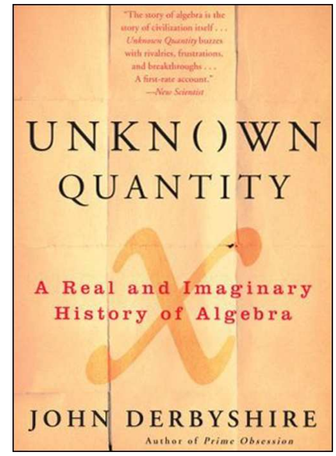
Unknown Quantity

Reviewed by Sanford L. Segal



Unknown Quantity: A Real and Imaginary History of Algebra
John Derbyshire
2007, Plume
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This is not a book to give to anyone mathematically knowledgeable. It *is* appropriate for someone who knows precious little about mathematics and would like to find out more. As the author says in the first sentence of his introduction: "This book is a history of algebra, written for the curious non-mathematician." The book is divided into three sections: The Unknown Quantity, Universal Arithmetic, Levels of Abstraction.

However, one has to know some mathematics to read such a history. Consequently, the author's fifteen chapters are interspersed with "Math Primers". The first of these precedes the first chapter of the book. It is perhaps good, first of all, to say what these "Math Primers" contain, since they are the mathematical substance of Derbyshire's book, before proceeding to his history of algebra. The first one, which opens the book is on Numbers and Polynomials and begins with the "five Russian dolls" $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$. There is an explanation of why we need to extend the natural numbers and others of the "Russian dolls", so $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$. What it means for $\mathbb{Q}$ to be dense is explained. The appearance of complex numbers in the sixteenth century is mentioned and $(1+i)/\sqrt{2}$ is demonstrated to be $\sqrt{i}$ (the distinction between i and $-i$ is not mentioned). A sequence of definitions explains polynomials.

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The second primer is on cubic and quartic equations (which, in fact, is where complex numbers made their necessary appearance—previously, in quadratic equations, complex solutions had been neglected under the assumption that "the solution does not exist"). This primer deals with the reduced cubic (called by the old term "depressed" by the author) and graphs (Derbyshire nowhere mentions that graphs did not exist for a long time, e.g., as late as Omar Khayyam). He explains the "irreducible case" and comments (in an endnote) how this is inappropriate terminology. A solution to the general cubic is given. Then comes the quartic solution (as in fact it did historically).

The primer on roots of unity includes Gauss's solution of the constructibility of the heptadecagon (but not a geometric construction of it, nor a mention of Gauss's diary entry). And there is a misprint: at the bottom of page 112, $k = 6$ should be $k = 5$.

The primer on Vector Spaces and Algebras is "an entirely modern treatment—using ideas and terms that began to be current around 1920". Addition of vectors is defined, as is linear dependence (and independence), and also dimension and basis. It is emphasized that the concept of vector space is purely algebraic. Linear transformations are defined, as are the notions of projection and embedding. The dual of a vector space is defined, as is inner product. An algebra is defined as a vector space with a multiplication.

The last two math primers are Field Theory and Algebraic Geometry. Field Theory precedes the first chapter of Levels of Abstraction, Part III of the text. It mentions finite fields (with an example) and extension fields, pointing out that an extension field, say of $\mathbb{R}$, is a vector space over $\mathbb{R}$. Galois groups are defined (the ensuing chapter will talk further about Galois).

The final math primer is on algebraic geometry and begins first with conic sections and the eccentricity of an ellipse, to introduce the idea of invariant. All conic sections (including degenerate ones) are discussed. Points at infinity and projective geometry with its line at infinity and homogeneous coordinates are introduced. The topological distinction between the Euclidean plane, the projective plane, and the surface of a sphere is discussed. Also, the fact that the lines and points are in a certain sense interchangeable is discussed.

Derbyshire's book is written in a light informal style. Included in this style is a sort of potted history with random remarks by Derbyshire reflecting his historical prejudices. For example, on the *Rubaiyat* of Omar Khayyam, he says that Edward Fitzgerald's "translation" "was a great favorite all over the English-speaking world up to World War I" and is a "sort of death-haunted hedonism with an alcoholic threat somewhat prefiguring A. E. Housman." I do not know of an "alcoholic threat" in Housman, and Fitzgerald was read long after World War I. In fact, one disadvantage for Derbyshire's readers is that he has no bibliography. This, of course, allows his potted history of mathematical irrelevancies "to give the atmosphere" while remaining uncheckable.

