Visualizing the Sieve of Eratosthenes

David N. Cox

Every so often new technology is applied to an age-old problem to produce unexpected results. This article re-examines the Sieve of Eratosthenes. The Sieve is a one-dimensional device for finding prime numbers. The numbers from 2 to n are written as a single, long sequence. Then the multiples of 2 are crossed out but leaving 2. Then the multiples of 3 are crossed out but leaving 3. And so on until the only numbers remaining are the primes. This paper explores what happens if the procedure is converted from one dimension to two. Rather than a single sequence, a matrix is constructed such that in the first row every column is marked with a dot. In the second row, every other column is marked with a dot. In the third row, every third column is marked with a dot. In general, in the nth row, every n^{th} column is marked with a dot. In this fashion a two-dimensional image is built for all n^2 cells. Results of this procedure as generated by computer software are presented in this article. Despite the simplicity of this method, when enough dots are generated, the resulting image turns out to be stunning. This article demonstrates well that computerized visualization can shed new light on old subjects—even those more than 2,000 years old.

According to [1] Eratosthenes lived from 276 to 194 BC. Only fragments of Eratosthenes's original documents have survived. However, a description of his sieve method for finding prime numbers was described in "Introduction to Arithmetic" by Nicomedes written sometime prior to 210 BC [2].

To use the method imagine a written sequence of numbers from 2 to n. Starting at 2 cross off every other number in the sequence except for 2 itself. When done, repeat for 3 (which will be the next remaining number in the sequence) by crossing off every third number. When done, the next number remaining in the sequence will be 5. Repeat the process for every fifth number in the sequence. Continue with this process until you reach the end of the sequence. At the end of the process the numbers remaining in the sequence will be the primes.

This simple technique has been used for more than 2,000 years to find prime numbers. One would think that everything there is to know about the method has long since been discovered. Indeed, there are advanced sieve methods and optimization methods. However, these are significant variations of the original method and do not provide any additional characterization of the original method itself. After 2,000 years what more could be said?

This article explores the use of computerized visualization to further characterize the Sieve of Eratosthenes. After all, Eratosthenes didn't have a computer and computer graphics and visualization have only been widely available for the past 20 of those 2,000 years. With a simple extension of the Sieve we arrive at a novel result.

The Method

The method extends the Sieve of Eratosthenes from a one-dimensional sequence to a two-dimensional matrix. The method constructs a matrix of dots that can be easily viewed on a computer screen. The method is as follows. In the first row of the matrix every column contains a dot. This would correspond to crossing off every number in Eratosthenes's original sequence. In the original method this is not done. However, in two dimensions it proves useful.

In the second row of the matrix, every other column contains a dot starting with the second column. This corresponds to crossing off every other

