WHAT IS...
a Skein Module?

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Consider all formal linear sums of oriented links in a 3-manifold M. Coefficients are to be in $\mathbb{Z}[v^{-1}, v, z^{-1}, z]$, the Laurent polynomials in v and z. Impose on this all relations of the form $v^{-1}L_+ - vL_- = zL_0$, where L_+, L_-, and L_0 are three links identical except near a point where they are as in Figure 1. The result is the oriented linear skein module of M.

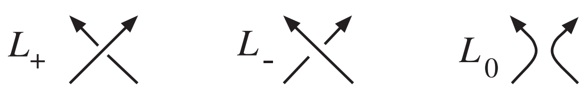

Figure 1. The only differences in L_+, L_-, and L_0.

A link is, of course, a finite collection of disjoint simple closed curves (strings) in M, two links being *the same* if one can be moved to become the other without breaking the strings (such a movement is technically an ambient isotopy). A fundamental theorem is that, when M is ordinary 3-space $\mathbb{R}^3$, or the 3-sphere S^3, this skein module is free of dimension one with base the unknot U. Thus the coordinate, or evaluation, of an oriented link L with respect to U is a polynomial invariant $P_L(v, z)$ of L.

Let T be the solid torus $A \times [0, 1]$, where A is an annulus. The union of a link in the solid torus $A \times [0, \frac{1}{2}]$ with a link in $A \times [\frac{1}{2}, 1]$ is a link in T. This defines a product, and the skein module of T becomes a commutative algebra. A short argument, using the above relation to "change crossings", shows this is generated as an algebra by $\{L_r : r \in \mathbb{Z}\}$, where L_r is a single string steadily encircling T, in as simple a way as possible, r times. Finding an easy description of the skein module of an arbitrary 3-manifold is usually not possible. However, any orientable 3-manifold M can be formed from S^3 by *surgery*, by removing and replacing differently solid tori fattening a link L in S^3. Evaluation of a certain carefully chosen element of the skein module of T placed around *every* component of L, can produce "quantum" invariants of M. The intricate details of this are in [3]. An evaluation of elements of skein modules, linear sums of links (or later, tangles) in some configuration, will always mean the multilinear sum of all evaluations of unions of summands.

Evaluating a link in terms of a base of a skein module can be seen as a partial calculation of a link invariant. In a relative version of the above, M has non-empty boundary ∂M containing $2N$ specified "fixed" points; N of these have an inwards orientation and N an outwards one. The theory proceeds as before using tangles instead of links. A tangle is a disjoint union of oriented simple closed curves with N oriented arcs joining in pairs the $2N$ specified points. Two tangles are the same if one can be moved to the other keeping ∂M fixed during the movement. In a famous example of J. H. Conway, M is a ball with four specified points. The skein module of M is then 2-dimensional with base the two tangles t_0 and t_∞ of Figure 2. Any 2-string tangle in the ball can be expressed as a linear combination of t_0 and t_∞. The tangle t_{2n} shown with $2n$ crossings is $v^{2n}t_0 + zv(v^{2n} - 1)(v^2 - 1)^{-1}t_\infty$, for example, and wherever a copy of t_{2n} occurs in a calculation this can be substituted. If a sphere S cuts a knot K contained in S^3 in four points, it divides K into a tangle in the ball inside S and another in the ball outside. The polynomial

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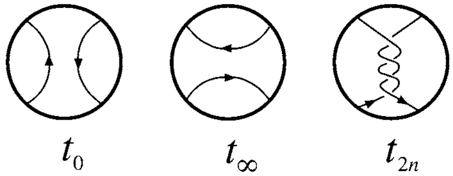


Figure 2. t_0 and t_∞ are a module base.

$P_K(v,z)$ can be found by expressing these tangles in terms of the bases of the two balls and combining the bases together with a simple 2×2 matrix. Now note that a π-rotation about one of the three coordinate axes of Figure 2, together with reversal of arrows if needed, leaves the two base elements unchanged. Thus, if K is changed to K' by removing the tangle inside S, giving it one of the three π-rotations and replacing it, $P_K(v,z)$ and $P_{K'}(v,z)$ are identical. The change from K to K' is Conway's *mutation*. He invented skein theory in the 1970s when the Alexander polynomial $\Delta_L(t)$ was the only relevant oriented link invariant (where $\Delta_L(t) = P_L(1, t^{-\frac{1}{2}} - t^{\frac{1}{2}})$). Conway noted that the only two (distinct) 11-crossing knots with $\Delta_K(t) = 1$ are related by mutation.

For a ring R and any 3-manifold M, a skein module is, then, just a quotient of the free R-module generated by all tangles in M. In practice the type of quotient must be chosen with care. A second linear skein module can be constructed from formal linear sums of *unoriented* links using the ring $\mathbb{Z}[z^{-1}, z]$ and all relations of the form shown in Figure 3. The same fundamental theorem then holds. Requiring links and tangles to be

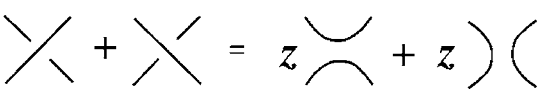


Figure 3. Skein relation without orientations.

