WHAT IS... an ∞-Category?

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One of the most celebrated invariants in algebraic topology is the *fundamental group*: given a topological space X with a base point x, the fundamental group $\pi_1(X, x)$ is defined to be the set of paths in X from x to itself, taken modulo homotopy. If we do not want to choose a base point $x \in X$, we can consider instead the *fundamental groupoid* $\pi_{\leq 1}X$: this is a category whose objects are the points of X, in which morphisms are given by paths between points (again taken modulo homotopy). The fundamental groupoid of X contains slightly more information than the fundamental group of X: it determines the set of path components $\pi_0 X$ (these are the isomorphism classes of objects in $\pi_{\leq 1}X$) and also the fundamental group of each component (the fundamental group $\pi_1(X, x)$ is the automorphism group of x in the category $\pi_{\leq 1}X$). The language of category theory allows us to package this information together in a very convenient form.

The fundamental groupoid $\pi_{\leq 1}X$ does not contain any information about the *higher* homotopy groups $\{\pi_n(X, x)\}_{n \geq 2}$. In order to recover information of this sort, it is useful to consider not only points and paths in X, but also the set of continuous maps $\mathrm{Sing}_n X = \mathrm{Hom}(\Delta^n, X)$ for every nonnegative integer n; here Δ^n denotes the topological n-simplex. This raises three questions:

- (A) What kind of a mathematical object is the collection of sets $\{\mathrm{Sing}_n X\}_{n \geq 0}$?
- (B) How much does this object know about X?
- (C) To what extent does this object behave like a category?

To address question (A), we note that $\mathrm{Sing}\, X = \{\mathrm{Sing}_n X\}_{n \geq 0}$ has the structure of a *simplicial set*. That is, for every nondecreasing function $f : \{0, 1, \dots, m\} \to \{0, 1, \dots, n\}$, there is an induced map $f^* : \mathrm{Sing}_n X \to \mathrm{Sing}_m X$, given by composition with a linear map of simplices $\Delta^m \to \Delta^n$. For example, if we take $f : \{0, 1, \dots, n-1\} \to \{0, 1, \dots, n\}$ to be the injective map that omits the value i, then we obtain a map $d_i : \mathrm{Sing}_n X \to \mathrm{Sing}_{n-1} X$, which carries an n-simplex of X to its ith face. The simplicial set $\mathrm{Sing}\, X$ is sometimes called the *singular complex* of the topological space X.

It turns out that the answer to question (B) is "essentially everything", at least to an algebraic topologist. More precisely, we can use the singular complex $\mathrm{Sing}\, X$ to construct a topological space that is (weakly) homotopy equivalent to the original space X. Consequently, no information is lost by discarding the original space X and working directly with the simplicial set $\mathrm{Sing}\, X$. In fact, it is possible to develop the theory of algebraic topology in entirely combinatorial terms, using simplicial sets as surrogates for topological spaces. However, not every simplicial set S behaves like the singular complex of a space; it is therefore necessary to single out a class of "good" simplicial sets to work with. For this, we need to introduce a bit of terminology.

Let $S = \{S_n\}_{n \geq 0}$ be a simplicial set. Here we imagine that S_n denotes the set of all continuous maps from an n-simplex Δ^n into a topological space X, although such a space need not exist. For $0 \leq i \leq n$, we define another set $\Lambda^n_i(S)$, which we think of as a set of partially defined maps from Δ^n into X: namely, maps whose domain consists of all faces of Δ^n that contain the ith vertex (this subset of Δ^n is sometimes called an *i-horn*). Formally, we define $\Lambda^n_i(S)$ to be the collection of all sequences $\{\sigma_j\}_{0 \leq j \leq n, j \neq i}$ of elements of S_{n-1}, which "fit together" in the following sense: if $0 \leq j < k \leq n$ and $j \neq i \neq k$, then $d_j \sigma_k = d_{k-1} \sigma_j \in S_{n-2}$. There is a restriction map $S_n \to \Lambda^n_i(S)$, given by the formula $\tau \mapsto (d_0\tau, d_1\tau, \dots, d_{i-1}\tau, d_{i+1}\tau, \dots d_n\tau)$.

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Definition 1. A simplicial set $S = \{S_n\}_{n \geq 0}$ is a *Kan complex* if the map $S_n \to \Lambda_i^n(S)$ is surjective for all $0 \leq i \leq n$.

In other words, S is a Kan complex if every horn $\sigma \in \Lambda_i^n(S)$ can be "filled" to an n-simplex of S. Roughly speaking, a Kan complex is a simplicial set that resembles the singular complex of a topological space. In particular, the singular complex $\operatorname{Sing} X$ of a topological space X is always a Kan complex.

