About the Cover

The Museum of Associahedra

This month's cover was suggested by Andrei Zelevinsky's article on clusters. As he mentions, certain convex polytopes arise in the theory of clusters that generalize the older construction of *associahedra*. The story of associahedra is in itself remarkable, although by now somewhat familiar.

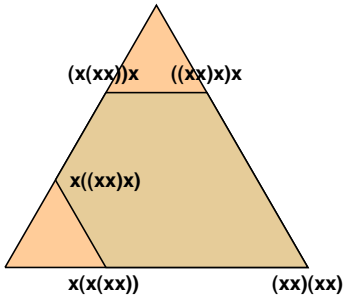
Start with a single product of n variables. Then make a list of all possible ways to insert a proper pair of balanced parentheses in this product. If n is 4, for example, we start with *xxxx* and get the list

(xx)xx
(xxx)x
x(xxx)
xx(xx)
x(xx)x.

Continue by inserting correctly other pairs of balanced parentheses. If n is 4 we get

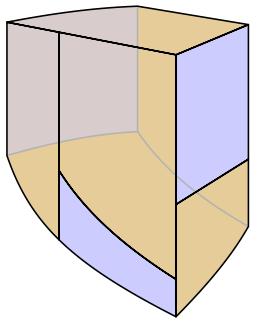
(x(xx))x
((xx)x)x
(xx)(xx)
x(x(xx))
x((xx)x)

It turns out that what we obtain finally is a cellular complex K_n topologically equivalent to an n-2 ball and its boundary sphere. Its vertices are the expressions which are saturated, and two vertices are joined by an edge if they differ by a single application of associativity. For example, (*xx*)(*xx*) and *x*(*x*(*xx*)) would be joined, and the entire cell for n=4 is a pentagon:



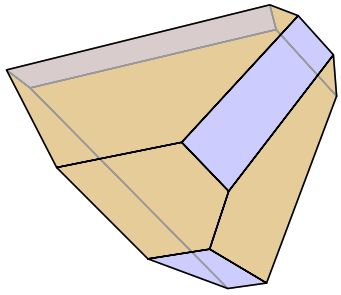
The associahedron for n=4, showing also the 2-simplex from which it is obtained by truncation.

In higher dimensions, this is a non-trivial fact. It was first observed by James Stasheff in his 1963 thesis on associativity and homotopy, but his cellular structure was, although ingenious, somewhat unsatisfying since it involved cells with curved boundaries.



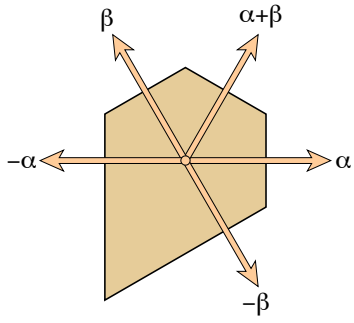
Stasheff's cell for n=5.

Since then numerous constructions of the associahedra as polytopes have been found, among the most recent one by Satyan Devadoss.



Devadoss' construction of K_5.

As Zelevinsky points out, these and similar polytopes occur in many incarnations, among them some related to root systems.



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References

JAMES STASHEFF, Homotopy associativity of H-spaces, *Ann. of Math.* **108**.

SATAYAN DEVADOSS, A realization of graph associahedra, preprint, 2007.

—Bill Casselman, Graphics Editor
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