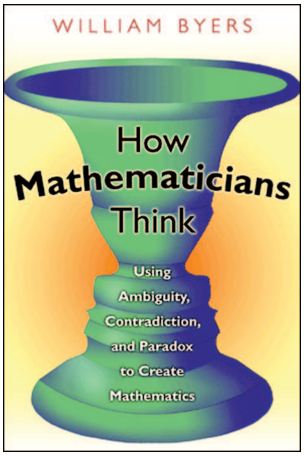

How Mathematicians Think

Reviewed by Reuben Hersh

How Mathematicians Think: Using Ambiguity, Contradiction, and Paradox to Create Mathematics
William Byers
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This book is a radically new account of mathematical discourse and mathematical thinking. It's addressed to everyone, from a lay reader who hasn't met complex numbers, up to a professional who appreciates Sarkovsky's theorem on cycles of iterated functions, or Goodstein's number-theoretic equivalent of Gödel's theorem for arithmetic with induction. No math preparation is presupposed, and everything is explained with complete clarity, yet deep contemporary issues are faced with no hesitancy. The discussion is free of pretentiousness or grandiosity. Byers straightforwardly explains the issues and clarifies them.

Starting with Imre Lakatos' 1976 *Proofs and Refutations*, some writers have been turning away from the search for a "foundation" for mathematics and instead, seeking to understand and clarify the actual practice of mathematics—*what real mathematicians really do.* Conferences toward this end have been held in Mexico, Belgium, Denmark, Italy, Spain, Sweden, and Hungary. In particular, I would mention books by Bettina Heinz, Carlo Cellucci, and Alexandre Borovik. My own anthology collects essays by mathematicians, philosophers, cognitive scientists, sociologists, a computer scientist, and an anthropologist.

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There's not much consensus, but at least one thing has been pretty generally taken for granted: mathematical thinking and discourse is supposed to be *precise*—that is to say, unambiguous. A mathematical statement is supposed to have a single definite meaning. What Byers's book reveals is that ambiguity is always present, from the most elementary to the most advanced level. In teaching school mathematics, it is an unacknowledged source of difficulty. At the level of research, it is often the key to growth and discovery.

Ambiguity can just mean vagueness. But also, it can mean, as Byers puts it, "a single situation or idea that is perceived in two self-consistent but mutually incompatible frames of reference." (p. 28) In fact, he makes a persuasive case that ambiguity is actually what makes mathematical ideas so powerful:

> Normally ambiguity in science and mathematics is seen as something to overcome, something that is due to an error in understanding and is removed by correcting that error. The ambiguity is rarely seen as having value in its own right, and the existence of ambiguity was often the very thing that spurred a particular development of mathematics and science.… The power of ideas reside in their ambiguity. Thus any project that would eliminate ambiguity from mathematics would destroy mathematics. (p. 24)

Familiar examples of ambiguity include: Negating Euclid's parallel postulate. Different sizes of infinite sets. Using logic to prove the limitations of logic. Infinitely rough curves, self-similar on infinitely many different scales. And on and on.

No surprise that there's ambiguity in "infinite" or "infinity". The philosophically inclined won't be surprised that there's ambiguity in "true" and even in "proof". But even in the simplest, most "elementary" mathematical steps, there is already deep, unacknowledged ambiguity.

An obvious example is: square roots of negative numbers. It takes effort simultaneously to know that "-1 has no square root" (on the real line) and "it has two of them" (in the plane.) The student must switch contexts as needed. Sometimes there is no square root, sometimes there are two. It all depends on what are you are talking about, what are you are trying to do! But a while back, the same effort was required regarding negative numbers. We have forgotten that for D'Alembert or De Morgan, it both made sense and didn't make sense, to contemplate a quantity less than "nothing".

Indeed, "zero" is ambiguous! Unlike D'Alembert or DeMorgan, today we don't say "nothing" when we mean "zero". Zero is *something*—it's a number. Yet, of course, "nothing" is what it means. Zero is a something, and what is stands for, what it means—is "nothing"! This is ambiguous, but we math teachers have buried the ambiguity so deeply, that if we ever have to talk to a student who is troubled by it, we can hardly understand what is her difficulty.

"One" is ambiguous! Frege's famous book, *Grundlagen der Arithmetik*, was motivated by mathematicians' inability to explain coherently what they meant by "one". Frege's answer was: "one" is the "concept" of singletons. But Dedekind and Peano had a different answer: "one" is just an undefined term, in the axioms for the relation of "successor". And still another answer is given in every elementary math classroom—"one" is a slash or a tally mark, which can be repeated to make "two", and repeated again, to make "three". If that's not enough ambiguity, there's still a deeper ambiguity in "one". When we choose to think about all the things that belong to some system (for example, all the counting numbers) and regard that collection as "one" set—when we make a *unity* out of a *multiplicity*—we are committing an ambiguity. An ambiguity, indeed, that is a central feature of mathematical thinking. (Notice how the word "universal", with the sense of "all-embracing", uses the primitive root, "uni", a single slash or tally mark.)

