Starting with the Group $SL_2(\mathbb{R})$

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The simplest objects with noncommutative multiplication may be 2×2 matrices with real entries. Such matrices *of determinant one* form a closed set under multiplication (since $\det(AB) = \det A \cdot \det B$), the identity matrix is among them, and any such matrix has an inverse (since $\det A \neq 0$). In other words those matrices form a group, the $SL_2(\mathbb{R})$ group [8]—one of the two most important Lie groups in analysis. The other group is the Heisenberg group [3]. By contrast the "$ax+b$"-group, which is often used to build wavelets, is a subgroup of $SL_2(\mathbb{R})$, see the numerator in (1).

The simplest nonlinear transformations of the real line—the linear-fractional or Möbius maps—may also be associated with 2×2 matrices [1, Ch. 13]:

(1)
$$g : x \mapsto g\cdot x = \frac{ax+b}{cx+d}, \text{ where } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, x \in \mathbb{R}.$$

An easy calculation shows that the composition of two transforms (1) with different matrices g_1 and g_2 is again a Möbius transform with matrix the product $g_1 g_2$. In other words (1) is a (left) action of $SL_2(\mathbb{R})$.

According to F. Klein's *Erlangen program* (which was influenced by S. Lie) any geometry is dealing with invariant properties under a certain group action. For example, we may ask: *What kinds of geometry are related to the $SL_2(\mathbb{R})$ action* (1)?

The Erlangen program has probably the highest rate $\frac{\text{praised}}{\text{actually used}}$ among mathematical theories, not only due to the big numerator but also due to the undeservedly small denominator. As we shall see below Klein's approach provides some surprising conclusions even for such over-studied objects as circles.

Make a Guess in Three Attempts

It is easy to see that the $SL_2(\mathbb{R})$ action (1) makes sense also as a map of complex numbers $z = x + \mathrm{i}y$, $\mathrm{i}^2 = -1$. Moreover, if $y > 0$ then $g\cdot z$ has a positive imaginary part as well, i.e., (1) defines a map from the upper half-plane to itself.

However there is no need to be restricted to the traditional route of complex numbers only. Less-known *double* and *dual* numbers [9, Suppl. C] also have the form $z = x + \mathrm{i}y$ but different assumptions on the imaginary unit i: $\mathrm{i}^2 = 0$ or $\mathrm{i}^2 = 1$ correspondingly. Although the arithmetic of dual and double numbers is different from the complex ones, e.g., they have divisors of zero, we are still able to define their transforms by (1) in most cases.

Three possible values -1, 0, and 1 of $\sigma := \mathrm{i}^2$ will be referred to here as *elliptic*, *parabolic*, and *hyperbolic* cases respectively. We repeatedly meet such a division of various mathematical objects into three classes. They are named by the historically first example—the classification of conic sections—however the pattern persistently reproduces itself in many different areas: equations, quadratic forms, metrics, manifolds, operators, etc. We will abbreviate this separation as *EPH-classification*. The *common origin* of this fundamental division can be seen from the simple picture of a coordinate line split by zero into

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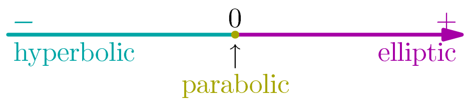
negative and positive half-axes:

(2)



Connections between different objects admitting the EPH-classification are not limited to this common source. There are many deep results linking, for example, ellipticity of quadratic forms, metrics and operators.

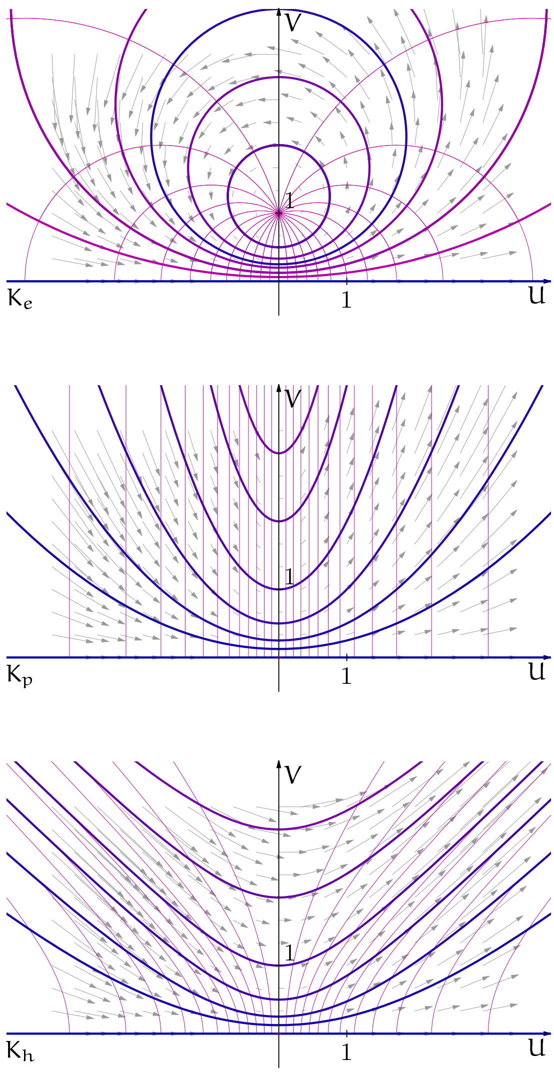


Figure 1. Action of the K subgroup. The corresponding orbits are circles, parabolas, and hyperbolas.

