Two Problems That Shaped a Century of Lattice Theory

George Grätzer

A century of lattice theory was shaped to a large extent by two problems. This introductory article defines the basic concepts, introduces these two problems, and describes their effect on lattice theory.

In Parts 1 and 3 there is a very brief introduction of the basic concepts. The reader may find a more detailed introduction in Part 1 of my 2006 book, *The Congruences of a Finite Lattice* [8], and complete coverage of the topic in my 1998 book, *General Lattice Theory,* second edition [7].

The two areas we discuss are

Uniquely complemented lattices: discussed in Part 2.

Congruence lattices of lattices: discussed in Part 4.

The two problems, the personalities, and the times are completely different for the solution of these two problems. But they also share a lot. Each problem has been around for half a century. It is not clear—or known—who first proposed the problem. Nevertheless, everybody knows about it. Everybody expects a positive solution. And then somebody overcomes the psychology of the problem and pushes really hard for a negative solution. However, the negative solution turns out to require groundbreaking new ideas and is technically very complicated.

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Part 1—Lattice Theory 101

Basic Concepts

Orders. An *order* $\mathbf{A} = \langle A, \leq \rangle$ (or A if $\leq$ is understood) consists of a nonempty set A and a binary relation $\leq$ on A (that is, a subset of A^2)—called an *ordering*—such that the relation $\leq$ is reflexive ($a \leq a$, for all $a \in L$), antisymmetric ($a \leq b$ and $b \leq a$ imply that $a = b$, for all $a, b \in L$), and transitive ($a \leq b$ and $b \leq c$ imply that $a \leq c$, for all $a, b, c \in L$). An order that is *linear* ($a \leq b$ or $b \leq a$, for all $a, b \in L$) is called a *chain*.

In an order P, the element u is an *upper bound* of $H \subseteq P$ iff $h \leq u$, for all $h \in H$. An upper bound u of H is the *least upper bound* of H iff, for any upper bound v of H, we have $u \leq v$. We shall write $u = \bigvee H$. The concepts of *lower bound* and *greatest lower bound* (denoted by $\bigwedge H$) are similarly defined. We use the notation $a \wedge b = \bigwedge\{a, b\}$ and $a \vee b = \bigvee\{a, b\}$ and call $\wedge$ the *meet* and $\vee$ the *join* of the elements a and b.

Lattices. An *order* $\mathbf{L}$ (or L if the $\wedge$ and $\vee$ are understood) *is a lattice* iff $a \wedge b$ and $a \vee b$ always exist. In lattices, the join and meet are both *binary operations*, which means that they can be applied to a pair of elements a, b of L to yield again an element of L. They are *idempotent* ($a \wedge a = a$, $a \vee a = a$, for all $a \in L$), *commutative*

($a \wedge b = b \wedge a$, $a \vee b = b \vee a$, for all $a, b \in L$), *associative* ($(a \wedge b) \wedge c = a \wedge (b \wedge c)$, $(a \vee b) \vee c = a \vee (b \vee c)$, for all $a, b, c \in L$), and together satisfy the *absorption identities* ($a \wedge (a \vee b) = a$, $a \vee (a \wedge b) = a$, for all $a, b \in L$).

An *algebra* $\mathbf{L} = \langle L, \wedge, \vee \rangle$ *is a lattice* iff L is a nonempty set; $\wedge$ and $\vee$ are binary operations on L; both $\wedge$ and $\vee$ are idempotent, commutative, and associative; and they jointly satisfy the two absorption identities.

It is the most intriguing aspect of lattice theory that lattices can be viewed as orders, so we can use order-theoretic concepts (such as completeness; see Part 3); and they are also algebras, so we can use algebraic concepts (such as free lattices).

It is easy to see that a lattice as an algebra and a lattice as an order are "equivalent" concepts. Starting with a poset $\mathbf{L} = \langle L, \leq \rangle$ which is a lattice, set $\mathbf{L}^a = \langle L, \wedge, \vee \rangle$; then $\mathbf{L}^a$ is a lattice. Starting with an algebra $\mathbf{L} = \langle L, \wedge, \vee \rangle$ which is a lattice, set $a \leq b$ iff $a \wedge b = a$; then $\mathbf{L}^p = \langle L, \leq \rangle$ is an order, and the order $\mathbf{L}^p$ is a lattice. In fact, for an order $\mathbf{L} = \langle L, \leq \rangle$ which is a lattice, $(\mathbf{L}^a)^p = \mathbf{L}$; and for an algebra $\mathbf{L} = \langle L, \wedge, \vee \rangle$ which is a lattice, $(\mathbf{L}^p)^a = \mathbf{L}$.

If K and L are lattices as algebras and φ maps K into L, then we call φ a *homomorphism* if $(a \vee b)\varphi = a\varphi \vee b\varphi$ and $(a \wedge b)\varphi = a\varphi \wedge b\varphi$. If the map is one-to-one and onto, it is called an *isomorphism*. If φ is one-to-one, it is called an *embedding*. If K and L are lattices as orders and φ maps K into L, then we call φ an *isomorphism* if it is one-to-one and onto and $a \leq b$ iff $a\varphi \leq b\varphi$. Note that the two isomorphism concepts are equivalent.

Semilattices. An *algebra* $\langle L, \wedge \rangle$ *is a meet-semilattice* iff L is a nonempty set; $\wedge$ is a binary operation on L; and $\wedge$ is idempotent, commutative, and associative. We can introduce meet-semilattices as orders and establish the equivalence of the two approaches as we did for lattices. Similarly, we can define a join-semilattice. A lattice is a meet-semilattice and a join-semilattice defined on the same set that jointly satisfy the two absorption identities.

Examples.

- All subsets of a set, ordered under inclusion; meet is intersection, and join is union.
- All closed subspaces of a topological space, ordered under inclusion; meet is intersection, and join is the closure of the union.
- All continuous functions on the real $[0, 1]$ interval, ordered componentwise.
- All subgroups of a group, ordered under inclusion; meet is intersection, and join the subgroup generated by the union. Similarly, for normal subgroups of a group, ideals of a ring.
- All subspaces of a geometry ordered under inclusion; meet is intersection, and join is the subspace spanned by the union.

And of course everybody knows Boolean algebras (lattices) from logic.

Let me give one more example. An *equivalence relation* ε on a set X is a reflexive, *symmetric* ($a \leq b$ iff $b \leq a$, for all $a, b \in L$), and transitive binary relation. If x and y are in relation ε, that is, $\langle x, y \rangle \in \varepsilon$, we write $x \; \varepsilon \; y$ or $x \equiv y \;\; (\varepsilon)$. On the set Part X of all equivalence relations on X, we can introduce an ordering: $\varepsilon_1 \leq \varepsilon_2$ if ε_1 is a refinement of ε_2; that is, $x \; \varepsilon_1 \; y$ implies that $x \; \varepsilon_2 \; y$. Then Part X is a lattice, called the *partition lattice* of X. Clearly, $\langle x, y \rangle \in \varepsilon_1 \wedge \varepsilon_2$ iff $\langle x, y \rangle \in \varepsilon_1$ and $\langle x, y \rangle \in \varepsilon_2$. The join, however, is more complicated to describe.

Diagrams. In the order P, *a is covered by b* (in notation, $a \prec b$) iff $a < b$ and $a < x < b$ holds for no x. The covering relation, $\prec$, determines the ordering in a finite order.

The *diagram* of an order P represents the elements with small circles. The circles representing two elements x, y are connected by a straight line iff one covers the other: if x is covered by y, then the circle representing y is higher than the circle representing x. Three examples are shown in Figure 1: the two-element chain, the four-element Boolean lattice, and the partition lattice on a four-element set.