Furthermore, Derbyshire does not take a great deal of effort to explain the mathematics not in his primers. For example, Omar Khayyam's use of the intersection of two quadratic curves to solve a cubic is noted on page 55. All he says, in discussing the cubic equation $2x^3 - 2x^2 + 2x - 1$, which he says has solution .647798871…but does not say where this comes from, is "Khayyam took an indirect approach, ending up with a slightly different cubic which he solved numerically via the intersection of two classic geometric curves." But he does not present this approach, nor even mention what the curves are. Thus we are left to marvel at Khayyam's ingenuity in solving somehow (not given) the geometric problem leading to a cubic equation (approximately, but not exactly (!) the one discussed). This sort of throwaway filler litters the book. Often where Derbyshire could say something intelligibly concrete to anyone, he avoids saying anything at all. He could, after all, say what Khayyam did instead of wasting sentences on what he did not do. This sort of remark extends itself to the copywriter of the dust jacket blurb who says: "Moving deftly from Abel's proof to the higher levels of abstraction developed millenia later by Galois…." (!) (As though Abel were in the Garden of Eden.)

Sometimes Derbyshire's "atmospheric" potted history leads him astray, e.g., he knows that Fibonacci means "filius Bonacci", but he does not mention that this name was first stuck on Leonardo of Pisa some six centuries after he lived (!), or that Eduard Lucas is responsible for the term Fibonacci Sequence. Derbyshire quotes Fibonacci's real root of $x^3 + 2x^2 + 10x = 20$, but does not show (as Leonardo did) that it could not be rational (which is surely not beyond his intended audience).

In explaining that Khayyam spent his life under the rule of Seljuk Turks ($2\frac{1}{2}$ pages that are mathematically irrelevant), Derbyshire manages to insert his version of the origin of the Crusades in 1095 as a throwaway remark.

Derbyshire, in discussing Bombelli (who was the first to give rules for multiplying complex numbers), talks about his solution of $x^3 = 15x + 4$. Using Cardano's method for solving the cubic, he gets

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}\,.$$

Then according to Derbyshire, "By some ingenious arithmetic he works out the cube roots to be $2 + \sqrt{-1}$ and $2 - \sqrt{-1}$, respectively." Adding of course produces 4. But he never produces the "ingenious arithmetic". Of course inspection shows that 4 is a solution. I do not know what Bombelli did, but cubing $a \pm ib$ and asking that a be 2 (since 4 was a solution) would give the required solution.

Derbyshire is not above presenting the usual idea of algebra as confusing, e.g., on page 90 or page 119. Similarly (endnote 30) he avoids giving the solution in general to Archimedes' cubic, though he does give the easy solution to one case.

In discussing analytic geometry, Derbyshire omits Fermat. In fact, while Descartes and Fermat both, in 1637, connected algebra and geometry, it was Fermat who took what seems today to be the more "modern approach" making the algebraic equation central.

Descartes brings us to the conclusion of Part I. Part II, entitled Universal Arithmetic, begins with Newton. Derbyshire's description of Newton, the person, leaves something to be desired. For example, it seems a bit much to call Newton's disputes with Hooke or Flamsteed "petty squabbles", and his persecution of Leibniz even beyond the grave does not sit well with the description of Newton as a "cold fish"; nor does his friendship with John Locke, nor his nervous breakdown in 1693, which followed frequent explosions of anger and depressions.

Chapter 7, entitled "The Assault on the Quintic" deals fairly with Ruffini, Lagrange, Vandermonde, and Abel. It ends with the declaration that the strictly chronological approach followed up to this point in the book will be dispensed with in the next few chapters. The chapter that follows deals reasonably with Hamilton and Grassman, though it does not mention the attraction that quaternions had for Maxwell, Tait, and their physics colleagues (though Maxwell is credited with using vectors to "fill out the math" for Faraday's lines of force). The fact that Faraday knew no mathematics is not mentioned.