To understand the action (1) in all EPH cases we use the Iwasawa decomposition [8] of $SL_2(\mathbb{R}) = ANK$ into *three* one-dimensional subgroups A, N, K:

(3)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}.$$

Subgroups A and N act in (1) irrespective of the value of σ: A makes a dilation by α^2, i.e., $z \mapsto \alpha^2 z$, and N shifts points to left by ν, i.e. $z \mapsto z + \nu$.

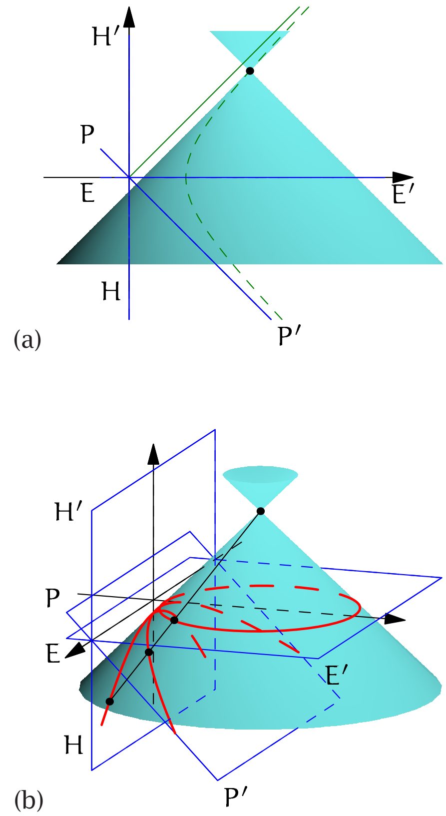


Figure 2. K-orbits as conic sections: circles are sections by the plane EE'; parabolas are sections by PP'; hyperbolas are sections by HH'. Points on the same generator of the cone correspond to the same value of ϕ.

By contrast, the action of the third matrix from the subgroup K sharply depends on σ, see Figure 1. In the elliptic, parabolic and hyperbolic cases K-orbits are circles, parabolas and (equilateral) hyperbolas correspondingly. Thin traversal lines in Figure 1 join points of orbits for the same values of ϕ and grey arrows represent "local velocities"—vector fields of derived representations.

Definition 1. The common name *cycle* [9] is used to denote circles, parabolas, and hyperbolas (as well as straight lines as their limits) in the respective EPH case.

It is well known that any cycle is a *conic section* and an interesting observation is that corresponding K-orbits are in fact sections of the same two-sided right-angle cone, see Figure 2. Moreover, each straight line generating the cone, see Figure 2(b), crosses corresponding EPH K-orbits at points with the same value of the parameter ϕ

from (3). In other words, all three types of orbits are generated by the rotations of this generator along the cone.

K-orbits are K-invariant in a trivial way. Moreover since actions of both A and N for any σ are extremely "shape-preserving" we find natural invariant objects of the Möbius map:

Theorem 2. *Cycles from Definition 1 are invariant under the action* (1).

Proof. We will show that for a given $g \in SL_2(\mathbb{R})$ and a cycle C its image gC is again a cycle. Figure 3 gives an illustration with C as a circle, but our reasoning works in all EPH cases.

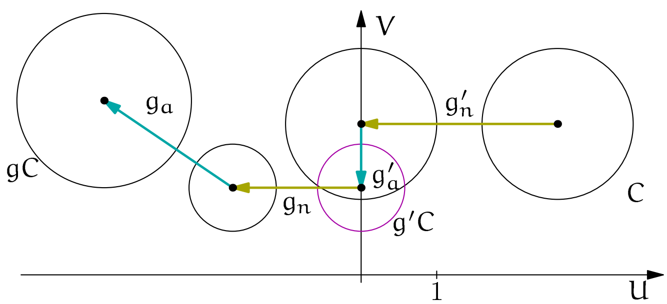


Figure 3. Decomposition of an arbitrary Möbius transformation g into a product $g = g_a g_n g_k g'_a g'_n$.