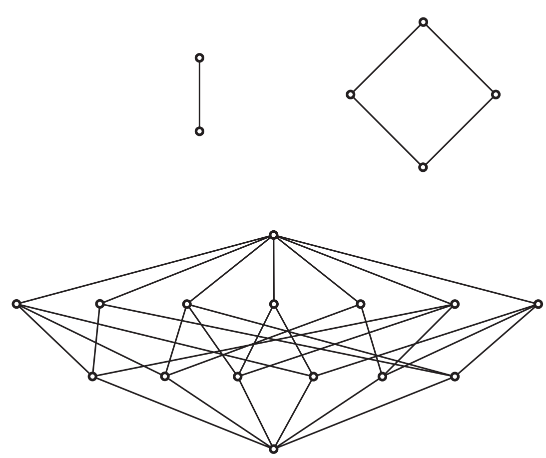

Figure 1. Three lattice diagrams.

Distributive Lattices. A lattice L is *distributive* if

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

holds for all $x, y, z \in L$; that is, $\wedge$ distributes over the $\vee$. It is an equivalent condition that $\vee$ distributes over the $\wedge$. *Modularity* is the same, except that we require the identity to hold only for $x \geq z$.

Distributive lattices are easy to construct. Take a set A and a nonempty collection L of subsets of A with the property that if $X, Y \in L$, then $X \cap Y$, $X \cup Y \in L$ (*a ring of sets*). Then L is a distributive lattice, since $\cap$ distributes over $\cup$. It is a result of G. Birkhoff, 1933, that the converse holds: *every distributive lattice is isomorphic to a ring of sets.*

A lattice has a *zero element*, 0, if $0 \leq x$, for all $x \in L$. A lattice has a *unit element,* 1, if $x \leq 1$, for all $x \in L$. A lattice is *bounded* if it has a zero and a unit. A bounded lattice L is *complemented* if for every $x \in L$, there is a $y \in L$ with $x \wedge y = 0$ and $x \vee y = 1$. A *Boolean lattice* is a distributive complemented lattice. Of course, all subsets of a set, ordered under inclusion, make up a Boolean lattice; the complement is the set complement.

Now let B be a Boolean lattice, and let y and z both be complements of the element x. Then $y = y \wedge 1 = y \wedge (x \vee z) = (y \wedge x) \vee (y \wedge z) = 0 \vee (y \wedge z) = y \wedge z$. Similarly, $z = y \wedge z$, and so $y = z$. A Boolean lattice is *uniquely complemented.* So we can consider a Boolean lattice B a *Boolean algebra* $\langle B, \wedge, \vee, ' \rangle$, where $'$ is a unary operation and a' is the (unique) complement of a.

How Lattice Theory Started

Garrett Birkhoff was the founder of modern lattice theory with his 1940 book *Lattice Theory* [1] and with the most influential second edition [2] in 1948 and the third edition [3] in 1967. The first edition was built on a large body of work in the mid- and late 1930s published by him, R. P. Dilworth, O. Frink, J. von Neumann, O. Ore, S. Mac Lane, and others.

The concept of a lattice comes from two sources. The first source is usually cited as R. Dedekind's two classic papers, 1897 and 1900. However, by tracing back the references in these, one can see that R. Dedekind was thinking (modular) lattice-theoretically for at least twenty years prior to that. R. Dedekind took notes at Dirichlet's lectures on number theory and wrote them up as a book with eleven "Supplements", which went through various revisions in the editions of 1863, 1871, 1879, and 1893. Section 169 in Supplement XI of the 1893 edition is about lattices, including the axioms, modular law, duality, and the free modular and distributive lattices on three generators—all developed as properties of modules and ideals. Furthermore, R. Dedekind points out that the lattice terminology (and the modular law) were already in an 1877 paper.

In his 1897 paper, R. Dedekind notes that general lattices were treated by E. Schröder in his famous book *Algebra der Logik* (1880, reprinted in English in 1966) and that this had led him to consider nonmodular lattices.

E. Schröder introduced—but did not name—lattices as orders exactly as we did above, of course, with different notation. There was a well-publicized debate in which C. S. Peirce claimed that all lattices were distributive, but counterexamples were provided by A. Korselt, 1894, R. Dedekind, and E. Schröder. Finally C. S. Peirce explained in a footnote in E. V. Huntington, 1904, that by a lattice he meant something somewhat different.

This debate had a profound effect. While discussing one of E. Schröder's axiom systems for Boolean algebras, E. V. Huntington, 1904, reproduced C. S. Peirce's proof, showing that distributivity can be derived from Schröder's axioms. Then he added the problem which we will state in the next section.

Part 2—Uniquely Complemented Lattices

The Problem

E. V. Huntington, 1904, stated the following problem:

Problem. *Is every uniquely complemented lattice distributive?*

Two series of papers appeared which strengthened the belief that the answer is a resounding YES. The first series was published by a reasonably large group of mathematicians interested in the axiomatics of Boolean algebras. They proved theorems of the type that if we make the complementation just a bit special, we get Boolean algebras. For instance, E. V. Huntington, 1904, published the result that if $\langle B, \wedge, \vee, ' \rangle$ is a uniquely complemented lattice with the property:

$$x \wedge y = 0 \text{ implies that } y \leq x',$$

then $\langle B, \wedge, \vee, ' \rangle$ is a Boolean algebra.

The second series of papers added a condition (P) to the lattices under consideration and concluded that a uniquely complemented lattice satisfying (P) is distributive. We call such properties *Huntington Properties.* The first may have been stated by J. von Neumann and G. Birkhoff with "P = modular". So the result is:

Theorem 1. *A uniquely complemented modular lattice is distributive.*

Here are some examples up to 1945:

- G. Bergman, 1929, "P = uniquely relatively complemented" (for all $a \leq b \leq c$, there is a unique d with $b \wedge d = a$ and $b \vee d = c$).
- G. Birkhoff and M. Ward, 1939, "P = complete and atomic" (a lattice L is *complete* if $\bigwedge X$ and $\bigvee X$ exist for any subset X of L; a lattice L is *atomic* if L has a 0 and for every $a \in L$, $0 < a$, there exists an element p such that $p \leq a$ and p is an atom, that is, if $0 \prec p$).
- R. P. Dilworth, 1940, re-proving G. Birkhoff and M. Ward for "P = finite dimensional".

Dilworth's Bombshell. In 1945, R. P. Dilworth announced a negative solution: there is a nondistributive uniquely complemented lattice. But what he published was so much more:

Theorem 2 (The Dilworth Theorem for Uniquely Complemented Lattices)**.** *Every lattice can be embedded in a uniquely complemented lattice.*

Dilworth was not very clear about the origin of the problem: "For several years one of the outstanding problems of lattice theory has been…" G. Birkhoff in *Mathematical Reviews* wrote: "It has been widely conjectured that…" Neither gives any references as to the origin of the problem. G. Birkhoff and M. Ward, 1933, reference E. V. Huntington, 1904, for the lattice axioms, which Huntington stated as being due to E. Schröder, but not for the problem. If the reader is surprised, I suggest he try to read the original paper of E. V. Huntington, and there he may find the clue. In my earlier papers on the subject, I reference only R. P. Dilworth, 1945, but in my lattice books (e.g., [7]) I give the correct reference. But I have no recollection of reading E. V. Huntington, 1904, until the preparation for this article.

The Solutions

Dilworth's Solution. After a year of attempting to obtain an affirmative solution, R. P. Dilworth decided to construct a counterexample. His paper, providing the result, has four sections.

Section 1 describes the free lattice generated by an order. To illustrate the concept, let us start with the order $P = \{a, b\}$, where a and b are *incomparable* (both $a \leq b$ and $b \leq a$ fail). Then we form $a \wedge b$ and $a \vee b$ and get the first lattice of Figure 2. There are some things to prove: for instance, why is it that we did not have to add the element $a \vee ((a \vee b) \wedge (a \wedge b))$? Answer: because the lattice axioms force that $a \vee ((a \vee b) \wedge (a \wedge b)) = a$.

