Encounters with Mischa Cotlar

John Horváth

Before I begin my tale, I want to call the reader's attention to the volume *Analysis and Partial Differential Equations* [20], edited by Cora Sadosky in honor of the seventieth birthday of Mischa Cotlar. There can be found the biography of Mischa, three essays about his personality, an analysis of his mathematical works, and the list of his publications until 1989.

The First Encounter

From 1948 to 1951 I lived in the *Colegio de España* at the University City in Paris. Since the Spanish civil war the college was under the administration of the French government and housed several refugees from Spain. Among them there were some mathematicians who had settled in Argentina, for instance Manuel Balanzat, professor at the University of Cuyo in San Luis and a disciple of Luis A. Santaló. I must add that I had already heard about Santaló in Budapest in 1946 when László Fejes-Tóth presented Santaló's proof of the isoperimetric inequality in his course on geometry.

There were also some Portuguese mathematicians, opposed to the Salazar regime and established in Brazil. It is from all these that I first heard the names of Antonio Monteiro, Leopoldo Nachbin, and Mischa Cotlar.

In May 1951, on my way from Paris to Bogotá, I visited the United States for a few weeks. With my fellow student Steve Gaal we traveled to New Haven to see Shizuo Kakutani at Yale University. Kakutani told us that Cotlar was in New Haven, supported by a Guggenheim fellowship and with the intention of obtaining a Ph.D. in mathematics from Yale. Kakutani gave us some information about Mischa's life: That he had arrived in Uruguay from Russia as a child, that later on he had earned a living playing the piano in the dives of the port of Montevideo, that he never went to school. Kakutani added that Mischa was a self-taught mathematician who had been publishing articles since 1936, when he was twenty-three years old. Cotlar's interest in ergodic theory had brought him to New Haven, to study with Kakutani. Indeed, Kakutani was considered at that time the foremost expert in the subject, so much so that he undertook to give a panoramic lecture on it at the International Congress of Mathematicians held in 1950 in Cambridge, Massachusetts [17]. I recall an amusing detail from those times before Xerox machines, before computers, and before email: Kakutani had a single copy of the text of his lecture, which had not yet been published. I asked him to lend it to me, which he did after I swore to return it.

Mischa Cotlar in Chicago, 1952.

Going back to May 1951, Cotlar came to dinner with us and afterwards, I witnessed the first example of his unbelievable humility. I had to return to New York, and the colleagues accompanied me to the railroad station. Cotlar, who was eleven years older than I, insisted on carrying my suitcase! After

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Mischa Cotlar died January 16, 2007, in Buenos Aires. He was an exceptional mathematician and human being. Generations of mathematicians in Venezuela, Argentina, and other Latin American countries grew under his guidance. He was one of the world experts in harmonic analysis and operator theory.

—Josefina Alvarez
New Mexico State University

I arrived in Bogotá, I learned that Yale suddenly had realized that it could not award a doctoral degree to a person without a high school diploma. When Marshall Stone, who had met Cotlar in Buenos Aires, found out about the problem, he suggested that Cotlar should go to the University of Chicago, where such official chicaneries were treated more lightly.

Thus, Cotlar arrived in the realm of Antoni Zygmund and Alberto P. Calderón, at the time when the object of study of the analysts was the theory of singular integrals. In a stroke of genius, Cotlar combined the topic of singular integrals with his own interest in ergodic theory, writing his doctoral dissertation, about which I will speak later, under the title "A unified theory of Hilbert transformations and of ergodic theorems".

Encounters in South America

My next personal meeting with Mischa was in July 1954 on the occasion of the Second Symposium on Some Mathematical Problems Which Are Being Studied in Latin America.

