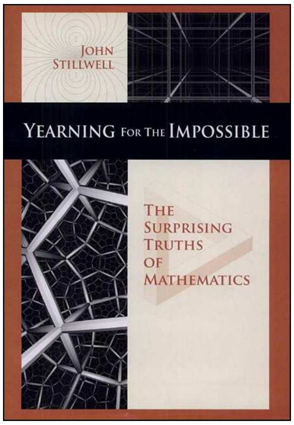
Book Review



Yearning for the Impossible

Reviewed by Daniel Biss

Yearning for the Impossible
John Stillwell
A K Peters, Ltd., 2006
$29.95, 244 pages, ISBN 978-1-56881-254-0

John Stillwell's *Yearning for the Impossible* is a new book that seeks to teach some mathematics to an audience of nonspecialists. Its title refers to the idea that mathematical advances tend to come about when researchers manage to understand ideas or constructions once thought to be impossible. So, for example, there are chapters about irrational numbers and imaginary numbers (after all, nobody used to think such things existed), chapters about 4-dimensional geometries and different orders of infinity (because nobody really knew whether one could make sense of such things), and a chapter about unique factorization via prime ideals (because unique factorization without prime ideals is, um, impossible).

The first thing I noticed when perusing this list is that the word "impossible" is being used awfully flexibly. The idea that irrational numbers were impossible was simply the incorrect statement that all lengths constructible via straightedge and compass would be expressible as fractions. In other words, there existed two notions of "number", namely, the set of rational numbers and the set of constructible numbers. Everybody assumed they were the same, but they aren't, and the impossibility lies in the difference between these two concepts. The idea that imaginary numbers were impossible was, by contrast, the perfectly consistent aesthetic decision that going beyond the real numbers and introducing *i* was simply absurd. The impossibility of unique factorization in arbitrary rings, of course, is a true mathematical theorem.

So, to recapitulate Stillwell, mathematics happens when we grapple with the impossible. The "impossible", in turn, is anything that is either hard to think about, hard to visualize, hard to interpret, or else literally undoable without changing some of the rules. In other words, mathematics is what happens when we're forced to be creative. What began as an intriguing and possibly even provocative theme has now revealed itself to be somewhat more pedestrian. Mathematics, Stillwell turns out to be saying, is sort of like art. And science. And technology. And snowboarding. And pretty much any other activity that requires its practitioners to do something they didn't already know how to do when they began.

Now, to be fair, Stillwell essentially admits this in his first paragraph:

> I hoped to show a general audience that mathematics is a discipline that demands imagination, perhaps even fantasy.

In other words, for all the fancy talk about yearning for the impossible, what's really being asserted is that you need to be imaginative to do math. While this is an important point to make, I do not believe that it is robust enough to structure a book around. Here's why: mathematics has become a highly jargonized discipline, and a book for nonmathematicians needs to circumvent a lot of that jargon and still get at the meaning. If you believe that at least some of the jargon is useful in communicating something, then the jargon-free model must find a communicative replacement for it. Consequently, it's essential that a prospective author of a math-book-for-nonmathematicians devise a useful strategy or theme around which to structure the book.

Daniel Biss is assistant professor of mathematics at the University of Chicago and a Research Fellow at the Clay Mathematics Institute. His email address is `daniel@math.uchicago.edu`.

What should such a theme accomplish? Well, its primary task is to focus the reader's attention and energy in a way that facilitates the communication of some mathematics. In other words, it must be some description or characterization of our subject that primes the reader to view mathematics in a new and useful way. Simply put, it must articulate some unique and telling quality of mathematics.

For this reason, the impossibility theme feels like a bit of a bait-and-switch. I wanted the book to have a theme that helped illustrate the fact that mathematics is an imaginative enterprise, but when examined closely, the purported theme turned out to be a restatement of this fact rather than an illustration of it.