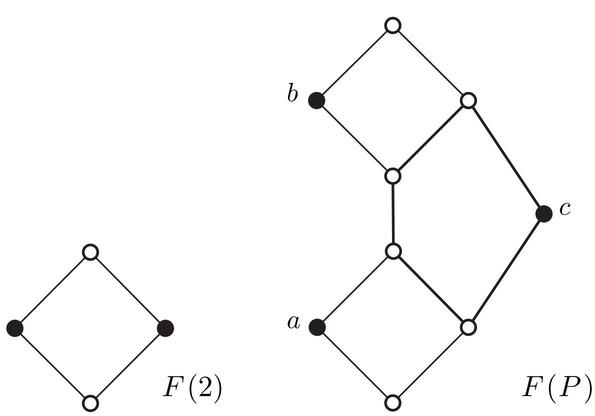

Figure 2. Two free lattices generated by orders.

Next take the order $P = \{a, b, c\}$, where $a < b$. Let us start by adding joins. Since $a \vee b = b$, we add only $b \vee c$ and $a \vee c$; since $\vee$ is idempotent, we do not have to bother with $x \vee x$. So now we have five elements: a, b, c, $a \vee c$, $b \vee c$, and we start forming meets; we get three new elements: $b \wedge (a \vee c)$, $b \wedge c$, and $a \wedge c$. Now we are back to joins, and we find that we get only one new element: $a \vee (b \wedge c)$. It is now an easy computation to show that the lattice axioms imply that this set of elements is closed under join and meet. We obtain the second lattice of Figure 2, the *free lattice generated by* P. A typical step in the verification is to prove that $((a \vee (b \wedge c)) \vee c = a \vee c$.

R. P. Dilworth starts with an order P, forms the *lattice polynomials*, and then describes when $p \leq q$ is forced by the lattice axioms, for the lattice polynomials p and q. Then introducing $p \equiv q$ if $p \leq q$ and $q \leq p$, he shows the equivalence classes form the free lattice over P.

To freely generate a uniquely complemented lattice, Dilworth needs a unary operation. He denotes it by *, and instead of lattice polynomials, he has to form *operator polynomials,* such as $((a \vee (b^* \wedge c)) \vee b$. He then describes when the equivalence of two operator polynomials p and q is forced by the lattice axioms. This is the hardest part of the paper: Section 2, the main result, needs twenty-three steps, some really technical and ingenious.

So now we have the free lattice with an operator, but of course it is no good, because $(p^*)^*$ is never (almost never) p, as it would be in a uniquely complemented lattice. So in Section 3 Dilworth comes up with a brilliant idea. Let N be the set of all operator polynomials that have *no subpolynomial of the form* $(p^*)^*$. It is easily seen that N defines a sublattice of the free operator lattice, but clearly N is not closed under *. So here is the idea: define a unary operation, $'$, on N. If p, $p^* \in N$, then $p' = p^*$. If $p^* \notin N$, then he gives a clever inductive definition of p' so that $p' \in N$. The main part of this section is this idea. To prove that it works is not that difficult. The heavy lifting was done in Section 2.

Section 4 deals with the free uniquely complemented lattices. The problem is that in the free algebra constructed in Section 3, we do not have $p \wedge p' = 0$ and $p \vee p' = 1$. The bad polynomials are those that contain (or are contained in) a p and p'. So here is the final idea: take those polynomials that have no such bad subpolynomials. It then takes only two pages to compute that these polynomials along with 0 and 1 define the free uniquely complemented lattice over P.

Newer Solutions. Dean's Theorem, 1964, extends Dilworth's free lattice generated by an order P to the free lattice generated by an order P *with any number of designated joins and meets;* we require that they be preserved. Dean's proof is the same complicated induction as the one in R. P. Dilworth, 1945. A greatly simplified proof can be found in H. Lakser's Ph.D. thesis, 1968; see also H. Lakser, 2007, manuscript.

It was around 1966 when I finished my book on universal algebra (it was published in 1968) and I started working on a book on lattice theory. It was clear to me that such a book should contain a proof of the Dilworth Theorem for Uniquely Complemented Lattices, but it was also clear that I would need a lattice-theoretic proof. And the most substantial parts of the original proof (Sections 2 and 3) are well beyond the reach of a book on lattices.

So D. Kelly, H. Lakser, C. Platt, J. Sichler, and I started systematically to work on free lattices and free products. The most important influence of R. P. Dilworth's paper came from the motivation and the ideas of Section 1, in particular the concept of covering. So when C. C. Chen came to Winnipeg in 1967, we had already developed a good understanding of this field.

With C. C. Chen our goal was to produce a proof of the Dilworth Theorem for Uniquely Complemented Lattices for my book. This we accomplished in C. C. Chen and G. Grätzer, 1969. We proved that the Dilworth Theorem for Uniquely Complemented Lattices can be proved with the ideas of Sections 1 and 4 of Dilworth's paper, completely eliminating Sections 2 and 3, which are very complicated and not lattice-theoretic. The proof is in two steps: the first is based on Section 1, and the second is based on Section 4 of R. P. Dilworth, 1945.

Actually, the result we proved is much stronger than the Dilworth Theorem for Uniquely Complemented Lattices. Let us call a lattice L *almost uniquely complemented* if it is bounded and every element has at most one complement. A $\{0,1\}$-*embedding* is an embedding that maps the 0 to 0 and the 1 to 1.

Theorem 3. *Let L be an almost uniquely complemented lattice. Then L can be $\{0,1\}$-embedded into a uniquely complemented lattice.*

R. P. Dilworth's embedding preserves no existing complement.

The method employed in C. C. Chen and G. Grätzer, 1969, was generalized to ($\mathcal{C}$-reduced) free products in G. Grätzer, 1971 and 1973, with some interesting applications in G. Grätzer and J. Sichler, 1970, 1974, 2000.

A completely different approach was taken by M. Adams and J. Sichler, 1978. They introduced *testing lattices* that allowed them to construct continuumly many such varieties (classes defined by identities) in which the Dilworth Theorem for Uniquely Complemented Lattices holds. See also V. Koubek, 1984.

Newest Solution. As opposed to the two steps of C. C. Chen and G. Grätzer, 1969, in G. Grätzer and H. Lakser, 2006, we provide a one-step solution.

Let K be a bounded lattice. Let $a \in K - \{0,1\}$, and let u be an element not in K. We extend the partial ordering $\leq$ of K to $Q = K \cup \{u\}$ by $0 \leq u \leq 1$. We extend the lattice operations $\wedge$ and $\vee$ to Q as *commutative partial meet and join operations*. For $x \leq y$ in Q, define $x \wedge y = x$ and $x \vee y = y$. In addition, let $a \wedge u = 0$ and $a \vee u = 1$; see Figure 3.

To construct and describe the lattice $F(Q)$ freely generated by Q, we repeatedly form joins and meets of elements of Q, obtaining the *polynomials* over Q, which will represent elements of $F(Q)$. For the polynomials A and B over Q, let $A \leq B$ denote the relation forced by the lattice axioms and the structure of Q. We observe that given any polynomial A, there is a largest element A_* of K with $A_* \leq A$ and a smallest element A^* of K with $A^* \geq A$.

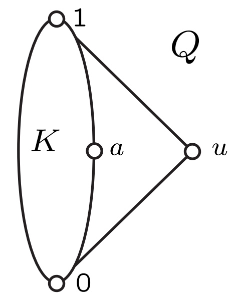

Figure 3. The partial lattice Q.

