WHAT IS...

Lehmer's Number?

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Lehmer's number, $\lambda \approx 1.17628$, is the largest real root of the polynomial

$$f_\lambda(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1.$$

This number appears in various contexts in number theory and topology as the (sometimes conjectural) answer to natural questions involving notions of "minimality" and "small complexity".

Its story begins within number theory. Lehmer's number λ is the conjectural answer to

What is the smallest size of an algebraic integer greater than one?

Since two algebraic integers are algebraically conjugate if they are roots of the same minimal polynomial, any natural notion of the size of an algebraic integer should be constant on conjugacy classes. Given an irreducible monic integer polynomial f, the Mahler measure of f, or $M(f)$, is the absolute value of the product of roots with norm greater than one. By size of an algebraic integer α we mean the Mahler measure of the minimal polynomial of α. The Mahler measure of α is one if and only if α is a root of unity. Since Lehmer's number λ is the only root of f_λ outside the unit circle, λ is its own Mahler measure.

A related notion of size is the maximal norm of algebraic integers conjugate to α, which we will call the length of α. By this definition, the length of an algebraic integer can be arbitrarily close to one (e.g., consider $\sqrt[n]{2}$ for n large). It is not known whether the same is true for Mahler measures. Lehmer in [1] formulated the problem in this way:

Given any $\delta > 0$, is there an algebraic integer whose Mahler measure is strictly between 1 and $1 + \delta$?

Algebraic integers with small Mahler measure were important to Lehmer in his study of prime number-generating functions. Using computing machines he built himself, he found the smallest Mahler measures for even degrees up to 10. Recent computer searches by D. Boyd, M. Mossinghoff, and G. Rhin verify that Lehmer's number λ is the smallest Mahler measure greater than one for all degrees up to 40 (see Mossinghoff's website [2]).

Lehmer's number λ has special number-theoretic properties. First, the coefficients of its minimal polynomial f_λ are the same when read from the left or from the right. We call such a polynomial reciprocal, since this implies that the set of algebraic conjugates of λ contains all its reciprocals. Second, Lehmer's number λ is the only one of its algebraic conjugates that lies outside the unit circle. Such an algebraic integer is called either a Salem number (reciprocal case) or a Pisot number (nonreciprocal case).

What is known about Lehmer's question restricted to Salem and Pisot numbers is similar to what is known for more general Mahler measures. For nonreciprocal polynomials f, C. Smyth showed in 1970 that

$$M(f) \geq M(x^3 - x - 1) = \theta(\approx 1.32472) > \lambda.$$

This generalizes A. Siegel's result that θ is the smallest Pisot number and shows that θ is also the smallest Mahler measure of nonreciprocal polynomials, reducing Lehmer's question to the reciprocal case. Similarly, Lehmer's number is both the smallest known Salem number and the smallest known Mahler measure greater than one.

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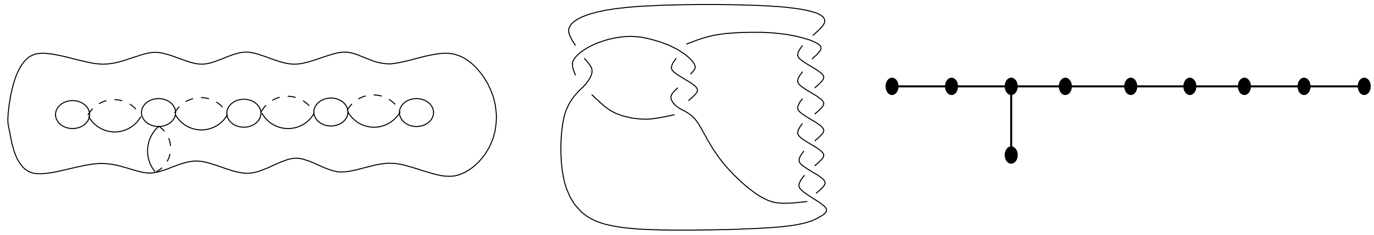

Figure 1. Manifestations of Lehmer's number as mapping class, pretzel knot, and Coxeter graph.

