WHAT IS... Forcing?

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What is *forcing*? Forcing is a remarkably powerful technique for the construction of models of set theory. It was invented in 1963 by Paul Cohen[1], who used it to prove the independence of the Continuum Hypothesis. He constructed a model of set theory in which the Continuum Hypothesis (CH) fails, thus showing that CH is not provable from the axioms of set theory.

What is the *Continuum Hypothesis*? In 1873 Georg Cantor proved that the continuum is uncountable: that there exists no mapping of the set **N** of all integers onto the set **R** of all real numbers. Since **R** contains **N**, we have $2^{\aleph_0} > \aleph_0$, where $2^{\aleph_0}$ and $\aleph_0$ are the cardinalities of **R** and **N**, respectively. A question arises whether $2^{\aleph_0}$ is equal to the cardinal $\aleph_1$, the immediate successor of $\aleph_0$. Cantor's conjecture that $2^{\aleph_0} = \aleph_1$ is the celebrated continuum hypothesis made famous by David Hilbert, who put it on the top of his list of major open problems in the year 1900. Cohen's solution of Cantor's problem does not prove $2^{\aleph_0} = \aleph_1$, nor does it prove $2^{\aleph_0} \neq \aleph_1$. The answer is that CH is undecidable.

What does it mean that a conjecture is *undecidable*? In mathematics, the accepted standard of establishing truth is to give a proof of a theorem. Thus one verifies a given conjecture by finding a proof of the conjecture, or one refutes it by finding a proof of its negation. But it is not necessary that every conjecture can be so decided: it may be the case that there exists no proof of the conjecture and no proof of its negation.

To make this vague discussion more precise we will first elaborate on the concepts of theorem and proof.

What are *theorems* and *proofs*? It is a useful fact that every mathematical statement can be expressed in the language of set theory. All mathematical objects can be regarded as sets, and relations between them can be reduced to expressions that use only the relation $\in$. It is not essential how it is done, but it can be done: For instance, integers are certain finite sets, rational numbers are pairs of integers, real numbers are identified with Dedekind cuts in the rationals, functions are some sets of pairs, etc. Moreover all "self-evident truths" used in mathematical proofs can be formally derived from the axioms of set theory. The accepted system of axioms of set theory is ZFC, the Zermelo-Fraenkel axioms plus the axiom of choice. As a consequence every mathematical theorem can be formulated and proved from the axioms of ZFC.

When we consider a well formulated mathematical statement (say, the Riemann Hypothesis) there is a priori no guarantee that there exists a proof of the statement or a proof of its negation. Does ZFC decide every statement? In other words, is ZFC complete? It turns out that not only is ZFC not complete, but it cannot be replaced by a complete system of axioms. This was Gödel's 1931 discovery known as the Incompleteness Theorem.

What is *Gödel's Incompleteness Theorem*? In short, every system of axioms that is (i) recursive and (ii) sufficiently expressive is incomplete. "Sufficiently expressive" means that it includes the

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[1]*Editor's Note: Paul Cohen died in March 2008, and an article containing reminiscences about him is in preparation for an upcoming issue of the* Notices.

"self-evident truths" about integers, and "recursive" means roughly that a computer program can decide whether a statement is an axiom or not. (ZFC is one such system, Peano's system of axioms for arithmetic is another such system, etc.)

It is of course one thing to know that, by Gödel's theorem, undecidable statements exist, and another to show that a particular conjecture is undecidable. One way to show that some statement is unprovable from given axioms is to find a model.

What is a *model*? A model of a theory interprets the language of the theory in such a way that the axioms of the theory are true in the model. Then all theorems of the theory are true in the model, and if a given statement is false in the model, then it cannot be proved from the axioms of the theory. A well known example is a model of non-Euclidean geometry.

A model of set theory is a collection M of sets with the property that the axioms of ZFC are satisfied under the interpretation that "sets" are only the sets belonging to M. We say that M satisfies ZFC. If, for instance, M also satisfies the negation of CH, then CH cannot be provable in ZFC.

Here we mention another result of Gödel, from 1938: the consistency of CH. Gödel constructed a model of ZFC, the constructible universe L, that satisfies CH. The model L is basically the minimal possible collection of sets that satisfies the axioms of ZFC. Since CH is true in L, it follows that CH cannot be refuted in ZFC. In other words, CH is consistent.

Cohen's accomplishment was that he found a method for constructing other models of ZFC. The idea is to start with a given model M (the *ground model*) and extend it by adjoining an object G, a sort of imaginary set. The resulting model $M[G]$ is more or less a minimal possible collection of sets that includes M, contains G, and most importantly, also satisfies ZFC. Cohen showed how to find (or imagine) the set G so that CH fails in $M[G]$. Thus CH is unprovable in ZFC, and, because CH is also consistent, it is *independent*, or *undecidable*.

A consequence of Gödel's theorem about L is that one cannot prove that there exists a set outside the minimal model L, so we have to pull G out of thin air. The genius of Cohen was to introduce so-called *forcing conditions* that give partial information about G and then to assume that G is a *generic* set. A generic set decides which forcing conditions are considered true. With Cohen's definition of forcing and generic sets it is possible to assume that, for any ground model M and any given set P of forcing conditions in the model M, a generic set exists. Moreover, if G is generic (for P over M), then $M[G]$ is a model of set theory.

To illustrate the method of forcing, let us consider the simplest possible example, and let us stipulate that G should be a set of integers. As forcing conditions we consider finite sets of expressions $a \in G$ and $a \notin G$ where a ranges over the set of all integers. (Therefore $\{1 \in G, 2 \notin G, 3 \in G, 4 \in G\}$ is a condition that forces $G \cap \{1,2,3,4\} = \{1,3,4\}$.) The genericity of G guarantees that the conditions in G are mutually compatible, and, more importantly, that every statement of the forcing language is decided one way or the other. We shall not define here what "generic" means precisely, but let me point out one important feature. Let us identify the above generic set of integers G with the set G of forcing conditions that describe the initial segments of G. Genericity implies that for any statement A of the forcing language, if every condition can be extended to a condition that forces A, then some condition p in G forces A (and so A is true in $M[G]$). For instance, if S is a set of integers and S is in the ground model M, then every condition p of the kind described above can be extended to a condition that forces $G \neq S$: simply add the expression "$a \in G$" for some $a \notin S$ (or "$a \notin G$" for some $a \in S$) where a is an integer not mentioned in p. It follows that the resulting set of integers G is not in M, no matter what the generic set is.

Cohen showed how to construct the set of forcing conditions so that CH fails in the resulting generic extension $M[G]$. Soon after Cohen's discovery his method was applied to other statements of set theory. The method is extremely versatile: every partially ordered set P can be taken as the set of forcing conditions, and when $G \subset P$ is a generic set then the model $M[G]$ is a model of ZFC. Moreover, properties of the model $M[G]$ can be deduced from the structure of P. In practice, the forcing P can be constructed with the independence result in mind; forcing conditions usually "approximate" the desired generic object G.

In the 45 years since Cohen's discovery, literally hundreds of applications of forcing have been discovered, giving a better picture of the universe of sets by producing examples of statements that are undecidable from the conventional axioms of mathematics.

A word of caution: If after reading this you entertain the idea that perhaps the Riemann Hypothesis could be solved by forcing, forget it. That conjecture belongs to a class of statements that, by virtue of their logical structure, are *absolute* for forcing extension: such a statement is true in the generic extension $M[G]$ if and only if it is true in the ground model M. This is the content of Shoenfield's Absoluteness Theorem.

And what is *Shoenfield's Absoluteness Theorem*? Well, that is for someone else to explain, some other time.