WHAT IS...

an Elliptic Genus?

Serge Ochanine

An elliptic genus is a special type of genus developed as a tool for dealing with questions related to quantum field theory. We first define the general notion of a genus and discuss Hirzebruch's theory of multiplicative genera, into which elliptic genera fit nicely.

Genera. A *multiplicative genus*, or simply a *genus*, is a rule that to every closed oriented smooth manifold M^n associates an element $\varphi(M^n)$ of a commutative unital $\mathbb{Q}$-algebra Λ and satisfies the following conditions:

(1) $\varphi(M^n \amalg N^n) = \varphi(M^n) + \varphi(N^n)$.
Here $M^n \amalg N^n$ is the disjoint union of two closed oriented manifolds of dimension n.

(2) $\varphi(M^n \times V^m) = \varphi(M^n)\varphi(V^m)$.

(3) $\varphi(M^n) = 0$,
if $M^n = \partial W^{n+1}$ is the oriented boundary of a compact oriented manifold W^{n+1}.

Properties (1) and (3) imply that if M^n and N^n are *cobordant*, i.e., if there is a compact oriented manifold W^{n+1} with boundary $M^n \amalg (-N^n)$, where $-N^n$ stands for N^n with reversed orientation, then $\varphi(M^n) = \varphi(N^n)$. In other words, $\varphi(M^n)$ depends only on the element $[M^n]$ represented by M^n in the *oriented cobordism ring* Ω_*^{SO}, and we may view φ as a ring homomorphism

$$\varphi : \Omega_*^{SO} \longrightarrow \Lambda.$$

The structure of Ω_*^{SO} is rather complicated. However, $\Omega_*^{SO} \otimes \mathbb{Q}$ is the polynomial ring $\mathbb{Q}[[\mathbb{C}P^2],[\mathbb{C}P^4],[\mathbb{C}P^6],\ldots]$ in the cobordism classes of the complex projective spaces $\mathbb{C}P^{2k}$. This implies that a genus vanishes on manifolds whose dimension is not divisible by 4 and is completely determined by its values on $\mathbb{C}P^{2k}$. The formal power series

$$g(u) = u + \frac{\varphi(\mathbb{C}P^2)}{3}u^3 + \frac{\varphi(\mathbb{C}P^4)}{5}u^5 + \cdots \in \Lambda[[u]]$$

is called the *logarithm* of φ. It satisfies

$$g(-u) = -g(u), \quad g(u) = u + o(u)$$

and completely determines φ. Conversely, every such series is the logarithm of a multiplicative genus.

Maybe the best known example of a genus is the *signature* $\sigma(M^n)$ of a closed oriented manifold of dimension $n = 4m$. It can be defined in terms of the de Rham cohomology $H^*_{\mathrm{DR}}(M^n)$ as follows: If α and β are closed $2m$-forms on M^{4m}, then the formula

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge \beta$$

defines a nonsingular symmetric bilinear form on the finite-dimensional vector space $H^{2m}_{\mathrm{DR}}(M^n)$. The index of this form is by definition the signature of M^{4m}. It follows from Poincaré duality that σ is a cobordism invariant. It can be viewed as a genus with logarithm

$$g(u) = u + \frac{u^3}{3} + \frac{u^5}{5} + \cdots = \tanh^{-1}(u).$$

Another important example of a genus is given by the *$\hat{A}$-genus* whose logarithm is $g(u) = 2\sinh^{-1}(u/2)$. The $\hat{A}$-genus has important connections with the *arithmetic genus* in algebraic geometry.

Hirzebruch's Formalism. In the early 1950s F. Hirzebruch discovered a beautiful way of expressing multiplicative genera in terms of other cobordism invariants, the Pontrjagin numbers. If M^n is a Riemannian manifold, the Pontrjagin class $p_i \in H^{4i}_{\mathrm{DR}}(M^n)$ is represented by a closed $4i$-form ρ_i extracted from the curvature tensor of M^n. If $n = 4m$ and $\omega = (i_1, i_2, \ldots, i_s)$ is a partition of m, then the Pontrjagin number $p_\omega[M^n]$ is defined by

$$p_\omega[M^n] = \int_M \rho_{i_1} \wedge \rho_{i_2} \wedge \ldots \wedge \rho_{i_s}.$$

R. Thom's pioneering work showed that any homomorphism $\Omega_n^{SO} \longrightarrow \Lambda$ is a linear combination (over Λ) of Pontrjagin numbers. This applies, in particular, to multiplicative genera. Let φ be a genus with logarithm $g(u)$, and let $s(u) \in \Lambda[[u]]$ be the formal

Serge Ochanine is professor of mathematics at the University of Kentucky. His email address is `ochanine@ms.uky.edu`.

functional inverse of $g(u)$, i.e., $g(s(u)) = u$. This series has properties similar to those of $g(u)$: $s(-u) = -s(u), s(u) = u + o(u)$. Consider the product

$$\prod_{i=1}^{N} \frac{u_i}{s(u_i)},$$

where $u_1, u_2, \ldots, u_N$ are some formal variables of weight 2 (N is assumed to be large). Since this is a symmetric expression in $u_1, u_2, \ldots, u_N$, and even in each variable, it can be expressed in terms of elementary symmetric functions of $u_1^2, u_2^2, \ldots, u_N^2$. Substitute p_i for the i-th elementary symmetric function and let $K_m(p_1, p_2, \ldots, p_m)$ be the part of the result that lies in $H_{\mathrm{DR}}^{4m}(M)$. