Your Hit Parade: The Top Ten Most Fascinating Formulas in Ramanujan's Lost Notebook

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At 7:30 on a Saturday evening in March 1956, the first author sat down in an easy chair in the living room of his parents' farm home ten miles east of Salem, Oregon, and turned the TV channel knob to NBC's *Your Hit Parade* to find out the Top Seven Songs of the week, as determined by a national "survey" and sheet music sales. Little did this teenager know that almost exactly twenty years later, he would be at Trinity College, Cambridge, to discover one of the biggest "hits" in mathematical history, Ramanujan's Lost Notebook. Meanwhile, at that same hour on that same Saturday night in Stevensville, Michigan, but at 9:30, the second author sat down in an overstuffed chair in front of the TV in his parents' farm home anxiously waiting to learn the identities of the Top Seven Songs, sung by *Your Hit Parade* singers, Russell Arms, Dorothy Collins (his favorite singer), Snooky Lanson, and Gisele MacKenzie. About twenty years later, that author's life would begin to be consumed by Ramanujan's mathematics, but more important than Ramanujan to him this evening was how long his parents would allow him to stay up to watch Saturday night wrestling after *Your Hit Parade* ended.

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Just as the authors anxiously waited for the identities of the Top Seven Songs of the week years ago, readers of this article must now be brimming with unbridled excitement to learn the identities of the Top Ten Most Fascinating Formulas from Ramanujan's Lost Notebook. The choices for the Top Ten Formulas were made by the authors. However, motivated by the practice of *Your Hit Parade*, but now extending the "survey" outside the boundaries of the U.S., we have taken an international "survey" to determine the proper order of fascination and amazement of these formulas. The survey panel of 34 renowned experts on Ramanujan's work includes Nayandeep Deka Baruah, S. Bhargava, Jonathan Borwein, Peter Borwein, Douglas Bowman, David Bradley, Kathrin Bringmann, Song Heng Chan, Robin Chapman, Youn-Seo Choi, Wenchang Chu, Shaun Cooper, Sylvie Corteel, Freeman Dyson, Ronald Evans, Philippe Flajolet, Christian Krattenthaler, Zhi-Guo Liu, Lisa Lorentzen, Jeremy Lovejoy, Jimmy McLaughlin, Steve Milne, Ken Ono, Peter Paule, Mizan Rahman, Anne Schilling, Michael Schlosser, Andrew Sills, Jaebum Sohn, S. Ole Warnaar, Kenneth Williams, Ae Ja Yee, Alexandru Zaharescu, and Doron Zeilberger. A summary of their rankings can be found in the last section of our paper. Just as the songs changed weekly on *Your Hit Parade*, the choices for the Top Ten Most Fascinating Formulas also change from week to week. The reason is simple. There are so many fascinating results in the lost notebook that thinking about a particular formula during the week will naturally generate increased appreciation for it, if not increased understanding, and vault it into the Top Ten, meanwhile shoving a not so recently contemplated formula out of the Top Ten.

It was the practice of *Your Hit Parade* to present the songs in reverse order of popularity to build up excitement about the identity of the Number One Song of the Week. (Of course, after the identification of the Number Two Song, you would have had to have slept through the program if you did not then immediately deduce the identity of the Number One Song to be announced about five minutes later.) However, it is the fate of "popular" songs to lose their popularity and fade off the charts. In fact, the immense popularity of *Your Hit Parade*, beginning on radio in 1935 and switching to TV in 1950, began to rapidly decline with the advent of Rock 'n Roll. On April 24, 1959, *Your Hit Parade* aired for the last time. It was said that Snooky Lanson could not compete with Elvis Presley in singing the latter's famous hit song, "Hound Dog". But although popular songs may fade away, Ramanujan's theorems do not fade away. They will remain as fascinating to mathematicians of future generations as they are to mathematicians of our present generation.

So here they are—The Top Ten Most Fascinating Formulas From Ramanujan's Lost Notebook—as determined by the authors, ordered by our panel of experts, and presented in reverse order of popularity among the members of the panel.

Notation

We employ the familiar notation in the theory of q-series. For each nonnegative integer n, let

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}), \tag{1}$$

with the convention that $(a;q)_0 := 1$. Set

$$(a;q)_\infty := \lim_{n\to\infty}(a;q)_n, \qquad |q|<1. \tag{2}$$

No. 10. A Remarkable Bilateral Hypergeometric Sum

On page 200 of his lost notebook [62], Ramanujan offers two results on certain bilateral hypergeometric series. The second follows from a theorem of J. Dougall [38], and we will not discuss it here. The first gives a formula for the derivative of a quotient of two certain bilateral hypergeometric series. Ramanujan's formula needs to be slightly corrected, but what is remarkable is that such a formula exists!

Theorem 1 (Corrected, p. 200)**.** *Define, for any real numbers θ and s and for any complex numbers α, β, γ, and δ, such that* $\mathrm{Re}(\alpha+\beta+\gamma+\delta)>3$,

$$\varphi_s(\theta) := \sum_{n=-\infty}^{\infty} \frac{e^{(n+s)i\theta}}{\Gamma(\alpha+s+n)\Gamma(\beta-s-n)\Gamma(\gamma+s+n)\Gamma(\delta-s-n)}.$$

Then

$$\frac{d}{d\theta}\frac{\varphi_s(\theta)}{\varphi_t(\theta)} = \frac{i\sin\{\pi(s-t)\}\left(2\sin(\tfrac{1}{2}\theta)\right)^{\alpha+\beta+\gamma+\delta-4} e^{i(\pi-\theta)(\alpha-\beta+\gamma-\delta+2s+2t)/2}}{\pi\varphi_t^2(\theta)\Gamma(\alpha+\beta-1)\Gamma(\beta+\gamma-1)\Gamma(\gamma+\delta-1)\Gamma(\delta+\alpha-1)}. \tag{3}$$

When a calculus student is asked to differentiate a quotient, she immediately turns to the familiar quotient rule from differential calculus. In this instance, as expected, the square of the denominator appears in the differentiated formula, and in the numerator she obtains the difference of two products of bilateral hypergeometric series. But, according to (3), we see that Ramanujan had the remarkable insight to see that this difference of products of series could be evaluated in closed form!

