Mathematical Omnibus: Thirty Lectures on Classic Mathematics *and* Roots to Research: A Vertical Development of Mathematical Problems

Reviewed by Harriet Pollatsek

Mathematical Omnibus: Thirty Lectures on Classic Mathematics
Dmitry Fuchs and *Serge Tabachnikov*
American Mathematical Society, 2007
US$59.00, 463 pages
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Roots to Research: A Vertical Development of Mathematical Problems
Judith D. Sally and *Paul J. Sally Jr.*
American Mathematical Society, 2007
US$49.00, 338 pages
ISBN-13: 978-0821844038

These are dangerous books! Perched on your desk, they will lure you away from duty with their lovely mathematics and engaging exposition. Seductiveness isn't all the books have in common. Both aspire to be of interest to an extraordinarily wide range of readers, from high school students to researchers "curious about results in fields other than their own". And both aim to traverse topics from school mathematics to the current research frontier. The inclusion of contemporary (since 1990) mathematics is especially notable, and in this respect these books differ from some others aimed at a similarly broad audience.

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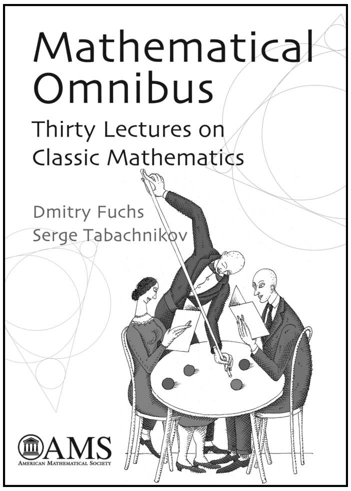
How would you use such a book? The Sallys write, "Our book ... can be used by teachers at all of the above-mentioned levels [high school through graduate school] for the enhancement of standard curriculum materials or extra-curricular projects." Fuchs and Tabachnikov write that their "book may be used for an undergraduate honors mathematics seminar (there is more than enough material for a full academic year), various topics courses, mathematics clubs at high school or college, or simply as a coffee table book to browse through at one's leisure."

These authors set lofty goals for their books (which we refer to as *Roots* and *Omnibus* for short). Do they achieve them? Yes they do, but using somewhat different strategies and with somewhat different strengths and weaknesses.

What do the books cover? The thirty lectures in *Omnibus* are organized into eight chapters (number of lectures in parentheses): Arithmetic and Combinatorics (3), Equations (5), Envelopes and Singularities (4), Developable Surfaces (3), Straight Lines (4), Polyhedra (6), Surprising Topological Constructions (2), and Ellipses and Ellipsoids (3). The over-arching themes are algebra and arithmetics for the first two chapters and geometry and topology for the rest. The authors say they have followed their eyes for beauty and have not attempted systematic development of ideas nor uniformity of length or difficulty in different lectures. They "do not assume much by way of preliminary knowledge: a standard calculus course will do in most cases, and quite often even calculus is not required."

The five problems in *Roots* are treated in five chapters, each divided into roughly a dozen sections: The Four Numbers Problem (the finiteness of an elementary arithmetic game—the only chapter that doesn't lead all the way to current mathematics research), Rational Right Triangles and the Congruent Number Problem, Lattice Point Geometry, Rational Approximation, and Dissection. The over-arching themes are similar to those in *Omnibus*. The prerequisites for different sections vary from school mathematics to topics in the upper-level undergraduate curriculum, and the beginning of each chapter specifies the mathematical background required for each section. The authors' goal is to "provide a source for the mathematics (from beginning to advanced) needed to understand the emergence and evolution" of these problems.