Right. So Stillwell's titular "impossibility" gambit is a little hollow. What's left to the book? Well, its table of contents reveals a fairly typical list of math-that-mathematicians-can't-figure-out-why-nonmathematicians-don't-know-and-love. In addition to the topics mentioned above, it touches on infinitesimals, curvature, projective geometry, and "periodic spaces" (things like tori and cylinders and so forth). As a mathematician, I can't figure out why nonmathematicians don't know and love these topics; this has always been a source of frustration for me. Consequently, I'm invariably eager to see whether a new attempt at explicating them will pass the Mother Test—in other words, whether I will send a copy of the book to my mother. (In recent months, the Mother Test has been threatened in prominence by the newly minted Wife Test, but it turns out that my wife has more than enough stuff to read already.) This book fails that test, but just barely. In other words, it rises nobly to the challenge of describing these topics to a genuine novice; some of the chapters, I think, are very clear, and others are less so. In some sense, a mathematician is the least qualified reviewer of the ultimate success of such a book, but I feel confident in saying that there is much to admire in Stillwell's attempt.

Of course, the task Stillwell sets for himself varies tremendously in difficulty from chapter to chapter. A discussion of irrational numbers, in addition to being entirely elementary, might even be familiar to some veterans of high school mathematics. Ideals and unique factorization, on the other hand, are totally new to practically everyone, and, moreover, they're pretty confusing and weird. (Come to think of it, I wonder if Stillwell really meant "weird" when he said "impossible".)

So let's discuss the chapter on ideals and factorization, remembering that it's probably one of the hardest to write. Stillwell begins by describing prime factorization of integers (not addressing uniqueness or anything, just saying what a prime factorization is). He then discusses division, remainders, and the Euclidean algorithm, which of course leads right into uniqueness of factorizations. He then, at some pains, introduces Gaussian integers and proves that they also have unique factorization. Finally, it's time to drop the bombshell, namely, the failure of unique factorization in $Z[\sqrt{(-5)}]$.

Now comes the hard part. Stillwell needs to introduce ideals, explain why they might be construed to constitute a "reasonable generalization" of the notion of number, and then address their unique factorization properties. He does a reasonable job, going on and on about the mind-blowing qualities of the equation $2 \times 3 = 6 = (1+\sqrt{(-5)}) \times (1-\sqrt{(-5)})$ for long enough that the reader begins to genuinely wish for some sort of resolution of this apparently nonunique factorization of the number 6, if only to pacify Stillwell. He then argues that such a resolution can be accomplished only if 2 and $1+\sqrt{(-5)}$ share some sort of "ideal" common factor. This is good stuff, and I found myself wishing that I'd been taught the definition of an ideal by Stillwell; he's accomplishing something very important and difficult here in demonstrating that there's some real struggle present in the process of mathematical discovery.

The magic still hasn't happened, though; that takes place when one decides to give the set of all numbers of the form $2M + (1+\sqrt{(-5)})N$, where M and N are integers, the status of an "ideal number" (or ideal) and then plays the common factorization game with these objects. It's a remarkable intellectual leap and a confusing one at that, and Stillwell labors mightily to make it seem natural. He nearly succeeds, but the truth is that it's almost a hopeless task, because of course the leap isn't natural at all—that's why it's so brilliant.

This brilliance, built mostly out of a childlike faith in the notion of uniquely factorable "ideal numbers", is the source of the real mathematics here; everything else is a combination of arithmetic and hot air. This, I suspect, is what Stillwell is getting at with his fixation on the impossible. And if the theme were a truly useful one and the book effective, then we'd be ready for the moment of brilliance. We'd realize that the impossible moment, that hallmark of mathematical achievement, was just around the corner, and we'd be eager to ride its wave to the promised shores.

Instead, the appearance of ideals 170-odd pages into the book just left me anxious and confused, full of questions. Is this as clear as it could be? Does it pass the Mother Test? Do I understand what an ideal is as well as I did before I read this chapter?

I don't know. If truth be told, I can't figure out why nonmathematicians don't know and love this topic, but I'm starting to wonder if it's just too hard to explain it to them. Maybe even impossible.