In fact, the relation of "equality" in general is ambiguous, for the entities on the two sides of the equals sign are usually not identical. ("$x = x$" is not usually interesting.) In an interesting equation, the entities on the left and the right are not identical, so the claim that they are "equal" is necessarily ambiguous, subject to different interpretations according to context. Using the simplest example imaginable, Byers elucidates the ambiguity inherent in the notion of equality:

> When we encounter "1+1=2", our first reaction is that the statement is clear and precise. We feel that we understand it completely and that there is nothing further to be said. But is that really true? The numbers "one" and "two" are in fact extremely deep and important ideas…The equation also contains an equal sign. Equality is another very basic idea whose meaning only grows the more you think about it. Then we have the equation itself, which states that the fundamental concepts of unity and duality have a relationship with one another that we represent by "equality"—that there is unity in duality and duality in unity. This deeper structure that is implicit in the equation is typical of a situation of ambiguity. Thus even the most elementary mathematical expressions have a profundity that may not be apparent on the surface level. (p. 27)
>
> A more advanced example of the ambiguity of the equals sign is $1 = .999\ldots$.
>
> What is the precise meaning of the "=" sign? It surely does not mean that the number 1 is identical to that which is meant by the notation .999…. There is a problem here, and the evidence is that, in my experience, most undergraduate math majors do not believe this statement… they all agreed that .999…was very close to 1. Some even said "infinitely close", but they were not absolutely sure what they meant by this…. This notation stands both for the process of adding this particular infinite sequence of fractions and for the object, the number that is the result of that process…. Now the number 1 is clearly a mathematical object, a number. Thus the equation $1 = .999\ldots$ is confusing because it seems to say that a process is equal (identical?) to an object. This appears to be a category error. How can a process, a verb, be equal to an object, a noun? Verbs and nouns are "incompatible contexts" and thus the equation is ambiguous…. I hasten to add that this ambiguity is a strength, not a weakness, of our way of writing decimals. To understand infinite decimals means to be able to move freely from one of these points of view to the other. That is, understanding involves the realization that there is "one single idea" that can be expressed as 1 or as $1 = .999\ldots$ that can be understood as the process of summing an infinite series or an endless process of successive approximation as well as a concrete object, a number. This kind of creative leap is required before one

> can say that one understands a real number as an infinite decimal. (p. 41)

Byers also discusses how students struggle with ideas that are less advanced than 1 = .999… For example, here he unravels the ambiguity of the "variable" x as students encounter it in the seventh grade:

> Does the "x" in "$x + 2 = 4$" refer to *any* number or does it refer to the number 2? The answer is, "Both and neither." At the beginning, x could be anything. At the end, x can only be 2. Yet at the end, we are saying that every number x NOT = 2 is *not* a solution, so the equation is also about all numbers. Thus at every stage, the x stands for *all* numbers but ALSO for the *specific* number 2. We are required to carry along this ambiguity throughout the entire procedure of solving the equation. It begins with something that could be anything and ends with a specific number that could not be anything else. What an exercise in subtle mental gymnastics this is! How could this way of thinking be called *merely* mechanical? No wonder children have difficulty with algebra. The difficulty is the ambiguity. The resolution of the ambiguity, solving the equation, does not involve eliminating the double context but rather being able to keep the two contexts simultaneously in mind and working within that double context, jumping from one point of view to the other as the situation warrants. (p. 42)

Mathematicians are accustomed to making use of multiple "representations" of "the same" thing. With the precise notion of "isomorphic equivalence", we are able legitimately and smoothly to use different representations simultaneously. The group of permutations on three letters is "the same thing" as the automorphism group of the equilateral triangle, or the group of functions under composition generated by $1/x$ and $1-x$, and so on and so on. And any graph is equivalent to, is virtually "the same", as its adjacency matrix. And any solution of Laplace's partial differential equation is an integral with a Green's function as kernel, and it is simultaneously the minimal solution of a certain variational problem, and it is simultaneously the limit of a sequence of solutions of difference equations, and it is simultaneously the expected value of the outcome of the random motion of a Brownian particle, as well as the equilibrium distribution of heat in a homogeneous medium, and also the potential of a distribution of gravitational mass or electrostatic charge. When we make simultaneous or alternate use of "different" representations or interpretations of "the same" structure, we are using ambiguity in a controlled, algorithmic way—using the multiple-meaningness of the concepts of group, or graph, or solution of a differential equation.

In discussions of the nature of mathematics, the notion of "abstraction" is often mentioned, but rarely clarified or explicated. Byers has a remarkable explanation of abstraction. "Abstraction consists essentially in the creation and utilization of ambiguity." For example, when *functions* are first introduced, either in the classroom or in the history of mathematics, they are active. The function transforms one number into another. Later, when we focus on differential operators, the functions are passive. The operator transforms one function into another. So which is it? Is a function active or passive, verb or noun? "The initial barrier to understanding, that a function can be considered simultaneously as process and object—as a rule that operates on numbers and as an object that is itself operated on by other processes—turns into the insight. That is, it is precisely the ambiguous way in which a function is viewed which is the insight." (p. 48)

Byers doesn't stop with mathematics itself. Not only mathematics, but even more, philosophy of mathematics, is inextricably tied up with ambiguity, paradox, and contradiction. "Do we create math or do we discover it?" "Is it in our minds, or is it out there"? Contradictions, *nicht wahr*?