To explain the origin of (3), we review some background about bilateral hypergeometric series. For every integer n, define, in contrast to the notation in the previous section and elsewhere in this paper,

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)}.$$

The bilateral hypergeometric series ${}_pH_p$ is defined for complex parameters $a_1, a_2, \ldots, a_p$ and $b_1, b_2, \ldots, b_p$ by

$${}_pH_p\left[\begin{matrix} a_1, a_2, \ldots, a_p; \\ b_1, b_2, \ldots, b_p; \end{matrix} z\right] := \sum_{n=-\infty}^{\infty} \frac{(a_1)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n(b_2)_n\cdots(b_p)_n} z^n.$$

With the use of D'Alembert's ratio test, it can be checked that ${}_pH_p$ converges only for $|z|=1$, provided that [65, p. 181, eq. (6.1.1.6)]

$$\mathrm{Re}(b_1+b_2+\cdots+b_p-a_1-a_2-\cdots-a_p)>1.$$

The series ${}_pH_p$ is said to be *well-poised* if

$$a_1+b_1 = a_2+b_2 = \cdots = a_p+b_p.$$

In 1907, Dougall [38] showed that a well-poised ${}_5H_5$ series could be evaluated at $z=1$. We do not provide here his evaluation in terms of gamma functions, but simply emphasize its importance in proving Theorem 1. Dougall [38] also evaluated in closed form a general ${}_2H_2$ at $z=1$, from which one can deduce the following bilateral form of the binomial series theorem. If a and c are complex numbers with $\mathrm{Re}(c-a)>1$ and z is a complex number with $|z|=1$, then

$${}_1H_1\left[\begin{matrix} a; \\ c; \end{matrix} z\right] = \frac{(1-z)^{c-a-1}}{(-z)^{c-1}}\frac{\Gamma(1-a)\Gamma(c)}{\Gamma(c-a)}. \tag{4}$$

For further remarks on (4), see the paper by the second author and W. Chu [26].

The proof of Theorem 1 now proceeds as follows. Use the familiar quotient rule to differentiate

$$\frac{d}{d\theta}\frac{\varphi_s(\theta)}{\varphi_t(\theta)}.$$

Combine the two products of series in the numerator into one double series. After some rearrangement, we find that the inner series of the resulting double series is surprisingly a well-poised ${}_5H_5$, which can be summed by Dougall's theorem. There remains a single bilateral sum, which we can evaluate by using (4). Theorem 1 now follows. See [26] for complete details.

No. 9. Some Challenging Integrals for Your Calculus Students

In his lost notebook [62], Ramanujan records approximately 15 equalities between two different types of integrals. On the left-hand sides are integrals of Dedekind eta-functions, $\eta(z)$, and on the right-hand sides are differences of two incomplete integrals of the first kind. To describe these identities, of a type never before seen, we introduce Ramanujan's notations for theta functions.

Define, following Ramanujan, for $|q| < 1$,

$$\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{5}$$

and, for $q = e^{2\pi iz}$ and $\operatorname{Im} z > 0$,

$$\begin{aligned} f(-q) &:= \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} \\ &= (q;q)_\infty =: e^{-2\pi iz/24}\eta(z). \end{aligned} \tag{6}$$

The product representations in (5) and (6) (and (35)) are instances of the Jacobi triple product identity. An incomplete elliptic integral of the first kind is an integral of the type

$$\int_0^\alpha \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}, \qquad 0 < \alpha \le \pi/2,$$

where k, $0 < k < 1$, is called the *modulus* of the integral. The complete elliptic integral of the first kind is that above when $\alpha = \pi/2$ and is denoted by $K(k)$, i.e.,

$$K(k) := \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}.$$

To illustrate Ramanujan's formulas, we record a triple of integral formulas found on page 52 in Ramanujan's lost notebook [62]. Recall first that the Rogers–Ramanujan continued fraction is defined by

$$\begin{aligned} R(q) &:= \frac{q^{1/5}}{1}\,\frac{q}{+\,1}\,\frac{q^2}{+\,1}\,\frac{q^3}{+\,1}+\cdots \\ &= q^{1/5}\frac{(q^2;q^5)_\infty(q^3;q^5)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty}, \qquad |q|<1, \end{aligned} \tag{7}$$

where the product representation is a consequence of the Rogers–Ramanujan identities (23) and (24).

Theorem 2. *With $f(-q)$, $\psi(q)$, and $R(q)$ defined by* (6), (5), *and* (7), *respectively, and with $\epsilon = (\sqrt{5}+1)/2$,*

$$5^{3/4}\int_0^q \frac{f^2(-t)f^2(-t^5)}{\sqrt{t}}dt \tag{8}$$

$$= 2\int_{\cos^{-1}\left((\epsilon R(q))^{5/2}\right)}^{\pi/2} \frac{d\varphi}{\sqrt{1-\epsilon^{-5}5^{-3/2}\sin^2\varphi}}$$

$$= \int_0^{2\tan^{-1}\left(5^{3/4}\sqrt{q}f^3(-q^5)/f^3(-q)\right)} \frac{d\varphi}{\sqrt{1-\epsilon^{-5}5^{-3/2}\sin^2\varphi}} \tag{9}$$

$$= \sqrt{5}\int_0^{2\tan^{-1}\left(5^{1/4}\sqrt{q}\psi(q^5)/\psi(q)\right)} \frac{d\varphi}{\sqrt{1-\epsilon 5^{-1/2}\sin^2\varphi}}. \tag{10}$$

Readers do not have to be told that these are rather uncommon integral formulas that would present a challenge to any integral zealot not steeped in the mathematics of Ramanujan surrounding theta functions. Theorem 2 and the other formulas in the lost notebook of this sort were first proved by S. Raghavan and S. S. Rangachari [57], but some of their proofs are probably not like those of Ramanujan, because they used the theory of modular forms. Motivated by this fact, Berndt, H. H. Chan, and S.–S. Huang [25] found proofs for all of Ramanujan's approximately 15 formulas employing only results from Ramanujan's notebooks [61] and lost notebook [62]. See also Chapter 15 of our book [10].

We briefly indicate some of the ingredients in the proofs of (8)–(10) and Ramanujan's further formulas of this sort.

First, to prove (8)–(10), three different transformation formulas for incomplete elliptic integrals are needed. One of them is the duplication formula given in the following lemma [21, p. 106, Entry 17(vi)], [61].

Lemma 3. *Suppose that $0 < \alpha, \frac{1}{2}\beta < \frac{1}{2}\pi$. If $\cot\alpha\ \tan(\frac{1}{2}\beta) = \sqrt{1-x\sin^2\alpha}$, then*

$$2\int_0^\alpha \frac{d\varphi}{\sqrt{1-x\sin^2\varphi}} = \int_0^\beta \frac{d\varphi}{\sqrt{1-x\sin^2\varphi}}. \tag{11}$$

Ramanujan found many transformations like (11). Some can be found in Entry 7 of Chapter 17 in [61], [21, pp. 104–114], but others are scattered throughout his notebooks.