David N. Cox is a Ph.D. student in the Computer Science Department, North Carolina State University. For freely available software to accompany the article and contact information, visit `http://www.guffy.net/sieve`*.*
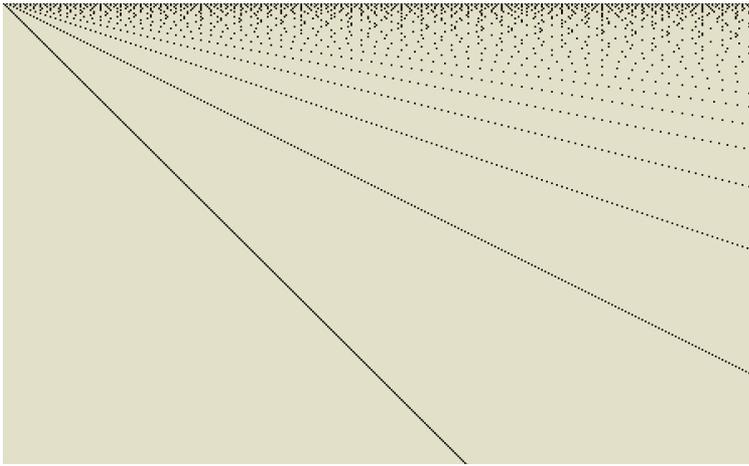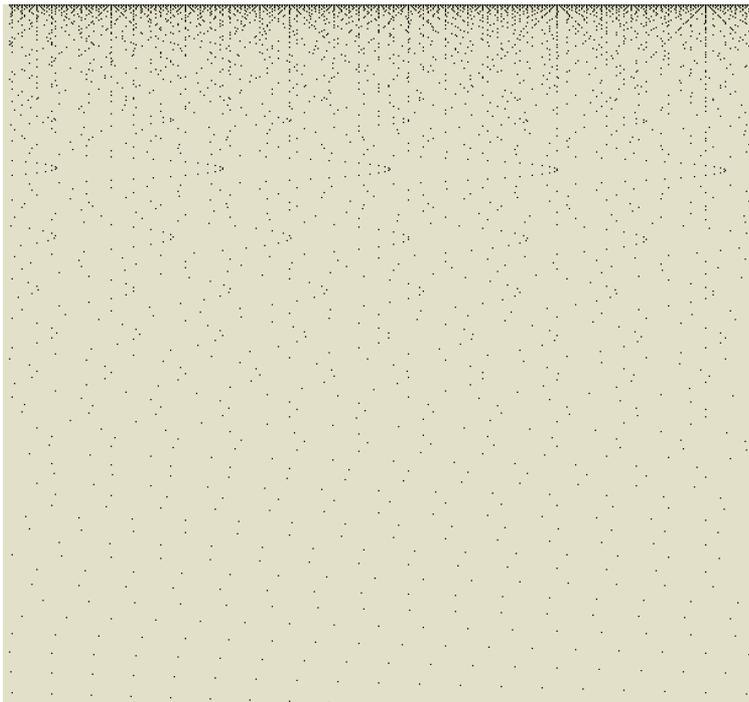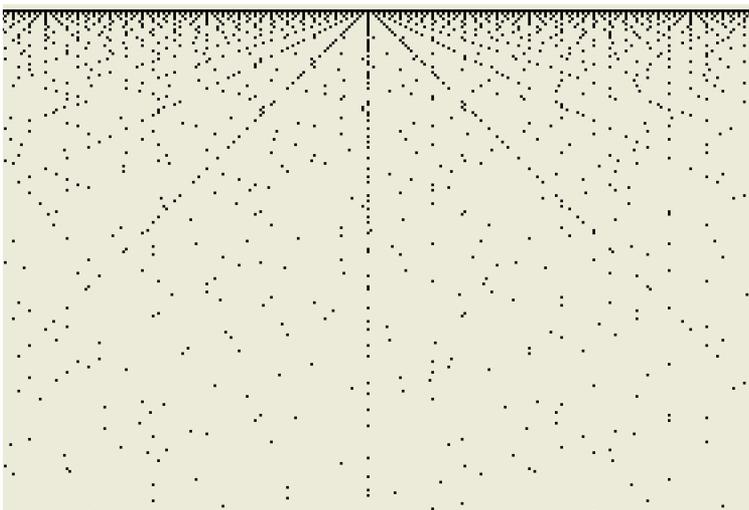
Figure 1.

Figure 2.

Figure 3.

number in the original sequence. In the third row, every third column contains a dot starting with the third column. Again, this corresponds to crossing off every third number in the original sequence. In general, in the nth row, every n^{th} column contains a dot starting with the nth column.

Aside from extending the one-dimensional sequence of length n to a two-dimensional matrix of size $n \times n$, the process of crossing off numbers (using dots) remains faithful to the original method with two differences. In the first row every number is marked with a dot. In the remaining rows every n^{th} number is marked with a dot including n itself. Consequently, even though 2 is not crossed off in Eratosthenes's original sequence, it is marked with a dot in row 2. The same holds for rows 3, 5, 7, etc. The implication of this is that a prime number corresponds to a column containing two dots—one in the first row (division by 1) and one in the nth row (division by the number itself).

Using this method a computer program was created to generate a matrix containing 1 million columns and the first 1,000 rows. The matrix is easily converted into a binary bitmap for viewing. Because it is not possible to view the entire matrix on a computer screen, a scrolling facility was provided to traverse through the columns and rows.

The computer algorithm builds the image of the matrix with the first row at the top of the screen and the first column at the left. The only reason for doing this is because most computers address pixels beginning with (0,0) at the top left. Other orientations would also work. As will be shown, the resulting image has several interesting features.