An easy computation (less than a page) shows that the following statements hold:

(i) *$u \leq u$. If $x \in K$, then $u \leq x$ iff $x = 1$.*
(ii) *$u_* = 0$. If $x \in K$, then $x_* = x$.*
(iii) *$u \leq A \wedge B$ iff $u \leq A$ and $u \leq B$.*
(iv) *$(A \wedge B)_* = A_* \wedge B_*$.*
(v) *$u \leq A \vee B$ iff either $u \leq A$ or $u \leq B$ or $A_* \vee B_* = 1$.*
(vi)
$$(A \vee B)_* = \begin{cases} 1 & \text{if } a \leq A_* \vee B_* \text{ and} \\ & \quad \text{either } u \leq A \text{ or } u \leq B; \\ A_* \vee B_* & \text{otherwise.} \end{cases}$$

The only complement of u is a.

Let A be a polynomial that defines a complement of u. Then
$$1 = (A \vee u)_* = \begin{cases} 1 & \text{if } a \leq A_* \vee u_* = A_*; \\ A_* & \text{otherwise.} \end{cases}$$
So either $a \leq A_*$ or $1 = A_*$; in either case, $a \leq A_*$. Dually, $a \geq A^*$. Thus
$$A \leq A^* \leq a \leq A_* \leq A,$$
and so $A \equiv a$.

Isn't this easy?

It is also easy to show that if K is almost uniquely complemented, then the only other complemented pairs in $F(Q)$ are the complemented pairs in K. Thus if a does not have a complement in K, we get an almost uniquely complemented $\{0,1\}$-extension in which a has a complement. By transfinite induction on the set of noncomplemented elements of K, we get an almost uniquely complemented $\{0,1\}$-extension K_1 of $K_0 = K$, where each element of K_0 has a complement. Then, by a countable induction, we get a uniquely complemented $\{0,1\}$-extension K_ω of $K_0 = K$.

This method has applications that previous techniques could not give (previous techniques can be used only to construct *complements*, not *relative complements*).

Theorem 4. *Let $[a,c] = \{x \in K \mid a \le x \le c\}$ be an interval of a lattice K. Let us assume further that every element in $[a,c]$ has at most one relative complement. Then K has an extension L such that $[a,c]$ in L, as a lattice, is uniquely complemented.*

See G. Grätzer and H. Lakser, 2005. There are many variants of the stronger results; we give only one more example. Let us say that a *bounded lattice* has *transitive complementation* if whenever b is a complement of a and c is a complement of b, either $a = c$ or c is a complement of a. Let us call a lattice L *n-complemented* if every element $a \ne 0, 1$ has *exactly n* complements. Similarly, a lattice L is *at most n-complemented* if every element $a \ne 0, 1$ has *at most n* complements; example: any at most uniquely complemented lattice. For instance, in a transitively 2-complemented lattice L, every element $a \ne 0, 1$ belongs to a (unique) sublattice M_3 (see Figure 4) so that every complement of a is in this sublattice. Then we have (G. Grätzer and H. Lakser, manuscript):

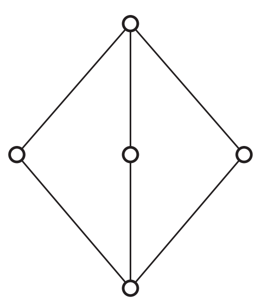

Figure 4. The lattice M_3.

Theorem 5. *Let K be an at most n-complemented and transitively complemented lattice. Then K can be $\{0,1\}$-embedded into a transitively n-complemented lattice.*

For $n = 1$, this gives the C. C. Chen and G. Grätzer, 1969, result and so, in turn, the Dilworth Theorem for Uniquely Complemented Lattices.

Huntington Properties and Varieties

We have seen a few examples of Huntington Properties. Among the structural properties, one of the nicest is from H.-J. Bandelt and R. Padmanabhan, 1979: *every interval contains a covering pair.*

A *Huntington Variety* is a lattice variety in which every uniquely complemented lattice is distributive. W. McCune, R. Padmanabhan, and B. Veroff, 2007, observed that there are continuumly many such varieties.

The forthcoming book of R. Padmanabhan and S. Rudeanu [9] has a chapter on Huntington Varieties. Here is an intriguing result from the book:

Theorem 6. *The variety* $\mathbf{M} \vee \mathbf{N}_5$ *is Huntington.*

$\mathbf{M} \vee \mathbf{N}_5$ is the smallest variety containing all modular lattices and the 5-element nonmodular lattice N_5.

We conclude this section with one more Huntington Property, an identity, from W. McCune, R. Padmanabhan, and B. Veroff, 2007:

Theorem 7. *Every uniquely complemented lattice satisfying*

$$x \wedge ((y \wedge (x \vee z)) \vee (z \wedge (x \vee y))) = (x \wedge y) \vee (x \wedge z)$$

is distributive.

Concluding Comments

It is interesting how little we know about a subject on which we have published so many papers. Ask any question and probably we do not know the answer. Let me mention a (very) few of my favorite ones.

All known examples of nondistributive uniquely complemented lattices are freely generated, one way or another. Is there a construction of a nondistributive uniquely complemented lattice that is different?

In the same vein, is there a "natural" example of a nondistributive uniquely complemented lattice from geometry, topology, or whatever else?

Is there a complete example of a nondistributive uniquely complemented lattice?

Part 3—Lattice Theory 201

Subalgebra Lattices of Algebras

Let $\mathbf{A} = \langle A, F\rangle$ be an *algebra*; that is, A is a nonempty set and every $f \in F$ is a finitary operation on A. If F is understood, we denote the algebra by A. In the case of lattices as algebras, $F = \{\wedge, \vee\}$, and both operations are binary. Boolean algebras are usually defined as algebras with $F = \{\wedge, \vee, ', 0, 1\}$, where $\wedge$ and $\vee$ are binary operations, $'$ is a unary operation, while 0 and 1 are nullary operations.

A *subalgebra* $B \subseteq A$ is a nonempty subset closed under all the operations in F: that is, if $f \in F$ is n-ary and $b_1, \ldots, b_n \in B$, then $f(b_1, \ldots, b_n)$ formed in A is in B. So we can regard $\mathbf{B} = \langle B, F\rangle$ as an algebra, a subalgebra of $\mathbf{A}$. The intersection of any set of subalgebras is a subalgebra again, provided that it is nonempty. This leads us to the following notation: $\operatorname{Sub} A$ is the set of subalgebras of A;

if the intersection of all subalgebras of A is the empty set, then we add $\emptyset$ to $\operatorname{Sub} A$.

$\operatorname{Sub} A$ under containment is an order, in fact, a *complete lattice*. $\operatorname{Sub} A$ is *meet-continuous*; that is,

$$X \wedge \bigvee \mathcal{C} = \bigvee (X \wedge C \mid C \in \mathcal{C}),$$

for any chain $\mathcal{C}$ in $\operatorname{Sub} A$.

For any nonempty subset X of A, there is a smallest subalgebra $[X]$ containing X, called the subalgebra generated by X. If a subalgebra B is of the form $[X]$, for a finite X, we call it a *finitely generated subalgebra*.

Following G. Birkhoff and O. Frink, 1948, in a complete lattice L, we call an element x *join-inaccessible* if whenever $x = \bigvee \mathcal{C}$ for a chain $\mathcal{C}$ in L, $x \in \mathcal{C}$. It is easy to see that finitely generated subalgebras are exactly the *join-inaccessible* elements of $\operatorname{Sub} A$. We thus arrive at the breakthrough definition of G. Birkhoff and O. Frink, 1948.