For a fixed C there is always a unique pair of transformations g'_n from the subgroup N and $g'_a \in A$ such that the cycle $g'_a g'_n C$ is exactly a K-orbit. We make a decomposition of $g(g'_a g'_n)^{-1}$ into a product as in (3):

$$g(g'_a g'_n)^{-1} = g_a g_n g_k.$$

Since $g'_a g'_n C$ is a K-orbit we have $g_k(g'_a g'_n C) = g'_a g'_n C$, then:

$$\begin{aligned} gC &= g(g'_a g'_n)^{-1} g'_a g'_n C = g_a g_n g_k g'_a g'_n C \\ &= g_a g_n g_k (g'_a g'_n C) = g_a g_n g'_a g'_n C. \end{aligned}$$

Since the subgroups A and N obviously preserve the shape of any cycle this finishes our proof. □

According to Erlangen ideology we should now study invariant properties of cycles.

Invariance of FSCc

Figure 2 suggests that we may get a unified treatment of cycles in each EPH case by consideration of higher-dimensional spaces. The standard mathematical method is to declare objects under investigation (cycles in our case, functions in functional analysis, etc.) to be simply points of some bigger space. This space should be equipped with an appropriate structure to hold externally information that was previously inner properties of our objects.

A generic cycle is the set of points $(u, v) \in \mathbb{R}^2$ defined for all values of σ by the equation

$$k(u^2 - \sigma v^2) - 2lu - 2nv + m = 0. \tag{4}$$

This equation (and the corresponding cycle) is defined by a point (k, l, n, m) from a projective space $\mathbb{P}^3$, since for a scaling factor $\lambda \neq 0$ the point $(\lambda k, \lambda l, \lambda n, \lambda m)$ defines the same equation (4). We call $\mathbb{P}^3$ the *cycle space* and refer to the initial $\mathbb{R}^2$ as the *point space*.

In order to get a connection with the Möbius action (1) we arrange the numbers (k, l, n, m) into the matrix

$$C^s_{\breve{\sigma}} = \begin{pmatrix} l + \breve{\imath}sn & -m \\ k & -l + \breve{\imath}sn \end{pmatrix}, \tag{5}$$

with a new imaginary unit $\breve{\imath}$ and an additional parameter s usually equal to ± 1. The values of $\breve{\sigma} := \breve{\imath}^2$ are -1, 0, or 1 independently of the value of σ. The matrix (5) is the cornerstone of the (extended) Fillmore–Springer–Cnops construction (FSCc) [2] and closely related to the technique recently used by A. A. Kirillov to study the Apollonian gasket [4].

The significance of FSCc in the Erlangen framework is provided by the following result:

Theorem 3. *The image $\tilde{C}^s_{\breve{\sigma}}$ of a cycle $C^s_{\breve{\sigma}}$ under transformation* (1) *with $g \in SL_2(\mathbb{R})$ is given by similarity of the matrix* (5)*:*

$$\tilde{C}^s_{\breve{\sigma}} = g C^s_{\breve{\sigma}} g^{-1}. \tag{6}$$

In other words FSCc (5) intertwines *Möbius action* (1) *on cycles with a linear map* (6).

There are several ways to prove (6): either by a brute-force calculation (fortunately performed by a CAS) [7] or through the related orthogonality of cycles [2]; see the end of the next section.

The important observation here is that FSCc (5) uses an imaginary unit $\breve{\imath}$ which is not related to i defining the appearance of cycles on plane. In other words any EPH type of geometry in the cycle space $\mathbb{P}^3$ allows one to draw cycles in the point space $\mathbb{R}^2$ as circles, parabolas, or hyperbolas. We may think of points of $\mathbb{P}^3$ as ideal cycles while their depictions on $\mathbb{R}^2$ are only their shadows on the wall of Plato's cave.

Figure 4(a) shows the same cycles drawn in different EPH styles. Points $c_{e,p,h} = (\frac{l}{k}, -\sigma\frac{n}{k})$ are their respective e/p/h-centers. They are related to each other through several identities:

$$c_e = \bar{c}_h, \quad c_p = \frac{1}{2}(c_e + c_h). \tag{7}$$

Figure 4(b) presents two cycles drawn as parabolas; they have the same focal length $\frac{n}{2k}$ and thus their e-centers are on the same level. In other words *concentric* parabolas are obtained by a vertical shift, not scaling as an analogy with circles or hyperbolas may suggest.