Results

Figure 1 shows several hundred columns and rows beginning with column 1 on the left. The most striking feature is the set of diagonals. Close inspection of these diagonals reveals a pattern. The main diagonal has a slope of 1 and consists of contiguous dots. The adjacent diagonal has a slope of 2 and has a dot in every other column. The third diagonal has a slope of 3 and has a dot in every third column. And so on. Remarkably, these diagonals are constructed in exactly the same manner as the rows from which the image was constructed.

Figure 2 shows a portion of the matrix beginning at number 17918. Diagonals are apparent at the bottom of the image. Towards the top the dots merge into other patterns. From row 1 there are smaller diagonals radiating out from the top row. Several rows below appear parabolic-like structures.

Some of the diagonals radiating out from the first row will be very prominent. An example is 327600 as shown in Figure 3. This prominence is related to the number of dots in the central column. The more dots, the more prominent the

diagonals. This is easy to understand by considering that a dot represents that column c is evenly divisible by row r. Consider two rows, r_1 and r_2, that both evenly divide column c. Clearly, if c is divisible by r_1, then so is $c + r_1$. Similarly, $c + r_2$ is divisible by r_2. Consequently a dot will be plotted at $(r_1, c + r_1)$ and another dot will be plotted at $(r_2, c + r_2)$. These points lie on a line with a slope of 1. The same reasoning can be extended to a dot at $(r_1, c + 2r_1)$ and another at $(r_2, c + 2r_2)$ which both lie on a line with a slope of 2.

Thus, while it is not immediately obvious, every column of points will have corresponding diagonals. What about the diagonals in Figure 1? What central column of points do they correspond to? If the procedure used to construct the matrix is extended to the left, we arrive at a result that is illustrated in Figure 4. There is a central column where every row contains a dot. The image to the left of this column is the mirror image of that to the right. Furthermore, the central column corresponds to the number 0.

Any binary image is easily represented numerically using 1's and 0's. Consequently, Figure 1 can be represented as in Figure 5.

Carrying this idea further, these numbers can be replaced by remainders. In other words, each cell will contain the value c mod r where c and r are the column and row respectively. Doing so gives the result in Figure 6.

The central column of red numbers are the remainders of 15 divided by $1, 2, 3, \ldots, 15$. Notice that these same remainders appear in the diagonals that are highlighted in red, yellow, and blue for easier reading. What is interesting is that every number will have a column of divisors that will be repeated in corresponding diagonals radiating out from the first row.

This simple method for visualizing the Sieve of Eratosthenes has resulted in surprisingly complex patterns. The set of dots in each column represents a set of divisors for that column. Extending out from each column is a set of diagonals with slopes of 1, 2, 3, etc. containing sets of dots that map 1-to-1 to the divisors in the corresponding column. This is true for every column.

Every column except for column 0 has a finite set of dots. Clearly every integer has a finite set of divisors while 0 is divisible by everything. Hence, column 0 contains an infinite set of dots.

The original Sieve is used to find prime numbers. In this method, the prime numbers are represented in the image as columns containing exactly two dots. Column c corresponds to a prime number if it contains a dot in row 1 and row c and nowhere else.

Alternatively, these images can be represented numerically using a matrix whose cells are filled with 1's and 0's. However, it is not necessary to limit the numerical representation to 1's and 0's.
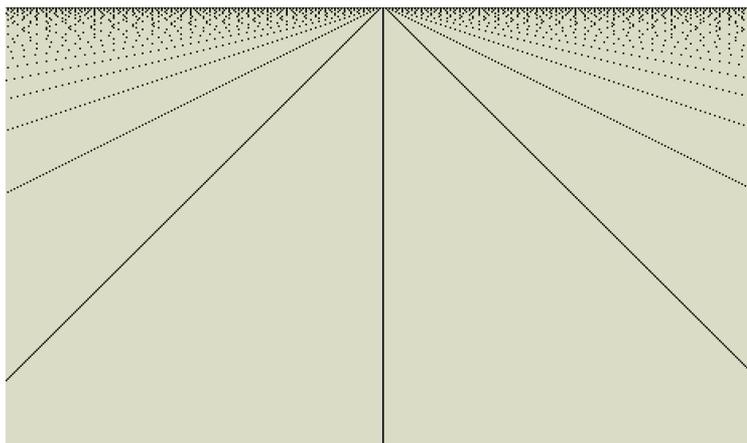

Figure 4.