A lattice L is called *compactly generated* if it is complete, meet-continuous, and every element is a (complete) join of join-inaccessible elements. G. Birkhoff and O. Frink, 1948, proved the following result:

Theorem 8 (The Birkhoff-Frink Theorem)**.** *A lattice L is isomorphic to the subalgebra lattice of a finitary algebra iff it is compactly generated.*

Congruence Lattices of Algebras

Let A (that is, $\mathbf{A} = \langle A, F \rangle$) be an algebra. An equivalence relation Θ is called a *congruence relation* if it has the *Substitution Property* for all $f \in F$. If f is n-ary, the Substitution Property for f is the following:

$$a_1 \equiv b_1(\Theta), \ \ldots, a_n \equiv b_n(\Theta)$$
$$\text{imply that} \quad f(a_1, \ldots, a_n) \equiv f(b_1, \ldots, b_n)(\Theta).$$

For $a \in A$, we denote by a/Θ the Θ-class containing a; that is,

$$a/\Theta = \{\, b \in A \mid a \equiv b(\Theta) \,\}$$

and A/Θ is the set of all a/Θ, $a \in A$. On A/Θ we can define the operations $f \in F$ by

$$f(a_1/\Theta, \ldots, a_n/\Theta) = f(a_1, \ldots, a_n)/\Theta,$$

and we get the *quotient algebra* A/Θ (that is, $\mathbf{A}/\Theta = \langle A/\Theta, F \rangle$). This is how we construct quotient groups, quotient rings, quotient lattices, and so on.

Let A be an algebra and let $a, b \in A$. Since the intersection of congruences is a congruence again, there is a smallest congruence $\operatorname{con}(a, b)$ such that $a \equiv b$. We call $\operatorname{con}(a, b)$ a *principal congruence*; they correspond to one-generated subalgebras. Finite joins of principal congruences were called by G. Birkhoff and O. Frink *finitely generated congruences* (today, we call them *compact congruences*); they are like the finitely generated subalgebras.

G. Birkhoff and O. Frink, 1948, observed:

Theorem 9. $\operatorname{Con} A$*, the congruences of the algebra A, form a compactly generated lattice.*

They raised the problem whether the converse is true. The problem is also raised in [2]. Interestingly, neither references G. Birkhoff, 1945, where the problem is first raised for algebras finitary or infinitary.

I remember when we first started thinking about this problem with E. T. Schmidt. Suppose for a compactly generated lattice L that we construct the algebra A. It really bothered me that I did not know how to utilize the meet-continuity of L in proving that L is isomorphic to $\operatorname{Con} A$.

Variants of Compactly Generated Lattices

In the late 1950s it became clear to a number of mathematicians that there are two important variants of the definition of compactly generated lattices. To state them, we need some elementary concepts.

Let L be a complete lattice and $c \in L$. Let us call c *compact* (L. Nachbin, 1959) if whenever $c \leq \bigvee X$, for some $X \subseteq L$, then $c \leq \bigvee X_1$, for some *finite* $X_1 \subseteq X$. We define an *algebraic lattice* as a complete lattice in which every element is a join of compact elements.

Let S be a join-semilattice with zero. An ideal I of S is a subset with three properties:

(i) $0 \in L$.
(ii) If $a, b \in L$, then $a \vee b \in L$.
(iii) If $a \in L$ and $x \leq a$, then $x \in L$.

The set $\operatorname{Id} S$ of all ideals of S is an order under containment. It is a compactly generated lattice in which the join-inaccessible elements are the principal ideals: ${\downarrow} a = \{\, x \in S \mid x \leq a \,\}$, for $a \in S$.

Theorem 10. *The following conditions on a lattice L are equivalent:*

(i) *L is a compactly generated lattice.*
(ii) *L is an algebraic lattice.*
(iii) *L can be represented as the ideal lattice of a join-semilattice with zero.*

The equivalence of the first two conditions was observed by G. Birkhoff [3], while the equivalence of the last two conditions is in L. Nachbin, 1959. Further references: A. Komatu, 1943; R. Büchi, 1952; and G. Grätzer, 1965.

Theorem 10 is quite trivial, but note that the Birkhoff-Frink Theorem can, as a result, be proved in a few lines. Let L be compactly generated. Then $L = \operatorname{Id} S$ for a join-semilattice S with zero. Define on S the $\vee$, and for all $a, b \in S$ define the unary operation $f_{a,b}$ as follows:

$$f_{a,b}(x) = \begin{cases} a \wedge b, & \text{for } x = a; \\ 0 & \text{otherwise.} \end{cases}$$

Set $F = \{\vee\} \cup \{\, f_{a,b} \mid a, b \in S \,\}$. Then the subalgebras of $\langle S, F \rangle$ are exactly the ideals of S;

hence $\mathrm{Sub}\langle S, F\rangle = L$, proving the Birkhoff-Frink Theorem.

Part 4—Congruence Lattices of Lattices

Congruence Lattices of Algebras

G. Birkhoff and O. Frink, 1948, raised the question whether congruence lattices of algebras can be characterized as compactly generated lattices. This problem was earlier raised by G. Birkhoff in a 1945 lecture and again in [2] as Problem 50. This problem was solved in G. Grätzer and E. T. Schmidt, 1963. (In this part, we use the acronym CLCT for the mouthful "Congruence Lattice Characterization Theorem".)

Theorem 11 (CLCT for Algebras)**.** *The congruence lattice of an algebra can be characterized as an algebraic lattice.*

An equivalent formulation is: *Every join-semilattice with zero can be represented as* $\mathrm{Con_c}\,\mathbf{A}$*, the join-semilattice with zero of compact congruences of an algebra* $\mathbf{A}$*.*

For a long time I had the nightmare that somebody would come along and present a brief construction of the algebra and a few-lines-long proof the way we did for the Birkhoff-Frink theorem. Any takers?

More polished versions of the original proof appeared in G. Grätzer [6]; W. A. Lampe, 1968, 1973; E. T. Schmidt, 1969; E. Nelson, 1980; and P. Pudlák, 1985. Other variants by B. Jónsson and R. N. McKenzie were written up, circulated, but not published.

All these proofs, for larger algebraic lattices L, construct algebras $\langle A, F\rangle$ with more and more operations. This cannot be avoided. R. Freese, W. A. Lampe, and W. Taylor, 1979, proved that if we take the algebraic lattice of all subspaces of an infinite-dimensional vector space over an uncountable field of cardinality $\mathfrak{m}$ and represent it as the congruence lattice of an algebra $\langle A, F\rangle$, then $\mathfrak{m} \leq |F|$.

G. Birkhoff, in his 1945 lecture, raised the question of how we can characterize congruence lattices of not necessarily finitary algebras. The congruence lattice is obviously complete, but we no longer have meet-continuity or compact elements. This was solved in G. Grätzer and W. A. Lampe, 1979:

Theorem 12 (CLCT for Infinitary Algebras)**.** *The congruence lattice of an infinitary algebra can be characterized as a complete lattice.*

In the early 1980s, R. Wille raised a related question (mentioned in K. Reuter and R. Wille, 1987). How can we characterize the lattice of *complete congruences* of a complete lattice? Again, this lattice is obviously complete.

I resolved Wille's problem in G. Grätzer, 1989:

Theorem 13 (CLCT for Complete Lattices)**.** *The lattice of complete congruences of a complete lattice can be characterized as a complete lattice.*

Note that Theorem 12 immediately follows from this result.

A large number of papers (G. Grätzer, 1989, 1990; G. Grätzer and H. Lakser, 1991, 1992; R. Freese, G. Grätzer, and E. T. Schmidt, 1991; G. Grätzer and E. T. Schmidt, 1993 (three papers), 1995 (three papers)) obtained better results. I quote just one from G. Grätzer and E. T. Schmidt, 1993:

Theorem 14 (CLCT for Complete and Distributive Lattices)**.** *The lattice of complete congruences of a complete and distributive lattice can be characterized as a complete lattice.*

I believe that the complete and distributive lattice constructed to prove this theorem is a candidate for the most complicated complete and distributive lattice ever constructed.