In the city of Mendoza, at the foothills of the Andes, the National University of Cuyo had established a mathematical institute, and Cotlar was appointed its director upon his return from the United States. The institute organized the symposium, and the inaugural lecture was given at the San Juan College of Mendoza by Julio Rey Pastor, a leading figure of Latin American mathematics. The following day we were transported by bus to an elegant hotel in Villavicencio, higher up in the mountains, where we were lodged and where the other talks of the symposium took place. Cotlar gave an excellent expository lecture titled "The Moment Problem and the Theory of Hermitian Operators" [6]. Let me mention that the first simposium had taken place in Punta del Este, Uruguay, in December 1951 and that Cotlar's lecture had been "On the Fundaments of the Ergodic Theory" [5].

The institute published its own journal, *Revista Matemática Cuyana.* The often cited second number of the first volume includes four contributions by Cotlar. The first three articles present auxiliary results, which are then applied in the fourth one. A footnote states that the essential parts of the last three articles are taken from the doctoral dissertation of the author, University of Chicago, 1953. The first article [8] contains one of the best known results of Cotlar and is the origin of the concept of *quasi-orthogonality*. Elias M. Stein consecrates to it a large part of the seventh chapter of his book [21]. The main theorem can be stated as follows:

> Let $\mathcal{A}$ be a commutative normed ring, let T_k, $1 \leq k \leq n$, be elements of $\mathcal{A}$, and set $T = \sum_{k=1}^{n} T_k$. If for $1 \leq i, j \leq n$ we have the quasi-orthogonality condition
>
> $$||T_i T_j|| \leq 2^{-|i-j|} \tag{1}$$
>
> and furthermore, $\|T_i\| \leq 1$, then $||T^k|| \leq 2^{3k} k k^{k^{3/4}} n$. An immediate consequence is that if $\{T_i\}_{i\in\mathbb{N}}$ is a sequence of hermitian operators on a prehilbert space that satisfy (1), and if $\|T_i\| \leq C$, then $\sum_{i=1}^{n} T_i$ converges in norm as $n \to \infty$ to an operator T such that $\|T\| \leq 5C$.

The original proof of this result, based on a combinatorial argument, is complicated. A little later, Béla Szőkefalvi-Nagy found a simpler proof [23]. Both Cotlar and Stein generalized the result to the case where the operators do not commute, and Stein, in collaboration with A. W. Knapp, used it in the theory of semi-simple Lie groups [22]. The inequality plays an important role in works by Calderón-Vaillancourt [3] and by Coifman-Meyer [4].

A classical theorem of Marcel Riesz has as a particular case the following:

> Let $\mathcal{D}$ be a function space that is dense in every space L^p, for instance the step functions, for $1 \leq p \leq \infty$. If $T : \mathcal{D} \to \mathcal{D}$ is a linear operator that satisfies the condition (C_r),
>
> $$\|Tf\|_r \leq M_r \|f\|_r$$
>
> for $r = 1$ and $r = p$, then (C_r) holds for all r between 1 and p. Furthermore, the optimal bound M_r is a logarithmically convex function of r, i.e.,
>
> $$M_r \leq M_1^{\frac{r-p}{1-p}} M_p^{\frac{r-1}{p-1}}.$$

It is because of this inequality that the theorem of Marcel Riesz used to be called "convexity theorem".

There are situations where the operator T does not satisfy (C_1) but nevertheless it satisfies (C_r) for $1 < r \leq p$; the most important example is the Hilbert transform,

$$Hf(x) = \frac{1}{\pi} vp \int_{\mathbb{R}} \frac{f(t)}{x - t} dt.$$

One can ask whether replacing $\|Tf\|_1$ in (C_1) by a smaller quantity, the condition so obtained does not still imply that (C_r) holds for $1 < r \le p$. An answer can be obtained from the inequality of Chebišov,

$$\|f\|_1 \ge \int_{\{|f|\ge\lambda\}} |f| \ge \lambda |\{|f| \ge \lambda\}|,$$

where $\{|f| \ge \lambda\} = \{x \in \mathbb{R} : |f(x)| \ge \lambda\}$ for $\lambda > 0$ and $|E|$ is the Lebesgue measure of the set E. This inequality suggests replacing (C_1) by (C_1^*),