The chapter on Lattice Point Geometry provides an illustrative sample of *Roots*. It begins with simple observations about lengths, angle measure, and areas of polygons with vertices in the lattice $\mathbb{Z}^2$. A reminder of the Two Squares Theorem in the preceding chapter leads to theorems characterizing integers that occur as areas of lattice squares and on the numbers of lattice points in and on particular circles. Dissection of lattice polygons into primitive lattice triangles leads to the algebraic structure of the lattice $\mathbb{Z}^2$ and its group of isometries. Then come two lovely proofs of Pick's Theorem, one independent of the preceding development and the other via Euler's formula. Applying Pick's Theorem leads to Farey sequences and to extension to bounded convex regions and to Minkowski's theorem, first in the plane and then in $\mathbb{R}^k$. The grand finale comes via the attempt to extend Pick's theorem to $\mathbb{R}^k$ for $k > 2$, leading to Ehrhart's theorem on the number of lattice points in a convex polytope in $\mathbb{Z}^k$.

The flavor of *Omnibus* shows, for example, in the titles of the lectures in the chapter on polyhedra: Curvature and Polyhedra, Non-inscribable Polyhedra, Can One Make a Tetrahedron Out of a Cube?, Impossible Tilings, Rigidity of Polyhedra, and Flexible Polyhedra. The lecture on making a tetrahedron from a cube opens with the statement of Hilbert's Third Problem as solved by Dehn and notes the (unique) omission of the Third Problem from the 1976 AMS collection *Mathematical Developments Arising from Hilbert Problems*—"no developments, no influence on mathematics, nothing to discuss", the authors say. Then they go on, "How strange it seemed just a couple of years later! Dehn's Theorem, Dehn's theory, Dehn's invariant became one of the hottest subjects in geometry." The lecture then shifts to a similar problem in the plane, moves on to a different planar problem with a similar solution (engagingly commenting on the roles of geometric and algebraic methods in the two planar problems), and culminates in Dehn's proof. The lecture ends with a brief discussion of the origin of the initial statement of Hilbert's Third Problem in the foundations of geometry and a statement of Sydler's 1965 result on polyhedra with equal volumes and equal Dehn invariants.

Where is the post-1990 mathematics, you may ask? Admittedly, not in this particular lecture, but somewhat similar themes recur in the lecture on flexible polyhedra, with a lovely description of Connelly's "courageous" search for and "breathtaking discovery" of a flexible, non-intersecting polyhedron, the improvement by Steffen, and the 1995 proof by Sabitov of the "bellows conjecture" that the volume inside a flexible polyhedron does not vary in the process of deformation.

The lectures in *Omnibus* have the enthusiasm and verve of a dynamic live presentation. The authors frequently editorialize about the beauty of the ideas or the cleverness of the arguments or their rationale for including one thing versus another. For example, they write on page 75, "This proof is convincing but it does not reveal the reasons for the existence of an expression for a, b, c via p, q, r. Let us try to explain these reasons." On page 225, they write "What is a surface? We would prefer to avoid answering this question honestly, but to prove theorems, we need precise definitions." The value of the book, especially for young readers, is enhanced by the authors' many side remarks, as on page 148, where they write, "Most mathematicians are brought up to believe that things like non-differentiable functions do not appear in 'real life' ... [but] Proposition 10.3 provides a perfectly natural example of such a situation."

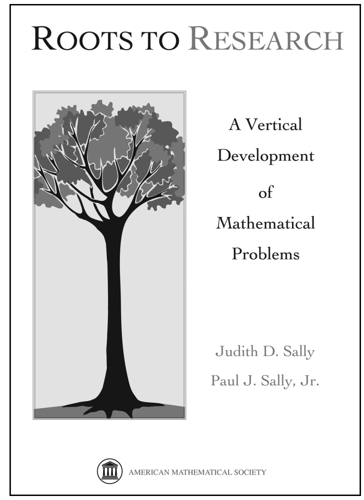
While the mathematical prerequisites for *Omnibus* are low, a high degree of mathematical sophistication and cleverness is often expected of the reader. Indeed, the authors warn in their preface that "it will take considerable effort from the reader to follow the details of the arguments." That was true for this reader. Some of the difficulties are avoidable, as in the confusing notation in section 8.4 or the failure to make clear at the start that much of Lecture 10 depends on a careful reading of Lecture 8. Also, I noticed more typos and careless errors in *Omnibus* than in *Roots*. Examples in the first few lectures include the omission of the exception $\alpha = 1/2 + n$ for n an integer on pages 9 and 10, the reversal of the interior and boundary points in Pick's formula on page 24, the reversal of p and q on page 39, and the incorrect arithmetic in the calculation of $\boldsymbol{p}(7)$ on page 51—none serious, but perhaps troubling for an inexperienced reader.