Secondly, modular equations play key roles in some proofs.

Thirdly, differential equations are sometimes necessary. For (8), only the simple differential equation

$$R'(q) = \frac{R(q)}{5q}\frac{f^5(-q)}{f(-q^5)}$$

is needed. However, for other integral identities, more difficult differential equations are required. We give one such example.

Lemma 4. *Let ν be defined by*

$$\nu := \nu(q) := q\left(\frac{f(-q)f(-q^{15})}{f(-q^3)f(-q^5)}\right)^3. \tag{12}$$

Then

$$\frac{d\nu}{dq} = f(-q)f(-q^3)f(-q^5)f(-q^{15}) \times \sqrt{1-10\nu-13\nu^2+10\nu^3+\nu^4}.$$

The differential equation of Lemma 4 is not easy to prove and is crucial in proving the following theorem of Ramanujan from page 51 in his lost notebook.

Theorem 5. *Let ν be defined by* (12)*, and let $\epsilon = (\sqrt{5}+1)/2$. Then*

$$\int_0^q f(-t)f(-t^3)f(-t^5)f(-t^{15})dt = \frac{1}{5}\int_{2\tan^{-1}\left(\frac{1}{\sqrt{5}}\sqrt{\frac{1-11\nu-\nu^2}{1+\nu-\nu^2}}\right)}^{2\tan^{-1}(1/\sqrt{5})} \frac{d\varphi}{\sqrt{1-\frac{9}{25}\sin^2\varphi}}. \tag{13}$$

Although we are able to prove Theorems 2 and 5, as well as Ramanujan's further claims about such integrals, we would not have been able to do so without knowing the formulas at the start. In other words, we do not know what led Ramanujan to believe that such formulas existed. A few further theorems were established in [25], but we know of no other formulas of this kind in the literature. As indicated at the end of [25], these formulas appear to be connected with elliptic curves. But, in summary, further study and understanding are necessary.

No. 8. A Double Sum of Bessel Functions and Sums of Two Squares

On page 335 in [62], Ramanujan records two identities, each involving a double series of Bessel functions. We discuss only one of the two identities.

The first identity involves the ordinary Bessel function $J_1(z)$, where

$$J_\nu(z) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu+n+1)}\left(\frac{z}{2}\right)^{\nu+2n}, \quad 0 < |z| < \infty, \nu \in \mathbb{C}.$$

To state Ramanujan's claim, we need to also define

$$F(x) = \begin{cases} [x], & \text{if } x \text{ is not an integer,} \\ x-\frac{1}{2}, & \text{if } x \text{ is an integer,} \end{cases} \tag{14}$$

where, as customary, $[x]$ is the greatest integer less than or equal to x.

Theorem 6. *If $0 < \theta < 1$ and $x > 0$, then*

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\sin(2\pi n\theta) = \pi x\left(\tfrac{1}{2}-\theta\right) - \tfrac{1}{4}\cot(\pi\theta) + \frac{1}{2}\sqrt{x}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{J_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}} - \frac{J_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}}\right\}. \tag{15}$$

Note that the series on the left-hand side of (15) is finite, and it is discontinuous if x is an integer. To examine the right-hand side, we recall that [73, p. 199], as $x \to \infty$,

$$J_\nu(x) \sim \left(\frac{2}{\pi x}\right)^{1/2}\cos\left(x - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi\right). \tag{16}$$

Hence, as $m, n \to \infty$, the terms of the double series on the right-hand side of (15) are asymptotically equal to

$$\frac{1}{\pi\sqrt{2}x^{1/4}m^{3/4}}\left(\frac{\cos\left(4\pi\sqrt{m(n+\theta)x}-\frac{3}{4}\pi\right)}{(n+\theta)^{3/4}} - \frac{\cos\left(4\pi\sqrt{m(n+1-\theta)x}-\frac{3}{4}\pi\right)}{(n+1-\theta)^{3/4}}\right).$$

Thus, if indeed the double series on the right side of (15) does converge, it converges conditionally and not absolutely.

We might ask what led Ramanujan to consider (15). Recall that Ramanujan visited G. H. Hardy in Cambridge during the years 1914–1919. Early in this stay, Hardy focused attention on the classical circle and divisor problems. We show that the first double series of Bessel functions (15) is related to the circle problem; the second series of Bessel functions is related to the divisor problem and a series of G. Voronoï [70].

Let $r_2(n)$ denote the number of representations of the positive integer n as a sum of two squares. Since each representation of an integer can be associated with a lattice point in the plane, we can write, with $r_2(0) := 1$,

$$\sum_{0\le n\le x}{}' r_2(n) = \pi x + P(x), \tag{17}$$

where the prime $'$ on the summation sign on the left side indicates that if x is an integer, only $\frac{1}{2}r_2(x)$ is counted. One of the most famous unsolved problems in the theory of numbers is to determine the correct order of magnitude of the "error term" $P(x)$ as $x \to \infty$. This is the "circle problem." It was shown by Gauss that $P(x) = O(\sqrt{x})$, as $x \to \infty$.

W. Sierpinski [64] in 1906, and then Hardy [43], [44, pp. 243–263] in 1915 proved that

(18)
$$\sum_{0\le n\le x}{}' r_2(n) = \pi x + \sum_{n=1}^{\infty} r_2(n)\left(\frac{x}{n}\right)^{1/2} J_1(2\pi\sqrt{nx}).$$

Thus, using (18) in (17), we can obtain a representation for $P(x)$ as an infinite series of Bessel functions. Observe that the series on the right-hand side of (18) is similar to the inner series on the right side of (15). In fact, one can derive the following corollary [29] of Theorem 6.

Corollary 7. *For any* $x > 0$,

(19)
$$\sum_{0\le n\le x}{}' r_2(n) = \pi x + 2\sqrt{x}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left\{\frac{J_1\left(4\pi\sqrt{m(n+\frac14)x}\right)}{\sqrt{m(n+\frac14)}} - \frac{J_1\left(4\pi\sqrt{m(n+\frac34)x}\right)}{\sqrt{m(n+\frac34)}}\right\}.$$

Ramanujan might therefore have derived (15) in anticipation of applying it to the circle problem. However, we have no record of any further work of Ramanujan on the circle problem. Because $r_2(n)$ does not arise on the right-hand side of (19), it may be that (19) is potentially more useful in the circle problem than (18). On the other hand, double series are usually more difficult to estimate than a single infinite series. We do not provide details, but it is not difficult to show that (19) can be derived from (18), and conversely.