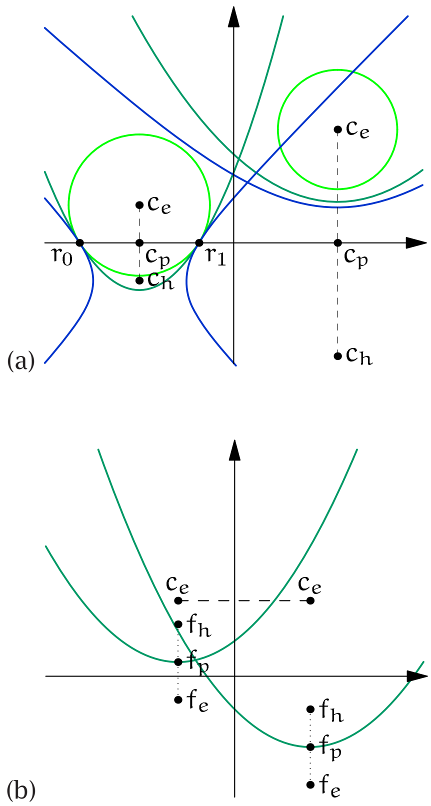


Figure 4. (a) Different EPH implementations of the same cycles defined by quadruples of numbers.
(b) Centers and foci of two parabolas with the same focal length.

Figure 4(b) also presents points, called e/p/h-foci:

$$f_{e,p,h} = \left(\frac{l}{k}, -\frac{\det C^s_{\breve{\sigma}}}{2nk}\right), \tag{8}$$

which are independent of the sign of s. If a cycle is depicted as a parabola then h-focus, p-focus, e-focus are correspondingly geometrical focus of the parabola, its vertex, and the point on the directrix nearest to the vertex.

As we will see, cf. Theorems 5 and 7, all three centers and three foci are useful attributes of a cycle even if it is drawn as a circle.

Invariants: Algebraic and Geometric

We use known algebraic invariants of matrices to build appropriate geometric invariants of cycles. It is yet another demonstration that any division of mathematics into subjects is only illusive.

For 2×2 matrices (and thus cycles) there are only two essentially different invariants under similarity (6) (and thus under Möbius action (1)): the *trace* and the *determinant*. The latter was already used in (8) to define a cycle's foci. However due to the projective nature of the cycle space $\mathbb{P}^3$ the absolute values of trace or determinant are irrelevant, unless they are zero.

Alternatively we may have a special arrangement for normalization of quadruples (k, l, n, m). For example, if $k \neq 0$ we may normalize the quadruple to $(1, \frac{l}{k}, \frac{n}{k}, \frac{m}{k})$ with the cycle's center highlighted. Moreover in this case $\det C^s_{\breve{\sigma}}$ is equal to the square of cycle's radius, cf. the next to last section below. Another normalization $\det C^s_{\breve{\sigma}} = 1$ is used in [4] to get a nice condition for touching circles.

We still get important characterizations even with non-normalized cycles, e.g., invariant classes (for different $\breve{\sigma}$) of cycles are defined by the condition $\det C^s_{\breve{\sigma}} = 0$. Such a class is parameterized by two real numbers and as such is easily attached to a certain point of $\mathbb{R}^2$. For example, the cycle $C^s_{\breve{\sigma}}$ with $\det C^s_{\breve{\sigma}} = 0$, $\breve{\sigma} = -1$ drawn elliptically represents a point $(\frac{l}{k}, \frac{n}{k})$, i.e., an (elliptic) zero-radius circle. The same condition with $\breve{\sigma} = 1$ in hyperbolic drawing produces a null-cone originating at a point $(\frac{l}{k}, \frac{n}{k})$:

$$(u - \frac{l}{k})^2 - (v - \frac{n}{k})^2 = 0,$$

i.e., a zero-radius cycle in the hyperbolic metric.

In general for every notion there are nine possibilities: three EPH cases in the cycle space times three EPH realizations in point space. The nine cases for "zero radius" cycles are shown in Figure 5. For example, p-zero-radius cycles in any implementation touch the real axis.

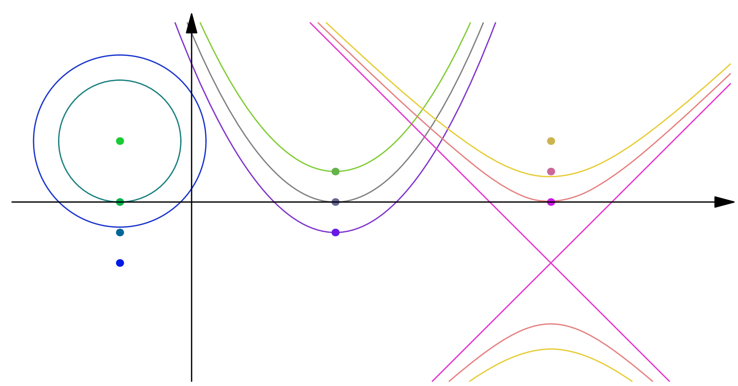


Figure 5. Different i-implementations of the same $\breve{\sigma}$-zero-radius cycles and corresponding foci.