Congruence Lattices of Finite Lattices

For a lattice L, the congruence lattice is distributive. This remarkable—but easy to prove—fact was published only in N. Funayama and T. Nakayama, 1942. About the same time R. P. Dilworth discovered—but did not publish—the even more remarkable converse:

Theorem 15 (The Dilworth Theorem for Finite Congruence Lattices)**.** *Every finite distributive lattice can be represented as the congruence lattice of a finite lattice.*

This result was made into an exercise (one with an asterisk, meaning difficult) in [2]. E. T. Schmidt and I got really interested in the result and inquired from G. Birkhoff where the result came from, but he did not know and encouraged us to write to R. P. Dilworth. Unfortunately, Dilworth was busy editing the proceedings of a lattice theory meeting, but eventually we got a response. Yes, he proved the result, and the proof was in his lecture notes. No, copies of his lecture notes were no longer available.

So we published a proof, G. Grätzer and E. T. Schmidt, 1963. In fact, we proved something much stronger. Let us call a lattice L with zero *sectionally complemented* if for every $a \leq b \in L$, there is a $c \in L$ with $a \wedge c = 0$ and $a \vee c = b$. This is what we proved:

Theorem 16 (CLCT for Finite Sectionally Complemented Lattices)**.** *Every finite distributive lattice can be represented as the congruence lattice of a finite sectionally complemented lattice.*

Proof Outline. An element a of a lattice D is *join-irreducible* if $a \neq 0$ and $a = x \vee y$ implies that $a = x$ or $a = y$. The join-irreducible elements of

D form an order $\mathrm{J}(D)$. If D is distributive, $\mathrm{J}(D)$ determines D and every finite order P is representable as the $\mathrm{J}(D)$ of a finite distributive lattice D. For a finite lattice L, let $\mathrm{Con_J}\,L$ denote the order of join-irreducible congruences.

A *chopped lattice* $\langle M, \wedge, \vee \rangle$ is a finite meet-semilattice $\langle M, \wedge \rangle$ in which $\vee$ is a *partial operation* so that whenever $a \vee b$ is defined it is $\bigvee\{a, b\}$. We can define ideals just as for join-semilattices with zero except that the second condition reads: *If a, $b \in I$ and $a \vee b$ exists, then $a \vee b \in I$.* The set $\mathrm{Id}\,M$ of all ideals of M is a lattice, and we can view M as a part of $\mathrm{Id}\,M$ by identifying $a \in M$ with $\downarrow a$.

Then we have (G. Grätzer and H. Lakser, 1968):

Theorem 17. *Every congruence of M has one and only one extension to $\mathrm{Id}\,M$. In particular, the congruence lattice of M is isomorphic to the congruence lattice of $\mathrm{Id}\,M$.*

So now we can rewrite the theorem we want to prove:

CLCT for Finite Lattices—order and chopped lattice version. *Every finite order P can be represented as $\mathrm{Con_J}\,M$ for a chopped lattice M.*

The basic gadget for the construction is the lattice $N_6 = N(p,q)$ of Figure 5. The lattice $N(p,q)$ has only one nontrivial congruence relation, Θ, where Θ is the congruence relation with congruence classes $\{0, q_1, q_2, q\}$ and $\{p_1, p(q)\}$, indicated by the dashed line. In other words, $p_1 \equiv 0$ "implies" that $q_1 \equiv 0$, but $q_1 \equiv 0$ "does not imply" that $p_1 \equiv 0$. We will use the $N_6 = N(p,q)$ to achieve such "congruence-forcing".

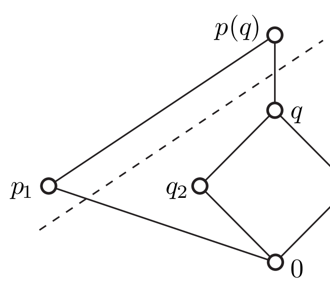

Figure 5. The lattice $N_6 = N(p,q)$ and the congruence Θ.

To convey the idea of the proof, we present two small examples in which we construct the chopped lattice M from copies of $N(p,q)$.

Let $P = \{a, b, c\}$ with $c \prec b \prec a$. We take two copies of the gadget, $N(a,b)$ and $N(b,c)$, and "merge" them to form the chopped lattice M, as shown in Figure 6.

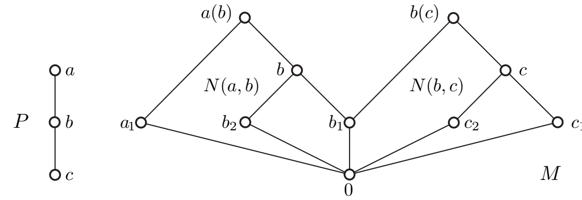

Figure 6. The chopped lattice M for $P = C_3$.

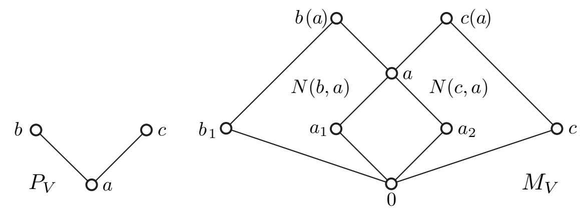

Figure 7. The chopped lattice for the order P_V.

Next consider the three-element order P_V of Figure 7. We take two copies of the gadget, $N(b,a)$ and $N(c,a)$, and "merge" them to form the chopped lattice M_V; see Figure 7.

The reader should now be able to picture the general proof. Instead of the few atoms in these examples, we start with M_0, which has enough atoms to reflect the structure of P. Whenever $b \prec a$ in P, we build a copy of $N(a,b)$ to obtain the chopped lattice M; see Figure 8.

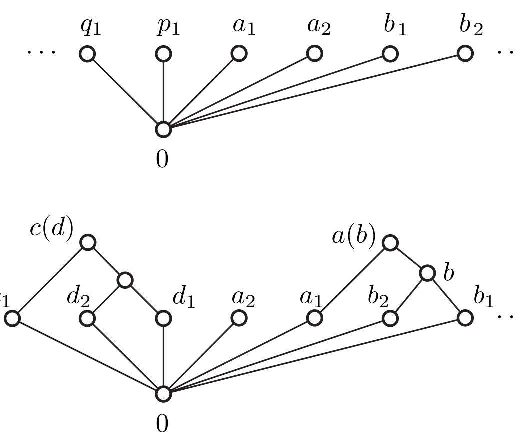

Figure 8. The chopped lattice M_0 and M.

Sectionally Complemented Lattices. Actually, the chopped lattices we construct are sectionally complemented. Unfortunately, the ideal lattice of a sectionally complemented chopped lattice is not necessarily sectionally complemented. We are not going to give the reason why it is sectionally complemented for this construction, but refer the reader to G. Grätzer and E. T. Schmidt, 1962; to the books [7] and [8]; and to the recent series of papers: G. Grätzer and H. Lakser, 2005 (two papers); G. Grätzer, H. Lakser, and M. Roddy, 2005; G. Grätzer and M. Roddy, 2007.

Other Special Properties for Finite Lattices. Let's see which properties (P) of finite lattices are such that

Every finite distributive lattice can be represented as the congruence lattice of a finite lattice with property (P).

So

P = sectionally complemented

is such. I refer the reader to my book [8] for a thorough survey of this field. Here is just one example. Let L be a finite lattice and let Θ be a congruence. We call Θ *isoform* if any two congruence classes of Θ are *isomorphic* as lattices. A lattice is *isoform* if all of its congruences are isoform. It was proved in G. Grätzer and E. T. Schmidt, 2003, that

P = isoform

is such a property. For stronger results on this property see G. Grätzer, R. W. Quackenbush, and E. T. Schmidt, 2004, and G. Grätzer and H. Lakser, two manuscripts.

