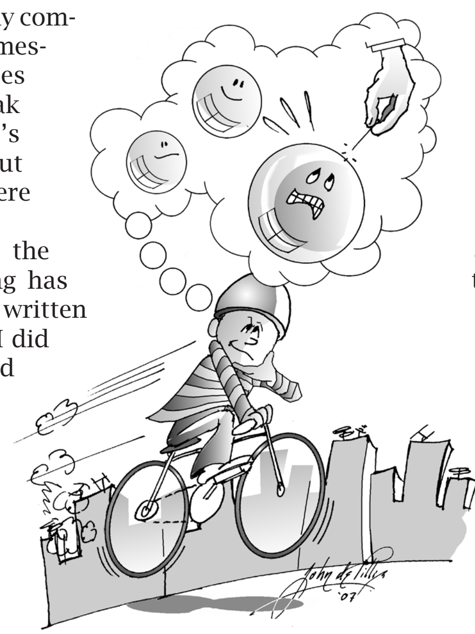
Popping Bubbles

Christopher Tuffley

It's Sunday evening on the first weekend of spring break, on the second of nine straight days without classes. Any other year I would have been home packing for the mountains, if not in them already, but this year I am not; I am in my office instead. In a little over a month I want to have a draft of my dissertation ready for my committee, and in any case this semester I am neither taking classes nor teaching, so spring break isn't really a holiday. There's work to be done, and guilt about work to be done, and I am here trying to do it.

Much as I enjoy teaching, the semester's break from TA'ing has been a real boon. I've finally written down carefully a calculation I did almost a year ago, and an old abandoned line of thought has come to life again. Yes, the technical difficulty that led me to drop it is still there, but only in three or more dimensions. The idea *does* work when $n = 2$, and I have finally broken the barrier between "one" and "many" that's plagued me for so long: where once I could speak only on circles and 1-complexes, now I can speak on surfaces and 2-complexes. The conjecture I'd hoped the idea would prove is true in two dimensions, and I'm one step closer to proving it in general—one step of infinitely many, it's true, but a step closer nonetheless.



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The words, though, are slow to catch up with the ideas, and I am struggling to find the best ones to pin my thoughts to the page. Around 8:30 p.m. I call it an evening and pack up to head down the hill for home and dinner. I take my bike from its usual spot beneath the blackboard, brush the chalk dust from the seat, and start down the hall to the lifts. Michael's light is off, so I don't bother knocking on his door and simply push the down button.

As I wait, my mind, now free to wander, reminds me that I already knew the conjecture was true for surfaces. By puncturing its 2-cell, popping it like a bubble, I can shrink a surface down onto its one-dimensional skeleton. Doing this in $k + 1$ places lets me use the one-dimensional case I proved in a tiny corner of a paper already submitted for publication. As the lift descends, I realise it doesn't have to be a surface; the argument works as well for any 2-complex with a single 2-cell. Turning the corner from Hearst onto Sacramento, I realise I'm not limited to a single 2-cell either; if I have more, I just need more punctures. Over dinner it occurs to me that I can pop a 3-, 4-, 5- or n-cell as easily as a 2-cell, and doesn't an exercise I did for homework my first year of grad school give me the rest of what I need to climb two, three, four, five, all the way up?

It's 10:30 p.m., and I am back on my bike up the hill. I want to know what that homework exercise said. If it says what I want it to, I'll have my conjecture in its full generality, and the proof will be so short, much shorter than the argument I finally pushed through in two dimensions, the argument I was struggling to write down earlier. It'll fit on a single sheet of paper, and I'll write it down there and then so that I can give it to my advisor tomorrow. I pedal hard up the familiar street, the arc that's connected me to campus for the last five years.

The lift doors open on the tenth floor. Michael is in this time, talking to Peter, and I stop for a while to chat—mathematics, climbing, work—then excuse myself. In my office I reach for the shelf lined with coloured binders—red, white, orange, purple, yellow, green—all unlabelled. Once I could find any class by the colour of its binder, but now I usually have to look through several. Algebraic topology, however, I know is in the blue one. I pull it down, leaf through to my homework.

The inequality points the wrong way. Where I want a less than, the exercise has a greater than. I check the text to be sure. It says the same thing. Did I bike back in vain?

Surely not. The lemma I need must still be true. If I put together a collection of simple spaces, making sure that all the overlaps are simple, then the result can't suddenly become more complicated—homology can't spring from nowhere like that. I think about the exercise, the example in it that appears to be my undoing, try to understand again what I wrote so long ago. I see that the example can't happen to me; I have an extra hypothesis I can use to rule it out. My instincts were right. The exercise itself doesn't say what I want it to, but my answer to it does. I can use it nearly unchanged, with only a little extra work, to get what I need. I pull out the chair at the computer, log on, and begin to TEX.

It's 3:30 a.m. and I'm finally done. The proof is written, and at one and a half pages it does indeed fit on a single sheet of paper. I log out and head home again, to bed this time.

Monday afternoon. I'm tired and have a headache from not enough sleep. My advisor isn't in. And the proof I stayed up late over is terribly written. The argument at least is still correct—it hasn't turned into a "Friday Theorem", one of those brilliant ideas that prove fatally flawed by the end of the weekend, if not earlier. The exposition just needs some work. I sit back down at the computer, and this time, when I'm rolling down the hill again, I'm satisfied and have emailed my advisor to tell him what I've done.