1	1	1	1	1	1	1	1	1	1	1
0	1	0	1	0	1	0	1	0	1	0
0	0	1	0	0	1	0	0	1	0	0
0	0	0	1	0	0	0	1	0	0	0
0	0	0	0	1	0	0	0	0	1	0
0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	1

Figure 5.

The cells of the matrix can also be filled with remainders found by dividing each column by each row. Doing so reveals copies of each number's divisors along diagonals extending out from the first row. These diagonals have slopes of 1, 2, 3, etc.

These images are instructive in that they reveal that the divisors are not distributed randomly. There is repetition of each number's divisors along the diagonals that radiate from each number. Although a number's divisors radiate diagonally and are intertwined with those of other numbers, they do not interfere with another number's divisors. Interestingly, the spacing of the divisors along these diagonals mirrors the spacing of dots used to create the initial images.

Finally, there are other parabolic-like patterns that emerge in these images. The points of several of these patterns have been checked and verified that they do indeed lie on a true parabola. For example, the point corresponding to column 17956 and row 134 is a vertex of a left-opening parabola.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
3	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0
4	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2
5	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0
6	0	1	2	3	4	5	0	1	2	3	4	5	0	1	2	3	4	5	0	1	2	3	4	5	0	1	2	3	4	5	0
7	0	1	2	3	4	5	6	0	1	2	3	4	5	6	0	1	2	3	4	5	6	0	1	2	3	4	5	6	0	1	2
8	0	1	2	3	4	5	6	7	0	1	2	3	4	5	6	7	0	1	2	3	4	5	6	7	0	1	2	3	4	5	6
9	0	1	2	3	4	5	6	7	8	0	1	2	3	4	5	6	7	8	0	1	2	3	4	5	6	7	8	0	1	2	3
10	0	1	2	3	4	5	6	7	8	9	0	1	2	3	4	5	6	7	8	9	0	1	2	3	4	5	6	7	8	9	0
11	0	1	2	3	4	5	6	7	8	9	10	0	1	2	3	4	5	6	7	8	9	10	0	1	2	3	4	5	6	7	8
12	0	1	2	3	4	5	6	7	8	9	10	11	0	1	2	3	4	5	6	7	8	9	10	11	0	1	2	3	4	5	6
13	0	1	2	3	4	5	6	7	8	9	10	11	12	0	1	2	3	4	5	6	7	8	9	10	11	12	0	1	2	3	4
14	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2	3	4	5	6	7	8	9	10	11	12	13	0	1	2
15	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	0
16	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
17	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	0	1	2	3	4	5	6	7	8	9	10	11	12	13
18	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	0	1	2	3	4	5	6	7	8	9	10	11	12
19	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	0	1	2	3	4	5	6	7	8	9	10	11
20	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	0	1	2	3	4	5	6	7	8	9	10
21	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	0	1	2	3	4	5	6	7	8	9
22	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	0	1	2	3	4	5	6	7	8
23	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	0	1	2	3	4	5	6	7
24	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	0	1	2	3	4	5	6
25	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	0	1	2	3	4	5
26	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	0	1	2	3	4
27	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	0	1	2	3
28	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	0	1	2
29	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	0	1
30	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	0

Figure 6.

The vertex of another left-opening parabola is nearby at column 18050 and row 95. In all cases up to number 1,000,000 the observed parabolas have all been left-opening. It remains to be proven that the points on all of these parabolic-like patterns do, in fact, lie on true parabolas and why. Furthermore, it remains to be proven if all such parabolas are left-opening and why.

Clearly, these images illustrate the wonderful nature of the integers. Moreover, these images illustrate that even with a method more than 2,000 years old, a surprising new way of viewing the results can be found through the use of computerized visualization.

References

[1] J. Gullberg, *Mathematics From the Birth of Numbers*, W. W. Norton & Company, New York, NY, 1997.

[2] A. Cojocaru and M. R. Murty, *An Introduction to Sieve Methods and Their Applications*, Cambridge University Press, 2005.

[3] C. S. Ogilvy and J. T. Anderson, *Excursions in Number Theory*, Dover Publications, Inc., New York, NY, 1966.

[4] R. Kaplan and E. Kaplan, *The Art of the Infinite: The Pleasures of Mathematics*, Oxford University Press, New York, NY, 2003.