Q^2 + 1 - 2y_Q}\,.$$

This has two real roots as long as

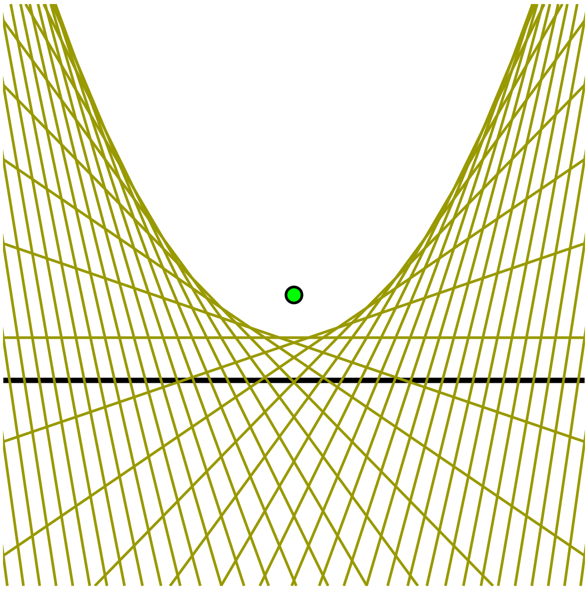
$$y_Q < (x_Q^2 + 1)/2\,,$$

or, equivalently, (x_Q, y_Q) lies outside the parabola $y = (x^2 + 1)/2$.



Fold lines are tangents to the parabola.

Geometrically, finding the line $y = mx + b$ that we are looking for amounts to finding a line through Q tangent to this parabola, which is both the envelope of all these lines and the parabola with focus P and directrix ℓ.



The parabola is the envelope of the fold lines.

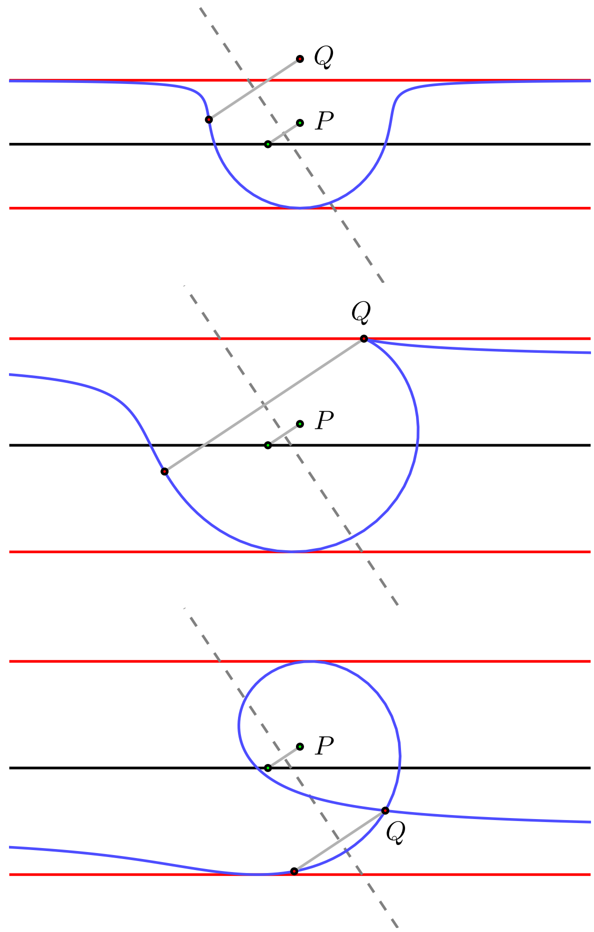
Since every point exterior to this parabola lies on two tangents, we can make more precise the construction stated somewhat imprecisely earlier: *Let P be a point and ℓ a line not containing P. If Q is a point not inside the parabola with focus P and directrix ℓ, suppose m to be a line through Q tangent to that parabola. Folding along m, taking P to a point on ℓ, is an allowable origami operation that, in effect, constructs m.*

It now becomes plausible that origami can calculate square roots, establishing that it is at least as potent as straight edge and compass.

Another allowable origami construction is more characteristic of origami, and more capable than any available by means of straight edge and compass. (4) **Suppose P and Q to be distinct points, ℓ and m two lines with P not on ℓ, Q not on m. The new operation folds P onto ℓ, Q onto m, in effect constructing the fold line, when this is possible.**

As before, we must answer some questions: *When is this operation possible? To what extent is it unique?* To answer, we first look at all possible operations folding P onto ℓ. This is simple, because if P' is a point on ℓ, then the axis of reflection taking P to P' must be the perpendicular bisector of PP'. Let that reflection be $\sigma_{P'}$. If we are given a further point Q and line m, we are reduced to asking: *Does one of the reflections $\sigma_{P'}$ take Q to m? If so, how many such reflections are there?* Another, equivalent, way to pose this question: let C be the image of Q with respect to the transformations $\sigma_{P'}$ as P' varies along ℓ. *Does C intersect m? In how many points?*

The following figures suggest what happens. In them, we fix $P = (0,1)$ and ℓ equal to the x-axis, which we may do without loss of generality, and plot the image of various points Q with respect to the various $\sigma_{P'}$. The red horizontal lines are those at distance $d(P,Q)$ from ℓ. The image must lie between them, since reflections are isometries.



The images of various fold maps.