In [43], Hardy proved that

(20)
$$P(x) \ne O\left(x^{1/4}\log^{1/4} x\right).$$

In the past 90 years, this result has been improved only once. In 2003, K. Soundararajan [67] proved that

$$P(x) \ne O\left(x^{1/4}\log^{1/4} x\frac{(\log\log x)^{3(2^{1/3}-1)/4}}{(\log\log\log x)^{5/8}}\right).$$

In fact, (18) was not employed by Hardy in his proof of (20). The identity (18) is more useful in obtaining an upper bound for $P(x)$, and it has been the starting point of almost all investigations on finding upper bounds for $P(x)$. In particular, Sierpinski [64] used (18) to prove that $P(x) = O(x^{1/3})$, giving the first improvement on Gauss's upper bound $P(x) = O(\sqrt{x})$. Since 1906, the exponent 1/3 has been gradually whittled down by a succession of several mathematicians. Currently, the best result,

(21)
$$P(x) = O\left(x^{131/408}(\log x)^{18,627/8320}\right)$$

is due to M. N. Huxley [49]. Note that $\frac{131}{408} = .3149\ldots$. It is conjectured that

$$P(x) = O\left(x^{1/4+\epsilon}\right),$$

for every $\epsilon > 0$. Thus, the theorems of Hardy and Soundararajan are thought to be much closer to the correct order of magnitude of $P(x)$ than (21).

Returning to (15), Berndt and A. Zaharescu [29] have found a long, difficult proof of (15), but with *the order of summation reversed.* However, they [30] recently found a completely different proof of (15) with the order of summation as prescribed by Ramanujan. A corollary of their proofs is that the order of summation in the double series in (15) can be reversed without affecting the equality. Can one directly prove that the order of summation in (15) can be reversed? This would seem to be an extremely difficult problem in view of the delicate convergence of the double series.

In this same paper [43], Hardy relates a beautiful identity of Ramanujan connected with $r_2(n)$, namely, for $a, b > 0$, [43, p. 283], [44, p. 263],

$$\sum_{n=0}^{\infty}\frac{r_2(n)}{\sqrt{n+a}}e^{-2\pi\sqrt{(n+a)b}} = \sum_{n=0}^{\infty}\frac{r_2(n)}{\sqrt{n+b}}e^{-2\pi\sqrt{(n+b)a}},$$

which is not given elsewhere in any of Ramanujan's published or unpublished work. This is further evidence that Hardy's work on sums of squares had captured Ramanujan's attention.

No. 7. Hadamard Products

On page 57 of Ramanujan's lost notebook [62], we find one of the most peculiar of all of Ramanujan's formulas

(22)
$$\sum_{n=0}^{\infty}\frac{a^n q^{n^2}}{(q;q)_n} = \prod_{n=1}^{\infty}\left(1+\frac{aq^{2n-1}}{1-q^n y_1 - q^{2n}y_2 - q^{3n}y_3 - \cdots}\right),$$

where

$$y_1 = \frac{1}{(1-q)\psi^2(q)},$$
$$y_2 = 0,$$
$$y_3 = y_1\frac{q+q^3}{(1-q^2)(1-q^3)} - y_1^3\sum_{n=0}^{\infty}\frac{(2n+1)q^{2n+1}}{1-q^{2n+1}},$$
$$y_4 = y_1 y_3,$$

and $\psi(q)$ is defined by (5). The most perplexing aspect of this formula, when first encountering it, is that the left-hand side is quite familiar. Indeed, it occurs in an identity originally published by L. J. Rogers and Ramanujan [63], namely,

$$\sum_{n=0}^{\infty}\frac{a^n q^{n^2}}{(q;q)_n} = \frac{1}{(aq;q)_\infty}\times\left(1+\sum_{j=1}^{\infty}\frac{(-1)^j a^{2j} q^{j(5j-1)/2}(1-aq^{2j})}{1-q^j}\frac{(aq;q)_{j-1}}{(q;q)_{j-1}}\right).$$

From this identity it is a simple exercise to deduce the celebrated Rogers–Ramanujan identities

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty} \tag{23}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{24}$$

However, the right-hand side of (22) is bizarre, since it does not appear to lie in the classical theory of q-hypergeometric series, even though it does contain familiar objects such as the classical theta function $\psi(q)$. Upon reflection, the central idea dawns. This is the Hadamard product for the entire function of the variable a

$$\sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q;q)_n},$$

where for our purposes here q is a fixed parameter with $0 < q < 1$.

The first published proof of (22) [7] by Andrews claims a larger domain of validity than is actually delivered by the argument of [7]. In fact, the claim that (22) is valid for $0 < q < \frac{1}{4}$ has to be modified to $0 < q < 0.00406$. However, empirical studies suggest that, in fact, (22) is valid for $0 < q < 1$.

Subsequently, several further papers have been written developing this topic. A companion identity found on page 26 of the lost notebook gives a Hadamard product expansion for the entire function of a

$$\sum_{n=0}^{\infty} a^n q^{n^2}. \tag{25}$$

This Hadamard product of (25) was examined in [8], and again it was claimed that the Hadamard expansion is valid for $0 < q < \frac{1}{4}$. Again (following standard methods from the theory of implicit functions [53]), this interval must be constricted to $0 < q < 0.00792$.

By completely different methods, W. Hayman [45] asymptotically established the formulas for the zeros of (22), as well as for those of a generalization

$$\sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q;q)_n (bq;q)_n}. \tag{26}$$

T. Huber [47], [48] generalized the methods of [7] to obtain results similar to Andrews for the series (26) as well as for

$$\sum_{n=0}^{\infty} \frac{(-c/(aq);q)_n a^n q^{n(n+1)/2}}{(q;q)_n (bq;q)_n}.$$

Finally, M. Ismail and C. Zhang [51], along completely new lines, developed a method involving elliptic integrals for proving (22) and related formulas. The power of their method allows them to prove that the domain $0 < q < 1$ is, at least, eventually valid for all zeros of sufficiently large modulus.

No. 6. An Integral with Character

With a modest change in notation, on page 207 in his lost notebook [62], Ramanujan offers the following formula for a character analogue of the Dedekind eta-function.

Theorem 8. *Let $\chi(n)$ denote the Legendre symbol $\left(\frac{n}{3}\right)$, and recall that $f(-q)$ is defined by* (6). *Then, for* $0 < q < 1$,

$$q^{1/9} \prod_{n=1}^{\infty} (1-q^n)^{n\chi(n)} = \exp\left(-C - \frac{1}{9}\int_q^1 \frac{f^9(-t)}{f^3(-t^3)}\frac{dt}{t}\right), \tag{27}$$

where

$$C := \frac{3\sqrt{3}}{4\pi} L(2,\chi) = L'(-1,\chi), \tag{28}$$

where $L(s,\chi)$ denotes the Dirichlet L-function associated with the character χ.