This "touching" property is a manifestation of the *boundary effect* in the upper-half plane geometry [7, Rem. 3.4]. The famous question on hearing the shape of a drum has a sister:

> *Can we see/feel the boundary from inside a domain?*

Both orthogonality relations described below are "boundary aware" as well. It is not surprising after all since the $SL_2(\mathbb{R})$ action on the upper-half plane was obtained as an extension of its action (1) on the boundary.

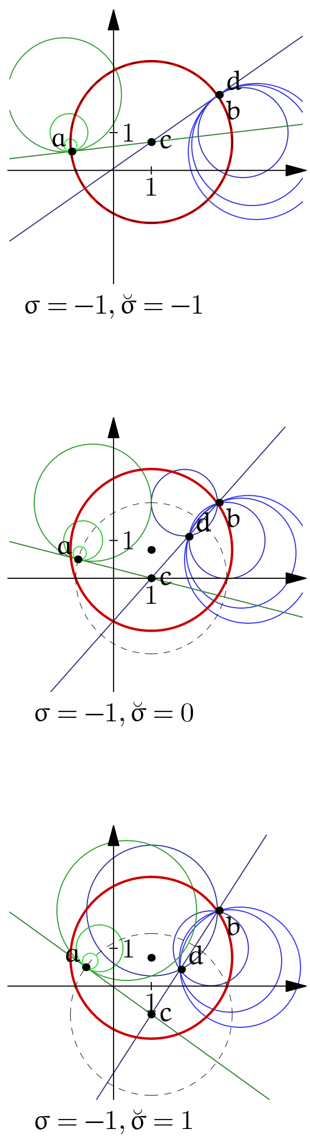


Figure 6. Orthogonality of the first kind in the elliptic point space.
Each picture presents two groups (green and blue) of cycles that are orthogonal to the red cycle $C^s_{\breve{\sigma}}$. Point b belongs to $C^s_{\breve{\sigma}}$ and the family of blue cycles passing through b is orthogonal to $C^s_{\breve{\sigma}}$. They all also intersect in the point d, which is the inverse of b in $C^s_{\breve{\sigma}}$. Any orthogonality is reduced to the usual orthogonality with a new ("ghost") cycle (shown by the dashed line), which may or may not coincide with $C^s_{\breve{\sigma}}$. For any point a on the "ghost" cycle the orthogonality is reduced to the local notion in the terms of tangent lines at the intersection point. Consequently such a point a is always the inverse of itself.

According to the categorical viewpoint internal properties of objects are of minor importance in comparison to their relations with other objects from the same class. As an illustration we may put the proof of Theorem 3 sketched at the end of the next section. Thus from now on we will look for invariant relations between two or more cycles.

Joint Invariants: Orthogonality

The most expected relation between cycles is based on the following Möbius invariant "inner product" built from the trace of the product of two cycles as matrices:

$$\left\langle C^s_{\breve{\sigma}}, \tilde{C}^s_{\breve{\sigma}} \right\rangle = tr(C^s_{\breve{\sigma}} \tilde{C}^s_{\breve{\sigma}}). \tag{9}$$

By the way, an inner product of this type is used, for example, in the GNS construction to make a Hilbert space out of C^*-algebras. The next standard move is given by the following definition.

Definition 4. Two cycles are called *$\breve{\sigma}$-orthogonal* if $\left\langle C^s_{\breve{\sigma}}, \tilde{C}^s_{\breve{\sigma}} \right\rangle = 0$.

For the case of $\breve{\sigma}\sigma = 1$, i.e., when geometries of the cycle and point spaces are both either elliptic or hyperbolic, such an orthogonality is the standard one, defined in terms of angles between tangent lines in the intersection points of two cycles. However in the remaining seven (= 9 − 2) cases the innocent-looking Definition 4 brings unexpected relations.

Elliptic (in the point space) realizations of Definition 4, i.e., $\sigma = -1$ are shown in Figure 6. The first picture corresponds to the elliptic cycle space, e.g., $\breve{\sigma} = -1$. The orthogonality between the red circle and any circle from the blue or green families is given in the usual Euclidean sense. The central (parabolic in the cycle space) and the last (hyperbolic) pictures show the non-local nature of the orthogonality. There are analogous pictures in parabolic and hyperbolic point spaces as well [7].

This orthogonality may still be expressed in the traditional sense if we associate to the red circle the corresponding "ghost" circle, which is shown by the dashed line in Figure 6. To describe the ghost cycle we need the *Heaviside function* $\chi(\sigma)$:

$$\chi(t) = \begin{cases} 1, & t \geq 0; \\ -1, & t < 0. \end{cases} \tag{10}$$

Theorem 5. *A cycle is $\breve{\sigma}$-orthogonal to the cycle $C^s_{\breve{\sigma}}$ if it is orthogonal in the usual sense to the σ-realization of the "ghost" cycle $\hat{C}^s_{\breve{\sigma}}$, which is defined by the following two conditions:*

(i) *The $\chi(\sigma)$-center of $\hat{C}^s_{\breve{\sigma}}$ coincides with the $\breve{\sigma}$-center of $C^s_{\breve{\sigma}}$.*
(ii) *The cycles $\hat{C}^s_{\breve{\sigma}}$ and $C^s_{\breve{\sigma}}$ have the same roots, moreover* $\det \hat{C}^1_{\sigma} = \det C^{\chi(\breve{\sigma})}_{\sigma}$.