The formula for the bisector of the segment between $(0,1)$ and $(t,0)$ is

$$y = tx + (1-t^2)/2$$

and that for $\sigma_{(t,0)}$ takes (x,y) to

$$(x,y) - 2\left(\frac{y - tx - (1/2)(1-t^2)}{t^2+1}\right)[-t,1].$$

As $y \to \pm\infty$ this tends asymptotically to the line at height $y - 1$. The image is therefore intersected by any line m with slope $\neq 0$. As for lines with slope $= 0$, the image spans the entire closed range $|y| \leq d(P,Q)$ except (as the first picture suggests) when Q lies on the y-axis. In all cases where m intersects ℓ there exists at least one point on m to which Q is mapped by some $\sigma_{P'}$. On the other hand, if m is parallel to ℓ then we must assume that $d(\ell,m) \leq d(P,Q)$ in order for the construction to be possible, and if $d(P,Q) = d(\ell,m)$ then m must be on the side of P opposite to Q. In all cases, there are only a finite number of possibilities. It turns out that finding these explicitly amounts to solving a cubic equation.

So, given our starting configuration and the rules so far given for constructing points and lines, what more can we construct?

• By method (1), from the three lines $x = 0, y = 0$, $x + y = 1$ we can construct the three points $(0, 0)$, $(1, 0)$, $(0, 1)$. This configuration of points and lines guarantees that every line constructed has at least two (constructed) points on it, and also that there exists at least one point not on it.

• We already know how to construct perpendiculars. Using this construction twice, we see that given a point P not on ℓ we can construct the line parallel to ℓ through P.

• Given a line ℓ and a point P, we can construct the reflection of P in ℓ—i.e., find an allowable fold along a line intersecting the perpendicular to ℓ through P.

• With these procedures in hand, we can now do anything straight edge and compass can do.

• More interesting, we can trisect angles.

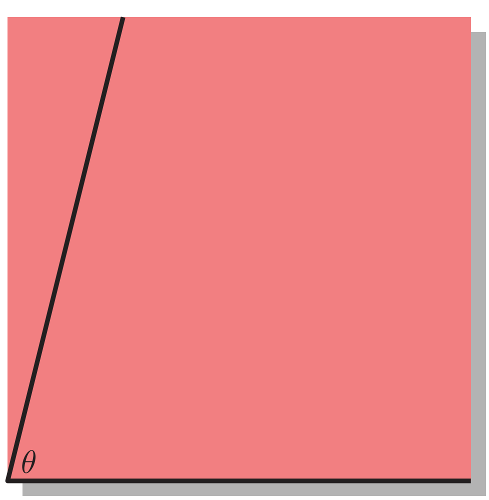
Since

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

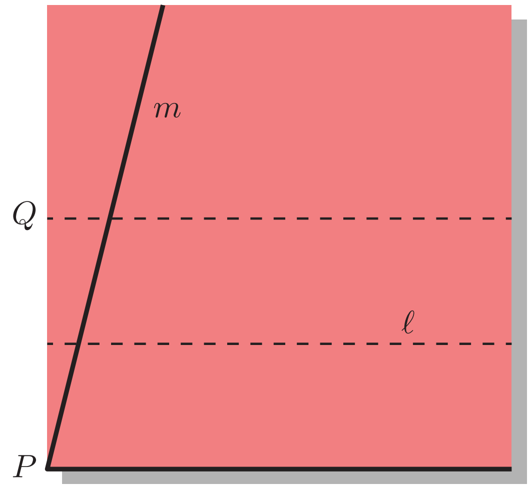
trisecting an angle is equivalent to solving a cubic equation. Activity 5 of Hull's book (see review of *Project Origami* on previous pages) explains how to construct $\sqrt[3]{2}$ (thus doubling the cube), and Activity 6 how to construct a root of an arbitrary cubic equation $x^3 + ax + b = 0$.

References

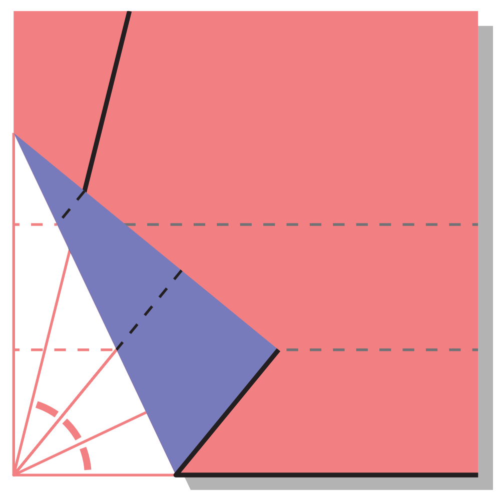
[1] David Cox, *Galois Theory*, John Wiley and Sons, 2004. Chapter 10 is concerned with geometric constructions, and §10.3 is concerned with origami.



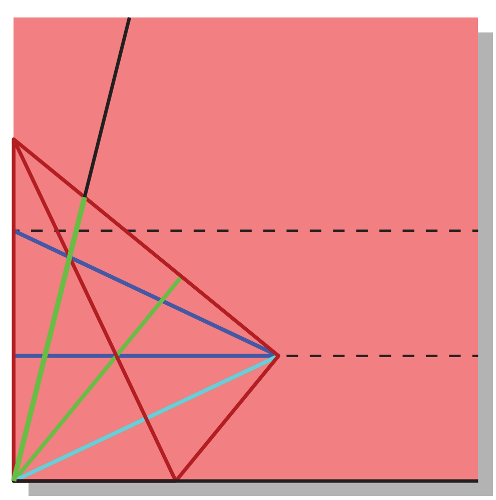
Trisection: 1. Start with an angle θ.



2. Construct two uniformly spaced horizontal lines.



3. Fold P and Q to ℓ and m.



4. The line from the corner to its reflection trisects the angle.