The second equality in (28) was not given by Ramanujan. In fact, after the first equality, Ramanujan recorded two question marks ??, evidently indicating some doubt about the truth of his formula. However, Ramanujan is indeed correct. Theorem 8 was first proved by S. H. Son [66] in an incomplete form, because the identity of C was not addressed. Berndt and Zaharescu [28] gave a completely different proof of Theorem 8 in which the value of C in (28) naturally emerged from their proof. A nagging question now loomed. Is (27) an isolated identity, or are there further examples of this sort?

It turns out that one should not think of Theorem 8 in terms of Ramanujan's function $f(-q)$, but instead in terms of Eisenstein series.

Suppose that χ is a nontrivial primitive character modulo N and that, for $n \geq 0$, $B_{n,\chi}$ denotes the nth generalized Bernoulli number defined by [69, p. 12]

$$\sum_{n=1}^{N} \frac{\chi(n) t e^{nt}}{e^{Nt}-1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{t^n}{n!}, \qquad |t| < 2\pi/N.$$

Let $k \geq 2$ be an integer, and choose a character χ such that $\chi(-1) = (-1)^k$. If $q := e^{2\pi i z}$, the function

$$E_{k,\chi}(z) := 1 - \frac{2k}{B_{k,\chi}} \sum_{n=1}^{\infty} \sum_{d|n} \chi(d) d^{k-1} q^n \tag{29}$$

is an Eisenstein series of weight k and character χ on the congruence subgroup $\Gamma_0(N)$ of the full modular group. We note that (with the correct choice of α) the integrand in (30) below is such an Eisenstein series.

Theorem 9. *Suppose that α is real, that $k \geq 2$ is an integer, and that χ is a nontrivial Dirichlet character that satisfies the condition* $\chi(-1) = (-1)^k$.

Then, for $0 < q < 1$,

$$q^{\alpha}\prod_{n=1}^{\infty}(1-q^n)^{\chi(n)n^{k-2}} \tag{30}$$
$$= \exp\left(-C - \int_q^1 \left\{\alpha - \sum_{n=1}^{\infty}\sum_{d|n}\chi(d)d^{k-1}t^n\right\}\frac{dt}{t}\right),$$

where

$$C = L'(2-k,\chi).$$

One consequence of Theorem 9 is that it provides an explanation of another identity found in Ramanujan's lost notebook and first proved by Andrews [3]. If we let $\chi(n) = \left(\frac{n}{5}\right)$, which is an even character, and $k = 2$, then it can be shown that Theorem 9 yields the identity

$$q^{1/5}\prod_{n=1}^{\infty}(1-q^n)^{\chi(n)} = \exp\left(-C_5 - \frac{1}{5}\int_q^1 \frac{f^5(-t)}{f(-t^5)}\frac{dt}{t}\right), \tag{31}$$

where

$$C_5 = L'(0,\chi).$$

Although we shall not provide details, which can be found in [1], (31) is equivalent to an identity for the Rogers–Ramanujan continued fraction $R(q)$ given by

$$R(q) = \frac{\sqrt{5}-1}{2}\exp\left(-\frac{1}{5}\int_q^1 \frac{f^5(-t)}{f(-t^5)}\frac{dt}{t}\right) \tag{32}$$

and found on page 46 in Ramanujan's lost notebook. To deduce the left-hand side of (32) from the left-hand side of (31), use the identity for $R(q)$ found in (7). Further examples of Theorem 9 can be found in [1]. Considerable generalizations of Theorem 9 have been established by Y. Yang [74] and R. Takloo-Bighash [68].

No. 5. Sums of Tails of Euler's Partition Products

In the middle of page 14 of the lost notebook [62] appear, at first glance, two of the strangest formulas in the entire volume, namely,

$$\sum_{n=0}^{\infty}(S(q)-(-q;q)_n) = S(q)D(q) + \frac{1}{2}R(q) \tag{33}$$

and

$$\sum_{n=0}^{\infty}\left(S(q) - \frac{1}{(q;q^2)_{n+1}}\right) = S(q)D(q^2) + \frac{1}{2}R(q), \tag{34}$$

where

$$S(q) := (-q;q)_{\infty} = \frac{1}{(q;q^2)_{\infty}},$$
$$D(q) := -\frac{1}{2} + \sum_{n=1}^{\infty}\frac{q^n}{1-q^n},$$

and

$$R(q) = 1 + \sum_{n=1}^{\infty}\frac{q^{n(n+1)/2}}{(-q;q)_n}.$$

Note that, in each of (33) and (34), Ramanujan is summing the difference between an infinite product, $S(q)$, and its nth partial product. The function $R(q)$ is the generating function for the excess of the number of partitions into distinct parts with even rank over those with odd rank. In [14], it is shown that almost all the coefficients of $R(q)$ are 0 (i.e., asymptotically 100%), and that every integer appears as a coefficient infinitely often.

One would hope that there is some general principle from which both (33) and (34) would emerge. However, the first proofs of (33) and (34) in [5] are essentially the long culmination of a long struggle with these identities and provide no general insight.

Subsequently, D. Zagier [75] proved a similar result

$$\sum_{n=0}^{\infty}((q;q)_{\infty}-(q;q)_n) = D(q)(q;q)_{\infty}$$
$$+\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{12}{n}\right)nq^{n^2/24},$$

where $\left(\frac{12}{n}\right)$ denotes the Jacobi symbol. It was subsequently discovered that N. J. Fine [41, p. 14] had earlier proved that

$$\sum_{n=0}^{\infty}\left(\frac{1}{(q;q)_{\infty}} - \frac{1}{(q;q)_n}\right) = \frac{1}{(q;q)_{\infty}}\left(D(q)+\frac{1}{2}\right).$$

All of these examples taken together strongly suggest that a common rationale lies behind all these discoveries. This turns out to be the case. In [18], the following lemma was proved.