"framed", by the giving at each point of a normal direction (which in diagrams will always point at the observer), enables the ring to be extended to become $\mathbb{Z}[a^{-1}, a, z^{-1}, z]$.

The next example is altogether more amenable. It has indeed been calculated for lens spaces and certain knot complements and generalised by "categorification". It is the Kauffman Bracket skein module of a 3-manifold M, all formal $\mathbb{Z}[A^{-1}, A]$ linear sums of framed unoriented tangles in M modulo all relations of the forms of Figure 4 (i) and (ii) (type (iii) is a consequence). Here (i) refers to the union of a tangle L with an unknot in a ball disjoint from L, and in (ii) the tangles are identical except where shown. In $\mathbb{R}^3$ one can project to a plane and use (ii) to lose crossings, then (i) to lose closed curves. The upshot is that the module for $\mathbb{R}^3$ is one-dimensional with base the untwisted unknot U. The U-coordinate of a link L is, up to a

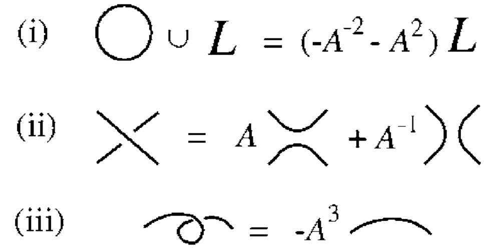


Figure 4. Kauffman bracket relations.

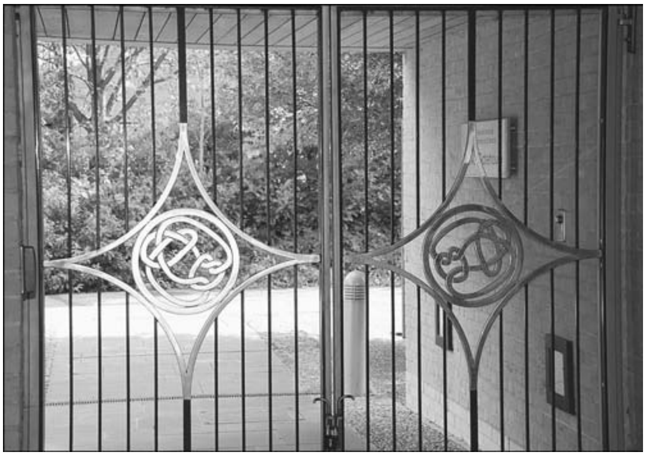
factor, the Jones polynomial $P_L(A^{-4}, A^{-2} - A^2)$ of L. This formulation of L. H. Kauffman [1] led to the main applications of the Jones polynomial.

The Kauffman Bracket skein module of the solid torus T is just the polynomial algebra over $\mathbb{Z}[A^{-1}, A]$, a generator being one untwisted curve, α say, encircling T once. Depict the ball B with $2n$ boundary points as a rectangle with n points on each of the left and right sides. Concatenation of rectangles makes the skein module of B into an algebra. It is the n^{th} Temperley-Lieb algebra of dimension $\frac{1}{n+1}\binom{2n}{n}$ but generated as an algebra by n elements $\{\mathbf{1}, e_1, e_2, \ldots, e_{n-1}\}$ shown in Figure 5. This algebra has a unique element $f^{(n)}$, the Jones-

Figure 5. Temperley-Lieb generators.

Wenzl idempotent, such that $f^{(n)} e_i = 0 = e_i f^{(n)}$ and $(f^{(n)} - \mathbf{1})$ belongs to the algebra generated by $\{e_1, e_2, ..., e_{n-1}\}$, (so $f^{(n)} f^{(n)} = f^{(n)}$). Placing $f^{(n)}$ in a solid torus T, and connecting the n points on the left to those on the right directly around T, produces $S_n(\alpha)$, a Chebyshev polynomial in α. If $S_n(\alpha)$ contained in T is placed, without twisting, around the unknot and interpreted in the skein module of S^3, its coordinate (with $\emptyset$ as base) is $\Delta_n = (-1)^n (A^{2(n+1)} - A^{-2(n+1)})(A^2 - A^{-2})^{-1}$. Thus $\Delta_n = 0$ if A is chosen to be a complex $(4n+4)^{\text{th}}$ root of unity. Then the (normalised) evaluation of $\sum_{r=0}^{n-1} \Delta_r S_r(\alpha)$ placed around each component of a surgery link defining a 3-manifold, gives a 3-manifold invariant.