The General Problem

Anybody familiar with the papers N. Funayama and T. Nakayama, 1942, and G. Birkhoff and O. Frink, 1948, would naturally raise the question:

Problem. *Can every distributive algebraic lattice L be represented as the congruence lattice of a lattice K?*

Surprisingly, this did not make it into G. Birkhoff and O. Frink, 1948, or [2]. When asked, Birkhoff and Frink in 1961 called it an oversight (personal communication). Certainly, R. P. Dilworth was aware of this problem. The first time it appeared in print was in G. Grätzer and E. T. Schmidt, 1962, but already in G. Grätzer and E. T. Schmidt, 1958, a partial positive solution is given. For sure, the second question raised in G. Grätzer and E. T. Schmidt, 1962, was new:

Problem. *Are further conditions on L necessary if we require K to be sectionally complemented?*

A join-semilattice with zero S is

(i) *distributive* if for all $a, b, c \in S$ with $c \le a \vee b$, there are $x \le a$ and $y \le b$ such that $c = x \vee y$;
(ii) *representable* if it is isomorphic to $\mathrm{Con}_c\, L$, for some lattice L.

Now we can state the semilattice formulation of the general problem:

Problem. *Is every distributive join-semilattice with zero representable?*

Positive Results. The first group of positive results started with two papers of E. T. Schmidt, 1968, 1981. To state Schmidt's results, we need some concepts.

A congruence Θ of a join-semilattice with zero S is *monomial* if any Θ-equivalence class has a largest element. A congruence of S is *distributive* if it is a join of monomial congruences.

A *generalized Boolean semilattice* is defined as the underlying join-semilattice of a sectionally complemented distributive lattice with zero. A join-semilattice with zero satisfies *Schmidt's Condition* if it is isomorphic to B/Θ for some distributive congruence Θ of a generalized Boolean semilattice B. One of the best results about the representability of distributive semilattices with zero is E. T. Schmidt, 1968:

Theorem 18. *Any semilattice with zero satisfying Schmidt's Condition is representable.*

Using this result, E. T. Schmidt, 1981, proved:

Theorem 19. *Every distributive lattice with zero is representable.*

In a recent paper, F. Wehrung, 2003, extended Schmidt's result:

Theorem 20. *Every countable union of distributive lattices with zero is representable.*

Surprisingly, Wehrung had to use methods inspired by set theory and forcing (Boolean-valued models) to prove this result. No elementary proof is known.

The second group of positive results is phrased in terms of the *cardinality* of the join-semilattice with zero. A. Huhn mentions in the introduction of his first 1989 paper (both Huhn papers of 1989 were prepared for publication by H. Dobbertin after A. Huhn passed away in 1985) that around 1980, H. Bauer proved a result (unpublished) implying the following:

Theorem 21. *Every countable distributive join-semilattice with zero is representable.*

An extended version of this result is proved by H. Dobbertin, 1986:

Theorem 22. *Every distributive join-semilattice with zero in which any principal ideal is countable is representable.*

A. Huhn, 1989, second paper, uses Schmidt's result to obtain:

Theorem 23. *Every distributive join-semilattice with zero of cardinality at most $\aleph_1$ is representable.*

An elementary proof of this result was published in G. Grätzer, H. Lakser, and F. Wehrung, 2000.

Wehrung's Bombshell

Last year, F. Wehrung announced that

Theorem 24. *There exists a distributive join-semilattice with zero of cardinality* $\aleph_{\omega+1}$ *that is not representable.*

The manuscript was widely circulated and is regarded by the experts as correct. In this section I outline how this semilattice is constructed. The reader should have no difficulty supplying the missing details. In the following two sections I discuss the background for this construction and provide some hints about the proof. These sections are a bit more technical than the rest of this exposition.

In this section we work with *partial join-semilattices*, defined as $\mathbf{P} = \langle P, \leq, 0, \vee \rangle$ such that $\langle P, \leq \rangle$ is an order; $0 \leq a$, for all $a \in P$; and $\vee$ is a partial binary operation such that if $a \vee b$ is defined, then $a \vee b$ is the least upper bound of a and b in the order $\langle P, \leq \rangle$; and if $a \leq b$ in $\langle P, \leq \rangle$, then $a \vee b = b = b \vee a$. The following is easy:

A partial join-semilattice P has a free extension $F(P)$ to a join-semilattice which contains P as a subsemilattice.

We start with the following partial join-semilattice. Let Ω and Ω' be disjoint sets, and let $' \colon \Omega \to \Omega'$ be a bijection. Let P_Ω be the disjoint union of $\{0, 1\}$, Ω, and Ω'. Define $0 < a$, $a' < 1$, for all $a \in \Omega$. Let $\mathbf{P}_\Omega = \langle P_\Omega, \leq, 0, \vee \rangle$ be the partial join-semilattice induced by defining $a \vee a' = 1$, for all $a \in \Omega$. Then $F(P_\Omega)$ is isomorphic to

$$\{\emptyset, \Omega \cup \Omega'\} \bigcup \{\, A \cup B' \mid A, B \subseteq \Omega, \quad |A|, |B| < \omega,\ A \cap B = \emptyset \,\},$$

ordered by containment. So now we have a join-semilattice and we have to make it distributive. It is easy to describe this process. For each $c \leq a \vee b$, add two elements a^+, b^+ with $a^+ \leq a$, $b^+ \leq b$ and declare that $a^+ \vee b^+ = c$; this yields a partial join-semilattice. Form the join-semilattice it generates. Now iterate this countably many times and take the union $\mathcal{G}(\Omega)$ of the join-semilattices formed.

The distributive join-semilattice with zero, $\mathcal{G}(\Omega)$, thus formed is F. Wehrung's bombshell: it is not representable provided that $\aleph_{\omega+1} \leq |\Omega|$.

Uniform Refinement Properties. Weakly distributive homomorphisms of join-semilattices with zero are defined in E. T. Schmidt, 1968. The following is a variant by F. Wehrung.

A $\{\vee, 0\}$-homomorphism $\mu \colon S \to T$ of the join-semilattices with zero is *weakly distributive* if for all $a, b \in S$ and all $c \in T$, $\mu(c) \leq a \vee b$ implies that there are $x, y \in S$ such that $c \leq x \vee y$, $\mu(x) \leq a$, and $\mu(y) \leq b$.

In most related works, the following "uniform refinement property" is used. It was introduced in F. Wehrung, 1998, 1999, and modified in M. Ploščica, J. Tůma, and F. Wehrung, 1998.

Let e be an element in a join-semilattice S with zero. We say that the *weak uniform refinement property*, WURP, holds at e if for all families $(a_i \mid i \in I)$ and $(b_i \mid i \in I)$ of elements of S such that $a_i \vee b_i = e$, for all $i \in I$, there exists a family $(c_{i,j} \mid \langle i, j \rangle \in I \times I)$ of elements of S such that the relations

(1) $c_{i,j} \leq a_i, b_j$,
(2) $c_{i,j} \vee a_j \vee b_i = e$,
(3) $c_{i,k} \leq c_{i,j} \vee c_{j,k}$

hold, for all i, j, $k \in I$. We say that S satisfies WURP if WURP holds at every element of S.

In M. Ploščica and J. Tůma, 1998, it is proved that WURP does not hold in $\mathcal{G}(\Omega)$, for any set Ω of cardinality at least $\aleph_2$. Hence $\mathcal{G}(\Omega)$ does not satisfy Schmidt's Condition. A similar result is proved in F. Wehrung, 1999.

However, the join-semilattices with zero used in these negative results are complicated. The following result, proved in M. Ploščica, J. Tůma, and F. Wehrung, 1998, is more striking, because it shows that a very well-known lattice, $F(\aleph_2)$, produces a *representable* semilattice that does not satisfy Schmidt's Condition.

Theorem 25. *Let $F(\aleph_2)$ be the free lattice on $\aleph_2$ generators. The join-semilattice with zero,* $\mathrm{Con_c}\, F(\aleph_2)$*, does not satisfy* WURP*. Consequently,* $\mathrm{Con_c}\, F(\aleph_2)$ *does not satisfy Schmidt's Condition.*

In fact, they prove a lot more. Let $\mathrm{F}_{\mathbf{V}}(\Omega)$ denote the free lattice on Ω in $\mathbf{V}$ for any nondistributive variety $\mathbf{V}$ of lattices.

The join-semilattice with zero, $\mathrm{Con_c}\, \mathrm{F}_{\mathbf{V}}(\Omega)$*, does not satisfy* WURP *for any set Ω of cardinality at least $\aleph_2$. Consequently,* $\mathrm{Con_c}\, \mathrm{F}_{\mathbf{V}}(\Omega)$ *does not satisfy Schmidt's Condition.*

It is proved in J. Tůma and F. Wehrung, 2001, that $\mathrm{Con_c}\, \mathrm{F}_{\mathbf{V}}(\Omega)$ *is not isomorphic to* $\mathrm{Con_c}\, L$*, for any lattice L with permutable congruences.* This is an important contribution to the second problem of G. Grätzer and E. T. Schmidt, 1963. By using a slight weakening of WURP, this result is extended to arbitrary *algebras* with permutable congruences in P. Růžička, J. Tůma, and F. Wehrung (to appear in *J. Algebra*). Hence, for example, *if Ω has at least $\aleph_2$ elements, then* $\mathrm{Con}\, \mathrm{F}_{\mathbf{V}}(\Omega)$ *is not isomorphic to the normal subgroup lattice of any group or the submodule lattice of any module.*

(Actually, to keep the exposition at an elementary level, I omitted a great deal. This work started with F. Wehrung, 1998, in which he attacked a problem of H. Dobbertin, 1983, on measure theory—Are there any nonmeasurable refinement monoids?—and a ring-theoretic problem of K. R. Goodearl, 1991—Is it the case that the positive cone of every dimension group with order-unit is isomorphic to $\mathcal{V}(R)$, for some regular ring R?

Unfortunately, I do not know how to present these topics in an introductory exposition within the space constraints I have.)

Solving CLP; the Erosion Lemma. Let us restate F. Wehrung's bombshell:

Theorem 26. *The join-semilattice with zero* $\mathcal{G}(\Omega)$ *is not isomorphic to* $\mathrm{Con_c}\, L$*, for any lattice* L*, whenever the set* Ω *has at least* $\aleph_{\omega+1}$ *elements.*

So the lattice that is a counterexample to CLP had been known for nearly ten years. All prior results about this lattice made use of some form of permutability of congruences. The novelty in F. Wehrung's approach was to find structure in congruence lattices of non-congruence-permutable lattices.

We shall denote by ε the "parity function" on the natural numbers, defined by the rule

$$\varepsilon(n) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd,} \end{cases} \quad \text{for any natural number } n.$$

Let L be an algebra possessing a structure of a semilattice $\langle L, \vee \rangle$ such that every congruence of L is also a congruence for $\vee$. We put

$$U \vee V = \{\, u \vee v \mid \langle u, v \rangle \in U \times V \,\}, \quad \text{for all } U, V \subseteq L,$$

and we denote by $\mathrm{Con_c^U}\, L$ the $\langle \vee, 0 \rangle$-subsemilattice of $\mathrm{Con_c}\, L$ generated by all principal congruences $\Theta_L(u, v)$, where $\langle u, v \rangle \in U \times U$. We put $\mathrm{con}_L^+(x, y) = \mathrm{con}_L(x \vee y, y)$, for any $x, y \in L$.

The Erosion Lemma. *Let* $x_0, x_1 \in L$*, and let* $Z = \{\, z_i \mid 0 \le i \le n \,\}$*, for a positive integer* n*, be a finite subset of* L *with* $\bigvee_{i<n} z_i \le z_n$*. Put*

$$\alpha_j = \bigvee (\mathrm{con}_L(z_i, z_{i+1}) \mid i < n,\ \varepsilon(i) = j), \quad \text{for all } j < 2.$$

Then there are congruences $\Theta_j \in \mathrm{Con_c^{\{x_j\} \vee Z}}\, L$*, for* $j < 2$*, such that*

$$z_0 \vee x_0 \vee x_1 \equiv z_n \vee x_0 \vee x_1 \ (\Theta_0 \vee \Theta_1) \quad \text{and}$$
$$\Theta_j \subseteq \alpha_j \cap \mathrm{con}_L^+(z_n, x_j), \text{ for all } j < 2.$$

The proof of Theorem 26 proceeds by establishing a "structure" theorem for congruence lattices of semilattices, namely, the Erosion Lemma, against "nonstructure" theorems for free distributive extensions $\mathcal{G}(\Omega)$, the main one being called the "Evaporation Lemma". While these are technically difficult, they are, in some sense, "predictable". In contrast, the proof of the Erosion Lemma is much easier.

The cardinality bound $\aleph_{\omega+1}$ has been improved to the optimal bound $\aleph_2$ by P. Růžička (manuscript).

Theorem 27. *The semilattice* $\mathcal{G}(\Omega)$ *is not isomorphic to* $\mathrm{Con_c}\, L$*, for any lattice* L*, whenever the set* Ω *has at least* $\aleph_2$ *elements.*

A key part in F. Wehrung's proof is a combinatorial argument of K. Kuratowski, 1951. Let $[X]^{<\omega}$ denote the set of all finite subsets of X and $[X]^n$ (for a positive integer n) the set of all n-element subsets of X. For a map $\Phi\colon [X]^n \to [X]^{<\omega}$, an $(n+1)$-element subset U of X is *free with respect to* Φ if $x \notin \Phi(U - \{x\})$ for all $x \in U$.

Theorem 28 (Kuratowski's Free Set Theorem)**.** *Let* n *be a natural number, and let* X *be a set of cardinality at least* $\aleph_n$*. For every map* $\Phi\colon [X]^n \to [X]^{<\omega}$*, there exists an* $(n+1)$*-element free subset of* X *with respect to* Φ*.*

See the book P. Erdös, A. Hajnal, A. Máté, and R. Rado [5], especially Chapter 10, for this result.

P. Růžička's proof follows the main lines of F. Wehrung's proof, except that it introduces an enhancement of Kuratowski's Free Set Theorem, called there *the existence of free trees*, which it uses in the final argument involving the Erosion Lemma.

A Short List of Books

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[2] ———, *Lattice Theory*, second edition, Amer. Math. Soc., Providence, RI, 1948.

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[4] K. P. Bogart, R. Freese, and J. P. S. Kung (editors), *The Dilworth Theorems. Selected Papers of Robert P. Dilworth*, Birkhäuser-Verlag, Basel–Boston–Berlin, 1990.

[5] P. Erdös, A. Hajnal, A. Máté, and R. Rado, *Combinatorial Set Theory: Partition Relations for Cardinals*, Studies in Logic and the Foundations of Mathematics, vol. 106, North-Holland Publishing Co., Amsterdam, 1984.

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[7] ———, *General Lattice Theory*, second edition, new appendices by the author with B. A. Davey, R. Freese, B. Ganter, M. Greferath, P. Jipsen, H. A. Priestley, H. Rose, E. T. Schmidt, S. E. Schmidt, F. Wehrung, and R. Wille; Birkhäuser-Verlag, Basel, 1998. Softcover edition, Birkhäuser-Verlag, Basel–Boston–Berlin, 2003.

[8] ———, *The Congruences of a Finite Lattice: A Proof-by-Picture Approach*, Birkhäuser Boston, 2006.

[9] R. Padmanabhan and S. Rudeanu, *Axioms for Lattices and Boolean Algebras*, manuscript.

[10] V. N. Saliĭ, *Lattices with Unique Complements*, translated from the Russian by G. A. Kandall, Translations of Mathematical Monographs, vol. 69, Amer. Math. Soc., Providence, RI, 1988.