Lemma 10. *Suppose that* $f(z) = \sum_{n=0}^{\infty}\alpha(n)z^n$ *is analytic for* $|z| < 1$. *If* α *is a complex number for which*

$$\sum_{n=0}^{\infty}|\alpha-\alpha(n)| < \infty$$

and

$$\lim_{n\to\infty} n(\alpha-\alpha(n)) = 0,$$

then

$$\lim_{z\to 1^-}\frac{d}{dz}(1-z)f(z) = \sum_{n=0}^{\infty}(\alpha-\alpha(n)).$$

This unleashed numerous results similar to (33) and (34) including seven further results in [18] and others by G. H. Coogan and Ono [37]. For example [18, p. 407], if

$$\varphi(-q) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} = \frac{(q;q)_{\infty}}{(-q;q)_{\infty}}, \tag{35}$$

then Andrews, Jimenez, and Ono [18] proved that

$$\sum_{n=0}^{\infty}\left(\varphi(q)-\frac{(q;q)_n}{(-q;q)_n}\right)$$

$$=-\frac{2}{\varphi(-q)}\sum_{n=1}^{\infty}\frac{(-q;q)_{n-1}}{(q;q)_{n-1}}\frac{q^n}{(1-q^n)^2}.$$

It should be noted that both Zagier [75] and the aforementioned authors were primarily concerned with the applications of such results to finding values of certain L-functions.

In [5], the first author asked for combinatorial proofs of (33) and (34). Recently, in a beautiful paper [36], W. Y. C. Chen and K. Q. Ji provided the combinatorial proofs that were sought by several researchers.

Finally, noting that Lemma 10 was a natural next step beyond Abel's Lemma, Andrews and P. Freitas [15] established an infinite family of extensions of Abel's Lemma and applied their results to obtain further q-series identities.

No. 4. A Continued Fraction with Three Limit Points

Let $\omega = e^{2\pi i/3}$. On page 45 in his lost notebook [62], Ramanujan writes, for $|q| < 1$,

(36)

$$\lim_{n\to\infty}\left(\frac{1}{1}-\frac{1}{1+q}-\frac{1}{1+q^2}-\cdots-\frac{1}{1+q^n+a}\right)$$

$$=-\omega^2\left(\frac{\Omega-\omega^{n+1}}{\Omega-\omega^{n-1}}\right)\cdot\frac{(q^2;q^3)_\infty}{(q;q^3)_\infty},$$

where

$$\Omega := \frac{1-a\omega^2}{1-a\omega}\,\frac{(\omega^2 q;q)_\infty}{(\omega q;q)_\infty}. \tag{37}$$

Of course, because of the appearance of the limiting variable n on the right side of (36), Ramanujan's claim is meaningless as it stands, but properly interpreted, the claim (36) is correct and interesting. Ramanujan is indicating that (36) has three limits, depending upon the residue class modulo 3 in which n lies. This should be compared to the classical theorem in the theory of continued fractions, which asserts that if all the elements of a divergent continued fraction are positive, then the even and odd approximants approach distinct limits [54, pp. 96–97]. Before further discussing why (36) belongs to the Top Ten in our Hit Parade of Ramanujan's fascinating formulas, we restate it in the more standard fashion in which it was first proved by Andrews, Berndt, J. Sohn, A. J. Yee, and A. Zaharescu [13]. See also, [10, Entry 8.2.2].

Theorem 11. *Let $N-1 = 3\nu+\epsilon$, where $\epsilon = 0$ or ± 1. Then*

(38)

$$\lim_{N\to\infty}\left(\frac{1}{1}-\frac{1}{1+q}-\frac{1}{1+q^2}-\cdots-\frac{1}{1+q^{N-1}+a}\right)$$

$$=-\omega^2\frac{\Omega-\omega^{\epsilon+1}}{\Omega-\omega^{\epsilon-1}}\frac{(q^2;q^3)_\infty}{(q;q^3)_\infty},$$

where Ω is defined in (37).

In his notebooks [61, Vol. 2, p. 290], for $0 < |q| < 1$, Ramanujan offered the continued fraction

$$\frac{(q^2;q^3)_\infty}{(q;q^3)_\infty}=\frac{1}{1}-\frac{q}{1+q}-\frac{q^3}{1+q^2}-\frac{q^5}{1+q^3}-\cdots$$

$$=\frac{1}{1}-\frac{1}{q^{-1}+1}-\frac{1}{q^{-2}+1}-\frac{1}{q^{-3}+1}-\cdots, \tag{39}$$

which was first proved by Andrews, Berndt, Jacobsen, and Lamphere [11], and later proved more simply by Andrews, Berndt, Sohn, Yee, and Zaharescu [12]. Thus, when $a = 0$, the continued fraction on the left side of (36) or (38) is the same as the continued fraction on the far right side of (39), but with q replaced by $1/q$.

Remarkably, $(q^2;q^3)_\infty/(q;q^3)_\infty$ appears in the three limits on the right side of (36) or (38). In this sense, Ramanujan's result (36) is analogous to his theorem on the divergence of the Rogers–Ramanujan continued fraction $R(q)$ (defined in (7)), found on pages 374 and 382 in his third notebook [61], and first proved by Andrews, Berndt, Jacobsen, and Lamphere [11], [22, p. 30, Entry 11]. In the latter result, Ramanujan explicitly determines the limits of the even and odd indexed approximants of the divergent Rogers–Ramanujan continued fraction for $|q| > 1$ and shows that these limits can be expressed in terms of $R(-1/q)$ and $R(1/q^4)$.

If $a \neq 0$, the "generalized" continued fraction in (36) converges in the sense that when n is confined to any one of the three residue classes modulo 3, the limit of the left side exists and is equal to that claimed on the right side of (36). Ramanujan's result is an example in the theory of the *general convergence* of continued fractions due to L. Jacobsen [52] in 1986; see also her book with H. Waadeland [54, pp. 41–44]. This is another illustration of Ramanujan having discovered a fundamental idea long ahead of his time.

In [13], the authors construct a large class of continued fractions with three limit points in the sense of Theorem 11. However, their theorem does not cover Theorem 11, which apparently lies at a deeper level. Ramanujan's identity (36) has been generalized by D. Bowman and J. McLaughlin [31, Theorem 3], who have established an identity having any number $n \geq 3$ limit points that reduces to

(36) when $n = 3$. Ramanujan's continued fraction (38) has limit period 1, i.e., is a limit 1-periodic continued fraction. The work of Bowman and McLaughlin in [31, especially Theorem 3] and [32, especially Theorem 7] significantly increases our understanding of limit 1-periodic continued fractions with n limits. A completely different proof of Theorem 11 arising from the theory of orthogonal polynomials has recently been given by M. E. H. Ismail and D. Stanton [50].