The above connection between various centers of cycles illustrates their meaningfulness within our approach.

One can easily check the following orthogonality properties of the zero-radius cycles defined in the previous section:

(i) Since $\langle C^s_{\breve\sigma}, C^s_{\breve\sigma}\rangle = \det C^s_{\breve\sigma}$ zero-radius cycles are self-orthogonal (isotropic) ones.

(ii) A cycle $C^s_{\breve\sigma}$ is σ-orthogonal to a zero-radius cycle $Z^s_{\breve\sigma}$ if and only if $C^s_{\breve\sigma}$ passes through the σ-center of $Z^s_{\breve\sigma}$.

Sketch of proof of Theorem 3. The validity of Theorem 3 for a zero-radius cycle

$$Z^s_{\breve\sigma} = \begin{pmatrix} z & -z\bar z \\ 1 & -\bar z \end{pmatrix} = \frac{1}{2}\begin{pmatrix} z & z \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -\bar z \\ 1 & -\bar z \end{pmatrix}$$

with center $z = x + iy$ is straightforward. This implies the result for a generic cycle with the help of Möbius invariance of the product (9) (and thus the orthogonality) and the above relation (ii) between the orthogonality and the incidence. See [2] for details. □

Higher Order Joint Invariants: s-Orthogonality

With appetite already whetted one may wish to build more joint invariants. Indeed for any homogeneous polynomial $p(x_1, x_2, \ldots, x_n)$ of several non-commuting variables one may define an invariant joint disposition of n cycles ${}^jC^s_{\breve\sigma}$ by the condition:

$$tr\, p({}^1C^s_{\breve\sigma}, {}^2C^s_{\breve\sigma}, \ldots, {}^nC^s_{\breve\sigma}) = 0.$$

However it is preferable to keep some geometrical meaning for constructed notions.

An interesting observation is that in the matrix similarity of cycles (6) one may replace the element $g \in SL_2(\mathbb{R})$ by an arbitrary matrix corresponding to another cycle. More precisely the product $C^s_{\breve\sigma}\tilde C^s_{\breve\sigma}C^s_{\breve\sigma}$ is again the matrix of the form (5) and thus may be associated to a cycle. This cycle may be considered as the reflection of $\tilde C^s_{\breve\sigma}$ in $C^s_{\breve\sigma}$.

Definition 6. A cycle $C^s_{\breve\sigma}$ is s-orthogonal *to* a cycle $\tilde C^s_{\breve\sigma}$ if the reflection of $\tilde C^s_{\breve\sigma}$ in $C^s_{\breve\sigma}$ is orthogonal (in the sense of Definition 4) to the real line. Analytically this is defined by:

$$tr(C^s_{\breve\sigma}\tilde C^s_{\breve\sigma}C^s_{\breve\sigma}R^s_{\breve\sigma}) = 0. \tag{11}$$

Due to invariance of all components in the above definition s-orthogonality is a Möbius invariant condition. Clearly this is not a symmetric relation: if $C^s_{\breve\sigma}$ is s-orthogonal to $\tilde C^s_{\breve\sigma}$ then $\tilde C^s_{\breve\sigma}$ is not necessarily s-orthogonal to $C^s_{\breve\sigma}$.

Figure 7 illustrates s-orthogonality in the elliptic point space. By contrast with Figure 6 it is not a local notion at the intersection points of cycles for all $\breve\sigma$. However it may be again clarified in terms of the appropriate s-ghost cycle, cf. Theorem 5.

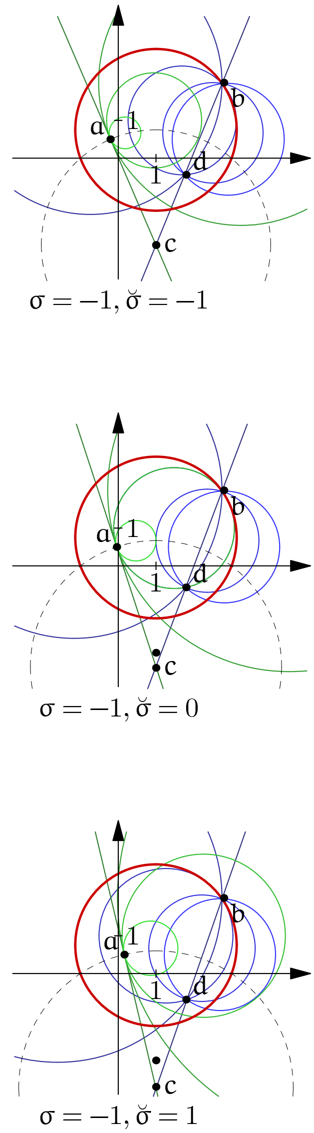


Figure 7. Orthogonality of the second kind for circles. To highlight both similarities and distinctions with the ordinary orthogonality we use the same notation as that in Figure 6.