One deep ambiguity is the double meaning of "exist". Does it mean something is "constructed" from already "constructed" entities, by some clearly understood notion of "construct"? Or does it rather mean something is contradiction-free, is "safe" to "postulate", because it doesn't crash into or interfere with other notions or facts that we don't want disturbed? This is just the stale old argument between intuitionist/constructivists and standard/classical mathematicians. For Byers, the point isn't to choose sides, to decide who is right and who is wrong. Rather, it is to perceive that this ambiguity of "exist" is intrinsic to our mathematical practice, and is fruitful. The clash of viewpoints arising from this ambiguity brings forth interesting mathematics.

Speaking of the often mentioned but rarely analyzed unreflective Platonism of the working mathematician, Byers writes:

> The ambiguity of an unsolved problem is mitigated somewhat by the Platonic attitude of the working mathematician. That is, she feels that it is objectively either true or false and that the job of the mathematician is "merely" to discover which of these a priori conditions applies. Psychologically, this Platonic point of view brings the ambiguity of the situation into enough control so that researchers have confidence the correct solution exists independent of their efforts. It moves the problem from the domain of

> "ambiguity as vagueness" in which anything could happen to the sort of incompatibility that has been discussed in this chapter where there are two conflicting frameworks, true or false.

"Contradictions demand resolution!" you may say. "To rise to the next level in philosophy of mathematics, we must overcome the contradiction, resolve it, not just pooh-pooh it!"

But Byers offers us an insight—this is the way it has to be! Live with it! Life is ambiguous and contradictory. Mathematics is part of life. Insofar as the philosophy of mathematics describes the total mathematical situation—process as well as content—naturally it's also bound to be ambiguous.

Well, that does in fact seem to be the case.

You might say that the work of the mathematician is to drive away ambiguity. "Precision" is what mathematics is all about. "Say what you mean, mean what you say, nothing is there except what is right on the page." Byers pushes us back, to the ambiguous situation that calls for mathematical explication. He makes us see that the ambiguity we insist on banishing is the source, the origin, of the mathematical work. "Logic moves in one direction, the direction of clarity, coherence and structure. Ambiguity moves in the other direction, that of fluidity, openness, and release. Mathematics moves back and forth between these two poles.... It is the interaction between these different aspects that gives mathematics its power." (p. 78) "Mathematical ideas are not right or wrong; they are organizers of mathematical situations. Ideas are not logical. In fact the inclusion should go the other way around—logic is not the absolute standard against which all ideas must be measured. In fact logic itself is an idea." (p. 257)

The normal mathematician—the philosopher's "working mathematician", the ordinary mathematician, the "mathematician in the street"?—may respond with a shrug and a, "So what?" We do our calculations and prove our theorems by following our noses, not by looking right or left to see where we are in the broader conceptual or "philosophic" realm. You don't need to know what is meant by "one" in order to know that one and one is two. But recall the old saying of Socrates, about the unexamined life. Most mathematical life, like most human life in general, is unexamined. Byers pulls away the covering habit and routine, to expose life-giving embarrassments hidden beneath.

You can't quite say that nobody has said this before. But nobody has said it before in this all-encompassing, coherent way, and in this readable, crystal clear style. The examples are well known and familiar, but it's something else to put them all together and say, "This is it! This is exactly what mathematics is all about, this is the very core and nature of mathematical thinking!"

Byers finds far-reaching consequences, beyond mathematics, for our very understanding of what it means to be human.

> Any great quest demands courage. It is a voyage into the unknown with no guaranteed results. What is the nature of this courage? It is the courage to open oneself up to the ambiguity of the specific situation. The whole thing may end up as a vast waste of time, that is, the possibility of failure is inevitably present... Our lives also have this quality of a quest, the attempt to resolve some fundamental but ill-posed question. In working on a mathematical conjecture, life's ambiguities solidify into a concrete problem. That is, the situation of doing research is isomorphic to some extent with the situation we face in our personal lives. This is one reason that working on mathematics is so satisfying. In resolving the mathematical problem we, for a while at least, resolve that large, existential problem that is consciously or unconsciously always with us.... Learners need support when they are encouraged to enter into new unexplored ambiguities. A new learning experience requires the learner to face the unknown, to face failure. Sticking with a true learning situation requires courage and teachers must respect the courage that students exhibit in facing these situations. Teachers should understand and sympathize with students' reluctance to enter into these murky waters. After all, the teacher's role as authority figure is often pleasing insofar as it enables the teacher to escape temporarily from their own ambiguities and vulnerability. Thus the value of learning potentially goes beyond the specific content or technique but in the largest sense is a lesson in life itself. (p. 57)

This book strikes me as profound, unpretentious, and courageous.

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