$$|\{|Tf| \ge \lambda\}| \le \frac{M_1}{\lambda} \|f\|_1 .$$

Indeed, jointly with many other similar conditions, Cotlar found that (C_1^*) together with (C_p) imply that (C_r) is valid for $1 < r < p$. When Zygmund read Cotlar's result, he told Cotlar that condition (C_1^*) had already been discovered by his Polish student Jozef Marcinkiewicz, who announced it without a proof in the *Comptes Rendus* of the Paris Academy [19] in 1939, a short time before the Red Army murdered him in the Katin forest. On this topic there is a letter of Cotlar to Jaak Peetre in [15], pp. 46-47. The proof of Marcinkiewicz's theorem was published by Zygmund only in 1956 [25]. I think that the note of Marcinkiewicz is the first place where the expression "interpolation of linear operators" occurs.

Similarly to the preceding, if the linear operator $T : \mathcal{D} \to \mathcal{D}$ satisfies the conditions

$$\|Tf\|_1 \le K_1 \|f\|_1 \tag{2}$$

$$\left|\{|Tf|^2 > \lambda\}\right| \le \frac{K_2}{\lambda} \|f\|_2^2 \tag{3}$$

for every $\lambda > 0$, then $\|Tf\|_p \le K_p \|f\|_p$ for $1 < p < 2$. Cotlar proceeds to weaken condition (2), replacing $\|Tf\|_1$ by a "modified norm", which is not a norm, and it is considerably smaller than $\|Tf\|_1$. To define it, he chooses $\mathcal{D}$ in a special manner and introduces the concept of generalized support $S_L(f)$ defined with the help of an operator $L : \mathcal{D} \to \mathcal{D}$, which possesses some of the properties of the identity operator. All this leads to the main theorem of [9].

The third article in the *Revista Matemática Cuyana* [10] deals with generalizations of inequalities concerning the maximal operator of Hardy and Littlewood,

$$\Lambda f(x) = \sup_{Q(x)} \frac{1}{|Q(x)|} \int_{Q(x)} |f(t)|\, dt,$$

where $Q(x)$ denotes cubes in $\mathbb{R}^n$ with center x and with edges parallel to the coordinate axes.

Let $M : \mathcal{D} \to \mathcal{D}$ be an operator that satisfies

$$|M(f+g)(x)| \le |M(f)(x)| + |Mg(x)|,$$

and let $T : \mathcal{D} \to \mathcal{D}$ be another operator that satisfies an analogous condition. Cotlar defines two local subordination conditions between M and T:

We write $|M| \preceq O_1 |T|$ if for $f \in \mathcal{D}$ and every $x \in \mathbb{R}^n$ there exists a cube $Q(x)$ such that

$$|Mf(x)| \le O_1 \frac{1}{|Q(x)|} \int_{Q(x)} |Tf(t)| dt.$$

On the other hand, $|M| \ll O_1|T|$ means that

$$|Mf(x)| \le O_1 \frac{1}{|Q(x)|} \int_{Q(x)} |T(\varphi_{Q(x)} f)(t)|\, dt,$$

where φ_E is the characteristic function of the set E.

Using the notation $|M_\alpha f| = |Mf|^\alpha$ and $|T_\alpha f| = |Tf|^\alpha$, Cotlar proves that if $|M_\alpha| \ll |T_\alpha|$ and if $\|Tf\|_p \le K_p \|f\|_p$ for $p > \alpha$, then

$$\|Mf\|_q \le K_q \|f\|_q$$

for every $q > p$.

Furthermore, if

$$|\{|Tf| > \lambda\}| \le \frac{O_p}{\lambda^p} \int |f|^p\, dx$$

for $p > \alpha$, then this "weak condition" is also satisfied by M. Similar results are true also for the other condition of subordination.