Theorem 7. *A cycle is s-orthogonal to a cycle $C^s_{\breve\sigma}$ if it is orthogonal in the traditional sense to its s-ghost cycle $\tilde C^{\breve\sigma}_{\breve\sigma} = C^{\chi(\sigma)}_{\breve\sigma}\mathbb{R}^{\breve\sigma}_{\breve\sigma}C^{\chi(\sigma)}_{\breve\sigma}$, which is the reflection of the real line in $C^{\chi(\sigma)}_{\breve\sigma}$ and χ is the* Heaviside function (10). *Moreover*

(i) *The $\chi(\sigma)$-Center of $\tilde C^{\breve\sigma}_{\breve\sigma}$ coincides with the $\breve\sigma$-focus of $C^s_{\breve\sigma}$, consequently all lines s-orthogonal to $C^s_{\breve\sigma}$ are passing through the respective focus.*

(ii) *The cycles $C^s_{\breve\sigma}$ and $\tilde C^{\breve\sigma}_{\breve\sigma}$ have the same roots.*

Note the above intriguing interplay between a cycle's centers and foci. Although s-orthogonality may look exotic it will naturally appear again at the end of the next section.

Of course, it is possible to define other interesting higher-order joint invariants of two or even more cycles.

Distance, Length, and Perpendicularity

Geometry in the plain meaning of this word deals with *distances* and *lengths*. Can we obtain them from cycles?

We mentioned already that for circles normalized by the condition $k = 1$ the value $\det C^s_{\breve{\sigma}} = \langle C^s_{\breve{\sigma}}, C^s_{\breve{\sigma}} \rangle$ produces the square of the traditional circle radius. Thus we may keep it as the definition of the *radius* for any cycle. But then we need to accept that in the parabolic case the radius is the (Euclidean) distance between (real) roots of the parabola, see Figure 8(a).

Having radii of circles already defined we may use them for other measurements in several different ways. For example, the following variational definition may be used:

Definition 8. The *distance* between two points is the extremum of diameters of all cycles passing through both points, see Figure 8(b).

If $\breve{\sigma} = \sigma$ this definition gives in all EPH cases the following expression, see Figure 8(b):

$$d_{e,p,h}(u, v)^2 = (u + \mathrm{i}v)(u - \mathrm{i}v) = u^2 - \sigma v^2. \tag{12}$$

The parabolic distance $d_p^2 = u^2$ algebraically sits between d_e and d_h according to the general principle (2) and is widely accepted [9]. However one may be unsatisfied by its degeneracy.

An alternative measurement is motivated by the fact that a circle is the set of equidistant points from its center. However the choice of "center" is now rich: it may be any point from among three centers (7) or three foci (8).

Definition 9. The *length* of a directed interval $\overrightarrow{AB}$ is the radius of the cycle with its *center* (denoted by $l_c(\overrightarrow{AB})$) or *focus* (denoted by $l_f(\overrightarrow{AB})$) at the point A that passes through B.

This definition is less common and has some unusual properties like non-symmetry: $l_f(\overrightarrow{AB}) \neq l_f(\overrightarrow{BA})$. However it comfortably fits the Erlangen program due to its $SL_2(\mathbb{R})$-*conformal invariance*:

Theorem 10 ([7]). *Let l denote either the EPH distances* (12) *or any length from Definition 9. Then for fixed y, $y' \in \mathbb{R}^\sigma$ the limit:*

$$\lim_{t\to 0} \frac{l(g\cdot y, g\cdot(y+ty'))}{l(y, y+ty')}, \quad \textit{where } g \in SL_2(\mathbb{R}),$$

exists and its value depends only on y and g and is independent of y'.