The evaluation of a link with $S_{c(r)}(\alpha)$ placed around its r^{th} component is the Jones polynomial of the link *coloured* by the function c. The form of the above 3-manifold invariant gives motivation to the calculation of coloured invariants. As $S_n(\alpha)$ is just $f^{(n)}$ with its end points joined around a solid torus, this leads to consideration of configurations of many $f^{(n)}$s joined together in subtle



Mathematical Gates. The main entrance to the Centre for Mathematical Sciences in Cambridge is sealed by iron gates. Medallions in these gates show the only two knots with at most eleven crossings, other than the unknot, that have trivial Alexander polynomial. One gate shows the knot studied by S. Kinoshita and H. Terasaka, whilst the other gate shows the knot discovered by J. H. Conway in his classification of eleven-crossing knots. Related as they are by one of Conway's mutations they cannot be distinguished by any invariant based on skein theory. The medallions were created by John Robinson and donated by Damon de Laszlo and Robert Hefner III.

ways. Such configurations are drawn as labelled planar trivalent graphs, possibly with crossings, where an integer $a \geq 0$ labelling an edge indicates the presence of a copy of $f^{(a)}$ in a strings parallel to the edge. A vertex, where edges labelled a, b, and c meet, denotes the element of the skein module of a ball with $a+b+c$ specified points shown in Figure 6 (i) where $x+y=c$, $y+z=a$, and $z+x=b$. For this to exist, $\{a,b,c\}$ must be

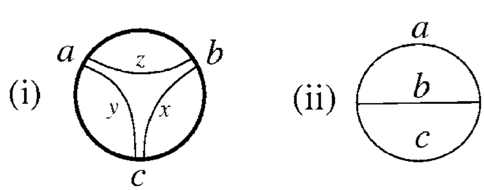


Figure 6. Triad and graph for $\theta(a,b,c)$.

compatible, meaning that $a+b+c$ is even, $a \leq b+c$, $b \leq c+a$, and $c \leq a+b$. The labelled graph of Figure 6 (ii) evaluates to

$$\theta(a,b,c) = \frac{\Delta_{x+y+z}!\Delta_{x-1}!\Delta_{y-1}!\Delta_{z-1}!}{\Delta_{y+z-1}!\Delta_{z+x-1}!\Delta_{x+y-1}!},$$

where $\Delta_n!$ denotes $\Delta_n\Delta_{n-1}\Delta_{n-2}...\Delta_1$.

In the Kauffman Bracket skein module of the ball with four specified boundary sets of a, b, c, and d points, consider the submodule of all skeins having $f^{(a)}$, $f^{(b)}$, $f^{(c)}$, and $f^{(d)}$ adjacent to these sets (see Figure 7). It is not hard to show that all the elements $[(a,b)j(c,d)]$, with both triples compatible, form a base of this submodule. So do the elements $[(b,c)i(d,a)]$. The terms in the change of base matrix, defined by $[(a,b)j(c,d)] = \sum_i \left\{\begin{matrix} a & b & i \\ c & d & j \end{matrix}\right\}[(b,c)i(d,a)]$, are called "6$j$-symbols". A cumbersome closed

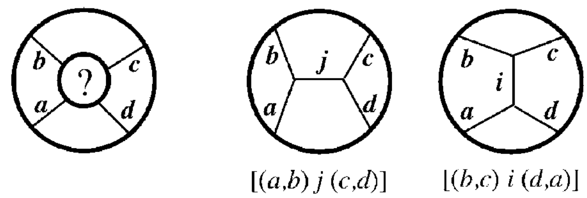


Figure 7. The ball with four point sets.

formula is known for them. An arc labelled a over-crossing one labelled b has j^{th} coordinate $(-1)^{\frac{a+b-j}{2}}\Delta_j\theta(a,b,j)^{-1}A^{a+b-j+(a^2+b^2-j^2)/2}$ with respect to base $\{[(a,b)j(a,b)]\}$. Using this, an expression for any coloured Jones polynomial can be derived in terms of the Δ_n, the $\theta(a,b,c)$, and the 6j symbols.

Finally, the evaluation τ of the edges of a tetrahedron labelled with $\{c,b,j\}$ around a face and $\{a,d,i\}$ on the edges opposite is $\Delta_i^{-1}\theta(i,b,c)\theta(i,a,d)\left\{\begin{matrix} a & b & i \\ c & d & j \end{matrix}\right\}$. When $A^{4n+4}=1$, the Turaev-Viro 3-manifold invariant is

$$k^v \sum_s \Big\{\prod_e \Delta_{se} \prod_f \theta(sf) \prod_t \tau(\widehat{st})\Big\},$$

where $k = (A^2 - A^{-2})^2/(-2n-2)$. Here e, f, and t are the edges, faces, and tetrahedra of a triangulation with v vertices, and s runs through edge-labellings compatible around any face, with face sums at most $2n-2$. A tetrahedron with labelling dual to s on t (three labels at a vertex now encircle a face) is denoted $\widehat{st}$.

Further Reading

[1] L. H. Kauffman, *Knots and Physics*, World Scientific, 1991.

[2] W. B. R. Lickorish, Quantum invariants of 3-manifolds, *Handbook of geometric topology*, (R. J. Daverman and R. B. Sher, eds.), Elsevier, 2002, pp. 707–734.

[3] Y. Yokota, Skeins and quantum SU(N) invariants of 3-maniolds, *Math. Ann.* **307** (1997), 109–138.