Cora Sadosky ([20], p. 772) writes: "The paper deals mainly with maximal operators that are to a given operator T what the Hardy-Littlewood maximal operator is to the identity operator I, and with other forms of 'localization' of operators. It also gives maximal theorems in product spaces, pioneering much later work in the subject." She mentions in particular the inequality

$$T_* f(x) \le C(Tf)^*(x) + \|T\| f^*(x),$$

which Coifman and Meyer ([4], p. 95) call "Cotlar's inequality" and say that it is the "heart of the proof" of their Theorem 21 on the convergence almost everywhere of an operator with Calderón-Zygmund kernel K. In this inequality, g^* is the Hardy-Littlewood function Λg and

$$T_* f(x) = \sup_{\varepsilon > 0} \left| \int_{|x-y|\ge\varepsilon} K(x,y) f(y) dy \right| .$$

Both Coifman-Meyer ([4], p. 102) and Sadosky emphasize the particular property of the inequality, namely that T figures on both sides.

The fourth and last part of this group of articles [11] uses the tools forged in the first three to deduce the most important contributions of this issue of the *Revista Matemática Cuyana*. Cotlar considers an integrable function K on $\mathbb{R}^n$ and defines

$$K_j(x) = \frac{1}{2^{nj}} K\left(\frac{x}{2^j}\right).$$

for $j \in \mathbb{Z}$. Cotlar considers as well $\Omega = \{P\}$, a "space" equipped with a measure μ and $\{\sigma_x : x \in \mathbb{R}^n\}$, a group of measure-preserving transformations, $\sigma_x : \Omega \to \Omega$, that is, transformations satisfying $\mu(\sigma_x(E)) = \mu(E)$ for every $x \in \mathbb{R}^n$ and $E \subset \Omega$ measurable. Moreover, he assumes that $\sigma_x \circ \sigma_y = \sigma_{x+y}$. If f is a μ-measurable function defined on Ω, Cotlar defines

$$H_m f(P) = \sum_{j=-m}^{m} \int_{\mathbb{R}^n} f(\sigma_x P) K_j(x) dx,$$

Mischa Cotlar and his wife, Yanny Frenkel, in Caracas, 2001.

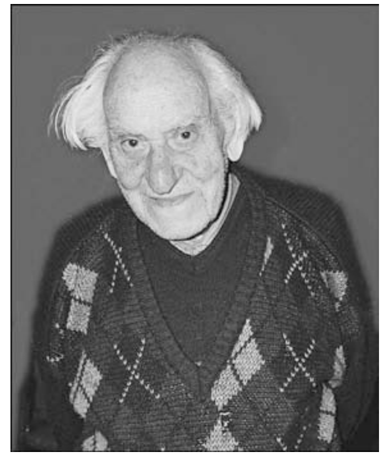
and asks whether $H_m f$ converges to a function Hf when $m \to \infty$, either at almost every point $P \in \Omega$ or in the mean.

When $\Omega = \mathbb{R}^n$, $\sigma_x(t) = x + t$, $K(x) = \omega(x)|x|^{-n}$ and $\int_{|x|=1} \omega(x)dx = 0$, the operator H_m is an n-dimensional generalization of the Hilbert transformation. When Ω is a general measure space, $K(x) = -1$ for $|x| < 1$ and $K(x) = 1$ for $1 \leq |x| \leq 2$, then H_m is the ergodic operator.

The function $H_m f$ converges to Hf in the mean as $m \to \infty$ when $f \in L^p(\Omega, \mu)$ and $p > 1$: this is the theorem of John von Neumann. The function $H_m f$ also converges to Hf at almost every point: this is the ergodic theorem of George Birkhoff. A section of the article studies the case when instead of $\mathbb{R}^n$ one considers a locally compact abelian group.

The mathematical institute of which Cotlar was the director was disbanded after less than two years; of the *Revista Matemática Cuyana* only three issues were published. In 1957 Mischa was appointed professor of mathematics in the School of Sciences of the University of Buenos Aires. With the collaboration of Cora Ratto de Sadosky, he published the series *Cursos y Seminarios de Matemática*, for which he obtained the contributions of a group of outstanding mathematicians, for instance Laurent Schwartz, Jean-Pierre Kahane, Alberto P. Calderón, Guido Weiss, and Stephen Vági.