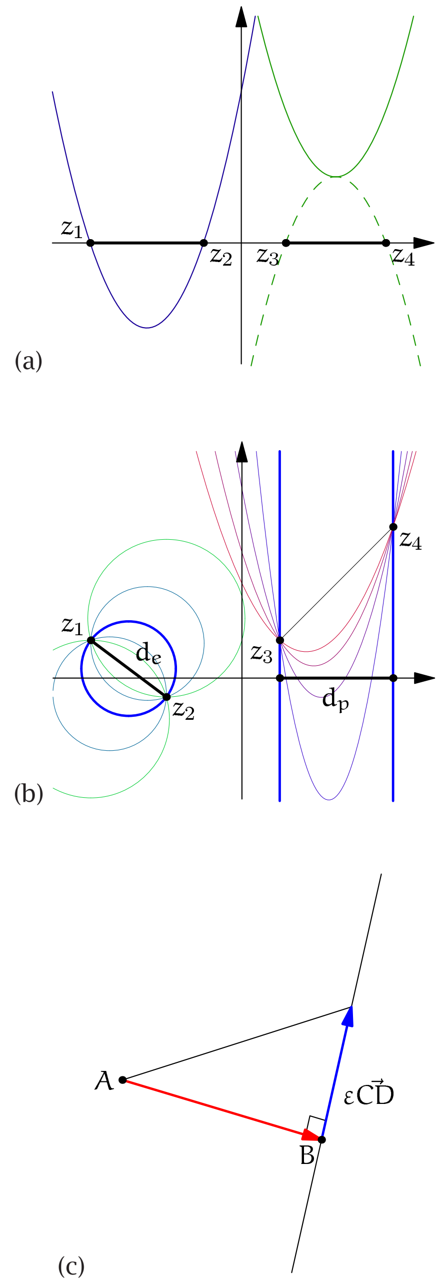


Figure 8. (a) The square of the parabolic diameter is the square of the distance between roots if they are real (z_1 and z_2), otherwise the negative square of the distance between the adjoint roots (z_3 and z_4). (b) Distance as extremum of diameters in elliptic (z_1 and z_2) and parabolic (z_3 and z_4) cases. (c) Perpendicular as the shortest route to a line.

We return from distances to angles recalling that in Euclidean space a perpendicular provides the shortest length from a point to a line, see Figure 8(c).

Definition 11. Let l be a length or distance. We say that a vector $\overrightarrow{AB}$ is *l-perpendicular* to a vector $\overrightarrow{CD}$ if the function $l(\overrightarrow{AB} + \varepsilon\overrightarrow{CD})$ of ε has a local extremum at $\varepsilon = 0$.

A pleasant surprise is that l_f-perpendicularity obtained through the length from focus (Definition 9) coincides with s-orthogonality already defined in the preceding section, as follows from Theorem 7(i).

All these notions are waiting to be generalized to higher dimensions, and Clifford algebras provide a suitable language for this [7].

Erlangen Program at Large

As we already mentioned the division of mathematics into areas is only apparent. Therefore it is unnatural to limit the Erlangen program only to "geometry". We may continue to look for $SL_2(\mathbb{R})$ invariant objects in other related fields. For example, transform (1) generates unitary representations on certain L_2 spaces, cf. (1):

$$g : f(x) \mapsto \frac{1}{(cx+d)^m} f\left(\frac{ax+b}{cx+d}\right). \tag{13}$$

For $m = 1, 2, \ldots$ the invariant subspaces of L_2 are Hardy and (weighted) Bergman spaces of complex analytic functions. All the main objects of *complex analysis* (Cauchy and Bergman integrals, Cauchy-Riemann and Laplace equations, Taylor series, etc.) may be obtained in terms of invariants of the *discrete series* representations of $SL_2(\mathbb{R})$ [5, § 3]. Moreover two other series (*principal* and *complimentary* [8]) play the similar roles for hyperbolic and parabolic cases [5, 7].

Moving further we may observe that transform (1) is defined also for an element x in any algebra $\mathfrak{A}$ with a unit $\mathbf{1}$ as soon as $(cx + d\mathbf{1}) \in \mathfrak{A}$ has an inverse. If $\mathfrak{A}$ is equipped with a topology, e.g., is a Banach algebra, then we may study a *functional calculus* for the element x [6] in this way. It is defined as an intertwining operator between the representation (13) in a space of analytic functions and a similar representation in a left $\mathfrak{A}$-module.

In the spirit of the Erlangen program such a functional calculus is still a geometry, since it is dealing with invariant properties under a group action. However even for a simplest non-normal operator, e.g., a Jordan block of the length k, the space obtained is not like a space of points but is rather a space of k-th *jets* [6]. Such non-point behavior is often attributed to *noncommutative geometry*, and the Erlangen program provides an important input on this fashionable topic [5].

Of course, there is no reason to limit the Erlangen program to $SL_2(\mathbb{R})$ only, other groups may be more suitable in different situations. However $SL_2(\mathbb{R})$ still possesses a big unexplored potential and is a good object to start with.

Note: Graphics for this article were created by the author with the help of Open Source Software:

MetaPost (`http://www.tug.org/metapost.html`), Asymptote (`http://asymptote.sourceforge.net/`), and GiNaC (`http://www.ginac.de/`).

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