The tone of *Roots* is somewhat more formal and restrained than that of *Omnibus*, but it is never dull. Although there is less editorializing, the ideas are carefully motivated, often with natural questions arising from examples and previous results. Difficulties are not dodged, but the arguments are laid out with great care, so that the reading is challenging, but never discouraging. I found *Roots* much easier to read than *Omnibus*. For example, maybe I'm just more comfortable with Farey sequences than continued fractions, but I found the proof of Hurwitz's theorem in *Roots* as clear as glass, while I had to struggle with the one in *Omnibus*. I could more easily imagine handing *Roots* to a student to read on his or her own, than *Omnibus*. Of course, in a seminar or other group setting, this difference would be less significant.

As noted above, when background beyond calculus is required in *Roots*, this is made explicit. Sometimes lucid summaries are provided of the prerequisite mathematics, as in the description of basic facts from field theory in the section on Liouville's theorem in Chapter 4. The handling of Roth's Theorem, on pages 227 and following, is a particularly nice example of making something beyond the reader's preparation comprehensible in a general way, but without hiding the difficulties. The authors write, "The proof…is immensely more complex than those of the theorems of Liouville and of Thue, but the framework is, in essence, the same ... [however] at every step, there are constants that are dependent only on α and δ, but are not specified until later stages ... The delicate balancing of these constants is the point of the proof, but is not even considered here." There then follows a discussion of the steps of the proof of Roth's Theorem modeled on the preceding proof of Thue's. Roth's result is also mentioned near the end of Lecture 1 in *Omnibus*. This brief description includes the fact that Roth was awarded a Fields Medal for it in 1958, an interesting tidbit not appearing in *Roots*.

Both books are rich in exercises, many of them challenging. This is a particular strength of *Omnibus*, where the exercises are more abundant. In *Roots* there are some missed opportunities for exercises to test a reader's understanding of a complex argument. Both books include hints (more in *Omnibus*) and references for some exercises. Only *Omnibus* includes selected solutions; the solutions are numerous and they are written out in full—a great strength. The absence of any solutions in *Roots* is a disadvantage for a solitary reader. Both books have generous bibliographies and extensive indexes.

Illustrations are also more abundant in *Omnibus* than in *Roots*. There are, the authors say, about four hundred figures in the book, and they are mathematically precise and invariably illuminating. The authors also include photographs (or drawings) of almost every mathematician they mention, more than eighty portraits, including more than twenty of mathematicians still living. Regrettably, none is a woman. Every lecture includes a drawing by Sergey Ivanov, formerly artist-in-chief of the magazine *Kvant* and now of its cousin *Quantum*. Many of these illustrations are witty, mathematically apt, and attractive. Unlike the pictures of mathematicians, many of Ivanov's drawings include women. Unfortunately, the women pictured are sometimes movie-star bosomy and revealingly clad; in particular, the use of female nudity (especially on pages 45 and 123) seems inappropriate. The authors make clear in their preface that they want to encourage young women as well as men (even using "(s)he" as a pronoun), but the absence of real women mathematicians and the presence of women who seem mainly decorative works against their goal.



To summarize, I loved both books. Both are filled with beautiful mathematics and sometimes surprising connections. Both show the authors' love for their subject and their eagerness to share their enthusiasm. Neither book occupies a standard niche, but each has many possible uses. I am already thinking about ways to sneak bits into my courses, students to whom I might give parts to read, and talks and projects for our Math/Stat Club that I might draw from them.