WHAT IS...

a Toric Variety?

Ezra Miller

A toric variety X_P is a certain algebraic variety—or, over the real or complex numbers, a differentiable manifold with some singularities allowed—modeled on a convex polyhedron P. Examples include all (products of) projective spaces, which are modeled on (products of) standard simplices. Algebraically, toric geometry is the study of sparse polynomials, whose nonzero coefficients are attached to specified monomials. In general, toric varieties admit equivalent descriptions arising naturally in many mathematical areas, including symplectic geometry, algebraic geometry, theoretical physics, and commutative algebra, as we shall see. These perspectives, combined with intimate connections to pure and applied topics as wide-ranging as integer programming, representation theory, geometric modeling, number theory, algebraic topology, and enumerative combinatorics, lend toric varieties their importance, especially in view of their concreteness as examples.

In the symplectic setting [2], the space X_P is constructed by specifying a surjection to the polyhedron $P \subseteq \mathbb{R}^n$. The faces of P are all assumed to possess normal vectors with rational numbers for coordinates. (Thus P could be a regular cube but not a regular icosahedron.) The fiber over any point $p \in P$ is declared to be a real compact torus T^d—a product of d circles. The dimension of this torus equals that of the smallest face of P containing p. As p moves to the boundary of this face, a certain subtorus of the fiber is required to shrink and, at the boundary, collapse. Set theoretically, then, X_P is a disjoint union, over all faces F of P, of products $F^\circ \times T^{\dim(F)}$, where F° is the relative interior of F.

Example 1. If P is an interval of length ℓ then X_P is a sphere of diameter ℓ. The moment map collapses the circles of latitude, which shrink toward the north and south poles as their collapsed images move to the endpoints of P; see the figure.

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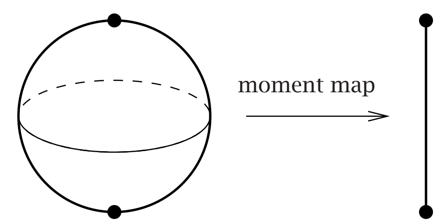

Example 2. If P is the positive orthant in $\mathbb{R}^n$ then X_P is the complex vector space $\mathbb{C}^n$. The moment map squashes each of the n copies of $\mathbb{C}$ to a ray by collapsing the concentric circles around the origin to points. The decomposition of X_P as a disjoint union over the faces of P is $\mathbb{C}^n = \bigcup_I (\mathbb{C}^\times)^I$, where $\mathbb{C}^\times$ is the group of nonzero complex numbers, and $(\mathbb{C}^\times)^I$ is the *algebraic torus* indexed by the subset $I \subseteq \{1, \ldots, n\}$.

As in the figure, the projection $X_P \to P$ is called the *moment map*. It is an instance of a general construction wherein a particularly well-behaved group action on a symplectic manifold X induces a map from X to the dual of the group's Lie algebra.

When the vertices of P have integer coordinates, X_P is a disjoint union of algebraic tori, one for each face of P, as is $\mathbb{C}^n$. If P has dimension n, then X_P carries a global action of the algebraic torus $(\mathbb{C}^\times)^n$. Restricting to the piece of X_P corresponding to the interior P° yields the regular action of $(\mathbb{C}^\times)^n$ on itself. This description is definitive [3]: a *toric variety* over $\mathbb{C}$ is a complex algebraic variety with an action of $(\mathbb{C}^\times)^n$ and a dense open subset isomorphic to $(\mathbb{C}^\times)^n$ carrying the regular action. That is, a toric variety is an algebraic torus orbit closure. The same works for fields other than $\mathbb{C}$, such as the real numbers $\mathbb{R}$ or algebraically closed fields of positive characteristic. With this definition, the connection to polyhedra is a fundamental theorem: the quotient of a complex toric variety X_P by the global action of the compact torus $T^n \subseteq (\mathbb{C}^\times)^n$ is the moment map to P in the dual $\mathbb{R}^n$ to the Lie algebra of T^n.

The term *variety* indicates a relation to polynomials, which occurs in this integer-vertex case. Any subgroup of $(\mathbb{C}^\times)^n$ isomorphic to $(\mathbb{C}^\times)^d$ has an orbit closure in $\mathbb{C}^n$ through the point $(1,\dots,1)$. This *affine toric variety* is parametrized by monomials: the inclusion $(\mathbb{C}^\times)^d \hookrightarrow \mathbb{C}^n$ takes $(\tau_1,\dots,\tau_d)$ to $(\tau^{\mathbf{a}_1},\dots,\tau^{\mathbf{a}_n})$, where $\tau^{\mathbf{a}_i} = \tau_1^{a_{i1}}\cdots\tau_d^{a_{id}}$ is a monomial for each i. For example, the parabola in the plane $\mathbb{C}^2$ is the curve parametrized by $t \mapsto (t, t^2)$. Every toric variety has a finite open cover by affine toric varieties; hence torus orbit closures in $\mathbb{C}^n$ are, in a toric sense, locally universal.