Lehmer's question is equivalent to asking whether an algebraic integer with small length must have a correspondingly large number of algebraic conjugates outside the unit circle. The number of exterior conjugates can be thought of as the complexity of α.

Lehmer's question and its offshoots have natural analogs in geometry and topology. For example, D. Lind, K. Schmidt, T. Ward, and others have studied the logarithm of a multivariable version of Mahler measure as the topological entropy of an associated dynamical system on the n-dimensional torus. D. Silver and S. Williams showed that the Mahler measure of the Alexander polynomial of a knot or link complement is the growth rate of its classical torsion numbers.

There is evidence for the minimality of Lehmer's number among Mahler measures in the contexts of mapping classes, fibered links, and Coxeter systems. Lehmer's number itself can be found in the cross-section of these fields of study.

An irreducible mapping class is an isotopy class of homeomorphisms of a compact oriented surface to itself so that no power preserves a nontrivial subsurface. By the Thurston-Nielsen classification, irreducible mapping classes are either periodic (analogous to roots of unity) or are of a type called pseudo-Anosov. There is a natural notion of length greater than one for pseudo-Anosov mapping classes: if ϕ is pseudo-Anosov, the surface has a local Euclidean structure (with singularities) so that ϕ expands by a real number $\alpha > 1$ in one direction and contracts by α^{-1} in another. The number α is called the (geometric) dilatation of ϕ.

The dilatations α are special algebraic integers, called Perron numbers, and are roots of reciprocal monic integer polynomials. The logarithm of α is the length of a geodesic determined by ϕ in Teichmüller space. As with lengths of algebraic integers, the dilatations of mapping classes on surfaces of genus g can be made arbitrarily close to one as g grows large. More precisely, R. Penner showed that the minimal dilatation α_g for a genus g surface satisfies the asymptotic relation $\log(\alpha_g) \asymp \frac{1}{g}$.

One source of mapping classes comes from fibered knots and links. A knot or link K in S^3 is fibered if its complement is the mapping torus for a mapping class ϕ defined on a surface S that spans K in S^3. The Alexander polynomial of K is the characteristic polynomial of the action of ϕ on the first homology of S. Its largest root, the homological dilatation of ϕ, is bounded above by the geometric dilatation. By a theorem of T. Kanenobu, any reciprocal monic integer polynomial is the Alexander polynomial of a fibered link. In particular, Lehmer's number λ is the homological dilatation of the (-2,3,7)-pretzel knot (shown in Figure 1, center) and is the Mahler measure of its Alexander polynomial.

One can also associate mapping classes to simply laced Coxeter systems. Given a simple graph Γ with ordered vertices, there is an associated linear transformation called the Coxeter element of Γ. From bipartite graphs Γ that are neither spherical nor affine, W. Thurston constructed an associated pseudo-Anosov mapping class so that the homological and geometric dilatations are both equal to the spectral radius of the Coxeter element. The monodromy ϕ of the (-2,3,7)-pretzel knot is the mapping class associated to the Coxeter graph E_{10} (Figure 1, right) and is the product of positive Dehn twists along simple closed curves dual to E_{10} on a genus 5 surface (Figure 1, left). Thus, Lehmer's number is the geometric dilatation of ϕ and the spectral radius of the Coxeter element of E_{10}.

Results from graph theory imply that to find Coxeter elements with small spectral radius it suffices to look at simple extensions of spherical and affine Coxeter graphs. C. McMullen showed further that the spectral radius of any element of a Coxeter group is either one or greater than Lehmer's number λ. This answers Lehmer's question not only for Coxeter systems but also for the corresponding subclasses of mapping classes and fibered links.

Further Reading

[1] D. H. Lehmer, Factorization of certain cyclotomic functions, *Ann. of Math.* **34** (1933), 461–469.

[2] M. Mossinghoff, Lehmer's problem website, `http://www.cecm.sfu.ca/~mjm/Lehmer`, 2003.