No. 3. Cranks

As usual, define the Rogers–Ramanujan functions $G(q)$ and $H(q)$ by

(40)
$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} \quad \text{and} \quad H(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n}.$$

One of the formulas in the lost notebook [62, p. 20] that has had the most impact on subsequent research in the theory of partitions is given by

(41)
$$\frac{(q;q)_\infty}{(\zeta q;q)_\infty(\zeta^{-1}q;q)_\infty} = A(q^5) - q(\zeta+\zeta^{-1})^2 B(q^5)$$
$$+q^2(\zeta^2+\zeta^{-2})C(q^5) - q^3(\zeta+\zeta^{-1})D(q^5),$$

where ζ is any primitive fifth root of unity and

(42)
$$\begin{cases} A(q) &= \dfrac{(q^5;q^5)_\infty G^2(q)}{H(q)}, \\ B(q) &= (q^5;q^5)_\infty G(q), \\ C(q) &= (q^5;q^5)_\infty H(q), \\ D(q) &= \dfrac{(q^5;q^5)_\infty H^2(q)}{G(q)}. \end{cases}$$

Identity (41) was proved first by F. Garvan [42], who used it to give a new proof of Ramanujan's famous congruence for the partition function $p(n)$ [58], [60, pp. 210–213]

(43) $$p(5n+4) \equiv 0 \pmod 5, \qquad n \geq 0.$$

Further proofs of (41) were later given by A. B. Ekin [40] and Berndt, H. H. Chan, S. H. Chan, and W.–C. Liaw [23].

More important, however, was the use of (41) by Garvan [42] and subsequently by Andrews and Garvan [16] to provide the answer to a tantalizing question posed by F. J. Dyson [39]. Namely, is there a partition statistic that provides a combinatorial interpretation for [59], [60, p. 230]

$$p(11n+6) \equiv 0 \pmod{11}, \qquad n \geq 0,$$

in the same way that Dyson's rank provides such an interpretation for (43)? Dyson conjectured that such a statistic exists, and he named it the "crank". In [16], it was shown that if

(44) $$\frac{(q;q)_\infty}{(zq;q)_\infty(z^{-1}q;q)_\infty} = \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} c(m,n) z^m q^n,$$

then, except for $n = 1$, $c(m,n)$ is the number of partitions of n with crank m, where the crank is given as follows.

Definition 12. *For any partition π, let $\ell(\pi)$ denote the largest part of π, $\omega(\pi)$ denote the number of ones in π, and $\mu(\pi)$ denote the number of parts of π larger than $\omega(\pi)$. The crank $c(\pi)$ is then given by*

$$c(\pi) = \begin{cases} \ell(\pi), & \text{if } \omega(\pi) = 0, \\ \mu(\pi) - \omega(\pi), & \text{if } \omega(\pi) > 0. \end{cases}$$

The generating function (44) for cranks along with Ramanujan's identity (41) form the starting point for K. Mahlburg's deep and fascinating study [55] of a variety of congruence theorems for $c(m,n)$. Thus, the results of K. Ono [56] on congruences for $p(n)$ have now been refined by Mahlburg with congruences for the related $c(m,n)$.

There is compelling evidence that the last topic on which Ramanujan worked before he died was cranks [24] (although, of course, he would not have used this terminology).

No. 2. The Mock Theta Functions

Perhaps the greatest surprise for the first author, when he began to thoroughly examine the pages in the lost notebook in May 1976, was the appearance of formulas such as

(45) $$\phi_0(-q) = \frac{(-q^2;q^5)_\infty(-q^3;q^5)_\infty(q^5;q^5)_\infty}{(q^2;q^{10})_\infty(q^8;q^{10})_\infty}$$
$$+1 - \sum_{n=0}^{\infty} \frac{q^{5n^2}}{(q;q^5)_{n+1}(q^4;q^5)_n},$$

where

(46) $$\phi_0(q) := 1 + \sum_{n=1}^{\infty} q^{n^2}(-q;q^2)_n$$

is a fifth order mock theta function, and where we are utilizing the notation (2) and (1). This formula was initially published without proof in [2], giving an initial introduction to the lost notebook.

It turns out that there are five formulas in the lost notebook equivalent to (45); each is related to one of the fifth order mock theta functions connected with the first Rogers–Ramanujan function $G(q)$, defined by (40). Moreover, there are five further formulas associated with five other fifth order mock theta functions, which are related to the second Rogers–Ramanujan function $H(q)$, also defined by (40). In [17], Andrews and Garvan proved that the five identities within each class are equivalent. In other words, if one is true, they all are true; and if one is false, they all are false. Indeed, building on the work of Garvan [42], Andrews and Garvan [17] were able to show that (45) and the remaining four formulas from the first class were, in fact, equivalent to the following assertion about partitions [17, p. 243]:

> The number of partitions of $5n$ with rank congruent to 1 modulo 5 equals the number of partitions of $5n$ with rank congruent to 0 modulo 5 plus the number of partitions of n with unique smallest part and with all other parts $\leq$ twice the smallest part.

The mock theta conjectures became known as a contraction for the assertions that both sets of five formulas are indeed true.

All these results (typified by (45)) were completely unexpected, primarily owing to the following words of G. N. Watson [72, p. 274] from the introduction of his paper on the fifth order mock theta functions:

> ...but I have failed to construct a complete and exact transformation theory of the functions, on the lines of the transformation theory of functions of the third order, and in view of the complexity of all the series which are involved, I am becoming somewhat skeptical concerning the existence of an exact transformation theory for functions of the fifth order.

As noted in [17], a proof of the mock theta conjectures would allow the same treatment for fifth order functions that Watson himself provided for the third order mock theta functions [71].

D. Hickerson [46] proved *the mock theta conjectures*. Suffice it to say that his method of proof began with Hecke-like representations of the mock theta functions given in [4] and built from there magnificent theta series identities such as

$$\frac{z^2 j(-z,q) j(z,q^3)(q^2;q^2)_\infty}{j(z,q^2)} =$$

$$qf_0(q)\left\{zj(q^6z^5,q^{30}) + z^4 j(q^{24}z^5,q^{30})\right\}$$

$$+f_1(q)\left\{z^2 j(q^{12}z^5,q^{30}) + z^3 j(q^{18}z^5,q^{30})\right\}$$

$$+2\sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{15r^2+15r+3} z^{5r+5}}{1-zq^{6r+2}}$$

$$+2\sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{15r^2+15r+3} z^{-5r}}{1-z^{-1}q^{6r+2}},$$

where

$$j(z,q) = (z;q)_\infty (q/z;q)_\infty (q;q)_\infty,$$

$$f_0(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(-q;q)_n},$$

and

$$f_1(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2+n}}{(-q;q)_n}.$$

This, of course, is only the beginning of the story. S. P. Zwegers [76] has developed a fully general theory of mock theta functions, and K. Bringmann and Ono [33] have extended that work to a theory of Maass wave forms that has, among many other things, provided a proof of the Andrews–Dragonette conjecture [6].