The parametrized view of (not necessarily affine) toric varieties is key in applications to geometric modeling, because every polynomial parametrization of a space is the projection of a monomial one. Thus projections of toric varieties over the real numbers generalize *Bézier curves*, which come from rational normal curves X_P, where P is an interval of integer length. Geometrically, the moment map carries the positive real part of a toric variety X_P homeomorphically to P itself, and the wavy polyhedral patch $X_P^+(\mathbb{R})$ can be used for modeling purposes. When P is a lattice triangle, for instance, $X_P^+(\mathbb{R})$ is a *Bézier triangle* in a Veronese embedding of the projective plane.

In commutative algebra, monomial parametrizations give rise to simple implicit equations. As with any variety in $\mathbb{C}^n$, an affine toric variety can be expressed as the set of points in $\mathbb{C}^n$ where a family $f_1,\dots,f_r$ of polynomials in variables $x_1,\dots,x_n$ all simultaneously vanish. The crucial observation is that in the toric case, one can always choose all of the f_j to be *binomials*, of the form $x^{\mathbf{u}} - x^{\mathbf{v}}$ for some nonnegative integer vectors $\mathbf{u}$ and $\mathbf{v}$ of length n. The binomials can be interpreted as linear equations on the exponent vectors of the parametrizing monomials. In the parabola example, the parametrized curve $(x,y) = (t,t^2)$ is implicitly defined as the set of points where the binomial $y - x^2$ vanishes; this binomial says that the exponent on the second t-monomial is twice the exponent on the first one. In this way, the binomials for X_P encode crucial information about the lattice points in P, and the integer vectors joining them. Aside from endowing the vanishing ideal of X_P with particularly rich algebraic and combinatorial structure, the binomials thereby convert toric varieties into vehicles for investigating *integer programming*, where the goal is to find a (path to a) vertex of P that maximizes some given linear *cost function*.

The combinatorial structure of toric varieties makes various flavors of cohomology explicitly computable. These computations have surprisingly wide applications, the overarching idea being that the topology of X_P usefully distills the combinatorics of P itself, and vice-versa. For instance, *Brion's formula* interprets a statement in equivariant K-theory of toric varieties as a shockingly elegant expression for the sum of the monomials corresponding to the lattice points in P. The underlying geometry is that global sections of holomorphic line bundles on X_P correspond to lattice points in polytopes related to P. Brion's formula is the key to A. Barvinok's polynomial-time algorithms for enumerating lattice points in polytopes. Concrete cohomological computations also form the basis for R. Stanley's approach to the enumerative combinatorics of polytopes. The question is how to count faces of convex polytopes. Translating into the toric world, Morse theory indicates an efficient way to encode the numbers of faces, and the Hard Lefschetz theorem from algebraic geometry implies unimodality for the encoded face numbers.

In theoretical physics, toric varieties arise in the context of *gauged linear sigma models*. Quantum field theory considers maps from a Riemann surface into $\mathbb{C}^n$, which carries an action of a compact d-torus T^d. Ground states for this theory, obtained by setting the potential energy to zero, constitute a certain fiber of the moment map of $\mathbb{C}^n$; modulo gauge equivalence—the T^d-action—this results in a toric variety X_P. (When $n = 2$ and X_P is the complex projective line, gauge equivalence is the *Hopf fibration* $S^3 \to S^2$.) Duality for polytopes in this setting gives rise to the phenomenon known as *mirror symmetry*.

A huge amount of active research has ties to toric methods. The symplectic setting has seen increasingly deep Euler–Maclaurin type summation formulas. Generalizations of toric spaces are ubiquitous, including *log schemes*, which are toric étale locally; *quasitoric manifolds* and *torus manifolds*, which are more flexible topological versions of toric varieties; and *toric stacks*, which take geometric account of extra arithmetic data beyond the polyhedron P. For the purpose of pushing Stanley's enumerative combinatorics to the setting of nonrational polytopes, there has even been success in abstracting toric cohomological computations polyhedrally, without constructing any sort of toric space at all! The future will surely see other types of developments, as well.

References

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