After this, we swing back to the origins of matrices in ancient China. Derbyshire's comments at the end of 9.6 (page 174) are typical of his "light writing": "Matrices are, in short, the bee's knees. They are tremendously useful, and any modern algebra course quite rightly begins with a good comprehensive introduction to matrices." While anyone would agree that matrices are important, apparently Derbyshire has never heard of the late Paul Halmos' *Finite Dimensional Vector Spaces*.

In talking about Sylvester (page 175), Derbyshire hints at homoeroticism. Hirst, who is cited innocently here, was more than a "mathematical hanger-on" (among other things he was a president of the London Mathematical Society). This sort of hint, or sexual remark, occupies Derbyshire elsewhere. For example, we learn that Henri III of France, though married, was "flamboyantly gay" and assassinated "while sitting on his commode" (whatever that has to do with the history of algebra).

On page 206, nearly one hundred pages after the quintic, which was addressed on page 115 ff., we reach Galois. Derbyshire mentions the novel about Galois by Tom Petsinis (who teaches mathematics in Australia), and, of course, E. T. Bell, whose fictionalization is dismissed, and the website by Tony Rothman, which updates his 1982 *American Mathematical Monthly* article (unmentioned). He does not mention Mario Livio's *The Equation that Couldn't be Solved*, also written for the general reader, which covers some of the same ground as Derbyshire's book. Livio's book also has a bibliography (so that the interested reader can pursue matters further).

Derbyshire puts a question mark after Ernest (page 210) to query Ernest Armand Duchatelet's name, Galois' supposed opponent in the fatal duel. As the opponent is identified in a newspaper article in Lyon, a good distance from Paris, as "L. D.", he wants the first name to begin with L, as is clear on page 211. This is more potted history—if you don't like the facts, change them.

Derbyshire makes it clear in an endnote that Liouville's journal was founded many years before it published Galois' papers. On page 218, the First Sylow Theorem is (half) mentioned, since the idea of normal subgroups is not mentioned until later. The last sentence on page 218 is another example of Derbyshire's "light" writing: "...and it is infallibly the case, at any point in time that somewhere in the world is a university math department with a rock band calling themselves 'Sylow and his p-subgroup'. " I suppose this made-up irrelevancy assures the reader that Derbyshire is a "regular guy". On page 222, the size of the Fischer-Griess "monster" is given (presumably to horrify readers).

The next chapter is cutely titled "Lady of the Rings" who is, of course, Emmy Noether, though she only occupies the last two of nine sections of this chapter, which is mostly spent appropriately on Fermat's Last Theorem, Kummer, and Dedekind. Incidentally, on page 253 there is not a picture of a Klein bottle (which is not explained) in a mathematical exhibit (since a Klein bottle requires four dimensions); but only a model of a Klein bottle. We are then led through "Geometry Makes a Comeback" (which consists of brief mentions of classical algebraic geometry, projective geometry, varieties, the Nullstellensatz, Riemann, the Erlangen Program, Lie).

The next chapter: "Algebraic This, Algebraic That" points out that the Möbius strip should be named for Listing (who first used the word *Topologie*, though *Analysis Situs* was popular for quite a while). This chapter also mentions intuitionism (à la Brouwer, though I do not know what that has to do with the history of algebra), algebraic number theory and p-adic numbers (which are explained), and ends with Lefschetz and Zariski. André Weil's situation in World War II is mentioned but not Bourbaki (strangely for a book about algebra).

The last chapter "From Universal Arithmetic to Universal Algebra" ends essentially with Grothendieck. This is preceded by a description of category theory (where forming a field of fractions from an integral domain is described in terms of functors). For Grothendieck, Derbyshire gives one of the sources he rarely acknowledges, namely Allyn Jackson's superb "As If Summoned from the Void", *Notices*, October 2004 and November 2004.

While there are no mathematical errors visible in Derbyshire's book, as this review makes clear, there is not much mathematical substance either. The treatment of most individuals is fair and accurate, with the occasional shock. Derbyshire's book will be found excellent by those for whom it is written. They are not, however, mathematicians.