Most recently, Bringmann, Ono, and R. C. Rhodes [35] have used their development of the mock theta functions as the holomorphic parts of Maass wave forms to obtain a general theorem that has as corollaries the mock theta conjectures [46].

No. 1. The Panel's Top Choice—Ranks

One of the romantic episodes in the theory of partitions (and one of the events that most highlights Ramanujan's incredible insight) is Dyson's discovery of the *rank of a partition*. The rank of a partition π is defined to be the largest part of π minus the number of parts of π. It is not difficult to show [39], [20] that the generating function for $N(m,n)$, the number of partitions of n with rank m, is given by

(47)
$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} N(m,n) z^m q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq;q)_n (q/z;q)_n}.$$

Dyson's objective in defining the rank was to provide a combinatorial explanation of Ramanujan's congruence (43). In Dyson's words [39], "...although we can prove ...that the partitions of $5n+4$ can be divided into five equally numerous subclasses, it is unsatisfactory to receive from the proofs no concrete idea of how the division is to be made." He conjectured (among other things) that if $N(m,Q,n)$ denotes the number of partitions of n with rank congruent to m modulo Q, then, for $0 \leq m \leq 4$,

(48) $$N(m,5,5n+4) = \frac{1}{5} p(5n+4).$$

Thus, the partitions of $5n+4$ would be divided into five equinumerous classes.

In 1954, A. O. L. Atkin and H. P. F. Swinnerton-Dyer [20] proved all of Dyson's conjectures. Their proof of (48) is a magnificent *tour de force* in the theory of theta functions and related series.

Now comes the great surprise unearthed by Garvan from Ramanujan's lost notebook [62]. Let ζ be a primitive fifth root of unity, let

$$\phi(q) := -1 + \sum_{n=0}^{\infty} \frac{q^{5n^2}}{(q;q^5)_{n+1}(q^4;q^5)_n},$$

$$\psi(q) := -1 + \sum_{n=0}^{\infty} \frac{q^{5n^2}}{(q^2;q^5)_{n+1}(q^3;q^5)_n},$$

and let $A(q)$, $B(q)$, $C(q)$, and $D(q)$ be defined by (42). On page 20 of the lost notebook [62], we find the identity

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(\zeta q;q)_n(\zeta^{-1}q;q)_n} = \tag{49}$$

$$A(q^5) + (\zeta + \zeta^{-1} - 2)\phi(q^5) +$$

$$qB(q^5) + (\zeta + \zeta^{-1})q^2 C(q^5)$$

$$-(\zeta + \zeta^{-1})q^3 \left\{ D(q^5) - (\zeta^2 + \zeta^{-2} - 2)\frac{\psi(q^5)}{q^5} \right\}.$$

Garvan showed that all of the theorems proved by Atkin and Swinnerton-Dyer [20] for $N(m,5,n)$ can be deduced from (49). Thus the conjectures originating with Dyson in an effort to provide a combinatorial proof of (43) turn out to be provable via Ramanujan's formula (49).

This is not the end of the story. Bringmann and Ono [33] have undertaken a further analysis of (47) and (48) to yield many deeper results relating the rank to partition congruences. In addition, the rank has been extended in [9] to a more general class of partitions with applications to congruences for the Atkin–Garvan moments of ranks [19].

Analysis of the Voting of the Top Ten Entries

Your Hit Parade — The Top Ten Formulas

1. Ranks
2. Mock Theta Functions
3. Cranks
4. Continued Fraction with 3 Limit Points
5. Sums of Tails
6. Integral with Character
7. Hadamard Products
8. Double Sum of Bessel Functions
9. Some Challenging Integrals
10. Bilateral Hypergeometric Series

Eight of the Top Ten received votes for first place. It is interesting that the entry receiving the most first place votes was not our first place winner, but instead the **Mock Theta Functions** garnered 12 first place votes to outdistance **Ranks** in second place with 6 and **Cranks** with 5. Although finishing in 6th place, the **Integral with Character** received 4 first place votes indicating that some voters really appreciated the beauty of this unusual looking formula. Also, despite ranking in 7th place, three voters gave the **Hadamard Products** their top vote. **Challenging Integrals** in the 9th position received 2 first place votes, with the **Continued Fraction with 3 Limit Points** and the **Bilateral Hypergeometric Series**, despite being shoved to last place, garnering the remaining first place votes. The **Sums of Tails** in 5th place and the **Bessel Functions** in 8th place were unfortunately shut out of the first place rankings.

Ranks received uniformly high marks from most voters with 8 seconds to augment its half dozen first place votes. No one ranked **Ranks** lower than 8th place. In contrast, **Cranks** in third place received votes at all ten positions, with two voters ranking Cranks at number 9 and one at number 10. No one ranked **Mock Theta Functions** last, but every other place received at least one vote from the panel, with only four ranking **Mock Theta Functions** in the second position and two ranking them in the 9th slot. The three top vote getters clearly set themselves apart from the remaining seven entries, with **Mock Theta Functions** losing to **Ranks** by a total of only four points.

The **Continued Fraction with Three Limit Points** was appreciated by most voters. Although capturing only 1 first place vote, 2 second place votes, and 3 third place votes, it had 21 votes in the 4–7 range, which was enough, by just four points, to beat out **Sums of Tails**, which was hampered by each of 3 ninth and tenth place rankings. The **Continued Fraction** had votes at every position, while **Sums of Tails** had votes at every position except the top one.

The **Integral with Character** in sixth place prevailed over the **Hadamard Products** by one measly vote. As indicated above, several voters loved one of these two entries, but, on the other hand, the **Integral with Character** had 4 last place votes and the **Hadamard Products** had even more, namely, 5 last place votes. Strangely, no one voted **Hadamard Products** in the fifth position, while the **Integral with Character** captured votes at all positions.

Although receiving 8 votes in either the second or third positions, **Bessel Functions** received several in the lower echelons, including 5 in the tenth position. In contrast, the **Challenging Integrals** captured only 3 seconds and no thirds to finish six points behind **Bessel Functions**. Although receiving only 2 tenth place votes, **Challenging Integrals** received a large number of eighth and ninth place tallies to doom the integrals to ninth place.

With 9 votes at the seventh rung and 12 votes at the bottom rung, **Bilateral Hypergeometric Series** was relegated to the bottom position in the panelists' voting. However, some voters indeed did appreciate Ramanujan's remarkable **Bilateral Hypergeometric Series** identity, as it received at least one vote at each position.

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