We now address question (C): in what sense does the singular complex $\operatorname{Sing} X$ behave like a category? To answer this, we observe that there is a close connection between categories and simplicial sets. For every category C and every integer $n \geq 0$, let C_n denote the set of all composable chains of morphisms

$$C_0 \to C_1 \to \ldots \to C_n$$

of length n. The collection of sets $\{C_n\}_{n \geq 0}$ has the structure of a simplicial set, which is called the *nerve* of the category C, and it determines C up to isomorphism. For example, the objects of C are simply the elements of C_0, and the morphisms in C are the elements of C_1.

We may therefore view the notion of a simplicial set as a *generalization* of the notion of a category. How drastic is this generalization? In other words, how can we tell if a simplicial set arises as the nerve of a category? The following result provides an answer:

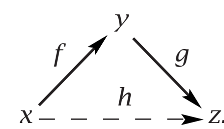
Proposition 2. *Let $S = \{S_n\}_{n \geq 0}$ be a simplicial set. Then S is isomorphic to the nerve of a category C if and only if, for each $0 < i < n$, the map $S_n \to \Lambda_i^n(S)$ bijective.*

The hypothesis of Proposition 2 resembles the definition of a Kan complex but is different in two important respects. Definition 1 requires that every horn $\sigma \in \Lambda_i^n(S)$ can be filled to an n-simplex of S. Proposition 2 requires this condition only in the case $0 < i < n$ but demands that the n-simplex be unique. Neither condition implies the other, but they admit a common generalization:

Definition 3. A simplicial set S is an *∞-category* if, for each $0 < i < n$, the map $S_n \to \Lambda_i^n(S)$ is surjective.

The notion of an ∞-category was originally introduced (under the name *weak Kan complex*) by Boardman and Vogt, in their work on homotopy invariant algebraic structures (see [1]). The theory has subsequently been developed more extensively by Joyal (who refers to ∞-categories as *quasicategories*); a good reference is [2].

Example 4. Let C be a category. Then the nerve $\{C_n\}_{n \geq 0}$ is an ∞-category, which determines C up to isomorphism. Consequently, we can think of an ∞-category $S = \{S_n\}_{n \geq 0}$ as a kind of generalized category. The *objects* of S are the elements of S_0, and the *morphisms* of S are the elements of S_1. Suppose we are given two morphisms $f, g \in S_1$ that are "composable" (that is, the source of g coincides with the target of f), as depicted in the following diagram



The morphisms f and g determine a horn $\sigma_0 \in \Lambda_1^2(S)$. If S is an ∞-category, then this horn can be filled to obtain a 2-simplex $\sigma \in S_2$. We can then define a new morphism $h : x \to z$ by passing to a face of σ; this morphism can be viewed as a composition of g and f. The 2-simplex σ is generally not unique, so that the composition $h = g \circ f$ is not unambigiously defined. However, one can use the horn-filling conditions of Definition 3 (for $n > 2$) to show that h is well-defined "up to homotopy". This turns out to be good enough: there is a robust theory of ∞-categories, which contains generalizations of most of the basic ideas of category theory (limits and colimits, adjoint functors, ...).

Definition 3 is really only a first step towards a much more ambitious generalization of classical category theory: the theory of *higher categories*. One can think of ∞-categories as higher categories in which every k-morphism is required to be invertible for $k > 1$.

Example 5. For every topological space X, the singular complex $\operatorname{Sing} X$ is an ∞-category. Since $\operatorname{Sing} X$ determines the space X up to (weak) homotopy equivalence, we can view the theory of ∞-categories as a generalization of classical homotopy theory.

Examples 4 and 5 together convey the spirit of the subject: the theory of ∞-categories can be viewed as a combination of category theory and homotopy theory and has the flavor of both. Where classical category theory provides a language for the study of algebraic structures (groups, rings, vector spaces, ...), the theory of ∞-categories provides an analogous language for describing their homotopy-theoretic counterparts (loop spaces, ring spectra, chain complexes, ...). The power of this language is only beginning to be exploited.

Further Reading

[1] J. M. Boardman and R. M. Vogt, *Homotopy Invariant Structures on Topological Spaces*, Lecture Notes in Mathematics 347, Springer-Verlag, Berlin and New York, 1973.

[2] A. Joyal, Quasi-categories and Kan complexes, *Journal of Pure and Applied Algebra* **175** (2002), 207-222.