Cotlar himself was the author of three volumes in the series. The first and thickest, at 353 pages, has the title *Continuity Conditions for Potential and Hilbert Operators* [12]. I had the pleasure of reviewing this work for *Zentralblatt für Mathematik* (Zbl. **99**, 377) and started my report with the words: "This is a vivid and highly readable account of the recent theory of potential operators and singular integrals, due mainly to Sobolev, Thorin, Calderón-Zygmund, and the author." I lent the volume to Jacques-Louis Lions, who was visiting the University of Maryland and who at that exact time was working in the theory of interpolation of linear operators. He returned it to me saying: "He who wrote this, knows very much." Cora Sadosky mentions that Béla Szőkefalvi-Nagy, who was an editor of the collection *Ergebnisse der Mathematik und Ihrer Grenzgebiete*, suggested publishing an English translation of the volume. It is regrettable that this project has never materialized.



Mischa Cotlar in Buenos Aires, 2006.

The other two volumes of the *Cursos y Seminarios de Matemática* written by Cotlar are Number 11, *Introduction to the Theory of Representation of Groups* ([13]) and Number 15, *Equipping with Hilbert Spaces* ([14]).

Encounters in the United States

In 1966 the military entered the University of Buenos Aires and brutally beat teachers as well as students. About four hundred faculty members resigned and the golden period of mathematics in Buenos Aires ended. Mischa first went to Montevideo, and in 1967 he was appointed professor at Rutgers University. His friends believed that this appointment would suit him because, among other reasons, his disciple and coauthor Ricardo Ricabarra was at that time professor at the nearby University of Delaware. Let me add that Dover is also near College Park, so I and my colleagues at the University of Maryland were happy to see Ricabarra frequently at our functional analysis seminar on Tuesday evenings.

During the time that Cotlar was a professor at Rutgers University, he also spent some time in Nice at the invitation of Jean Dieudonné. However, Cotlar was homesick for the Hispano-American atmosphere. In 1971 he went to Caracas, then spent two years in Argentina between the University of Buenos Aires and the University of La Plata, and he finally settled in Caracas in 1974.

In 1972 the Center for Research in Applied Mathematics, Systems and Services of the Autonomous National University of Mexico invited me to teach a summer course on locally convex spaces. Lucien Waelbroeck invited me to teach a course on the same subject at the Summer School on Topological Vector Spaces in Brussels in September. The courses given at this summer school have been published ([24] and [18]). The first problem I wanted to discuss in my course was the extension of the so-called Hahn-Banach theorem to ordered semi-groups. I consulted the article of Georg Aumann [1] which begins with the following sentence: "The theorems of M. Cotlar [7] on the extension of additive monotone functions on partially ordered semi-groups, can be generalized without essential change of the method of proof and so obtain a notable round out form." In my lecture I mentioned this reference to the work of Cotlar, and after the class Carlos Berenstein, who had just received his Ph.D. with Leon Ehrenpreis, came to talk with me. He told me that he was Argentinean and that there were other Argentineans in the audience who all were happy to hear Cotlar's name quoted. I must add that after this summer school, Benno Fuchssteiner furthered the theory of operators on ordered semi-groups. His results can be found in [16], as well as in the Lecture Notes ([24], pp. 45–46) and in the updated edition of the book on topological vector spaces by Bourbaki ([2], II, p. 78, Exerc. 7).

In 1975 the collaboration between Mischa Cotlar and Cora Sadosky began. Together, and occasionally with other coauthors, they published more than fifty papers. Due to their collaboration, Cotlar visited Washington almost every year, since Cora was and still is a professor at Howard University.

At the University of Maryland we profited from Cotlar's visits by inviting him to lecture in our departmental colloquium and in the seminar of Israel Gohberg, who for several years spent a considerable amount of time in College Park. Gohberg and Cotlar had many common interests, in particular Toeplitz operators. When we invited Cotlar to lecture to us, the usual answer of this very humble man was: "Why do you people invite such an ignorant person to lecture?"

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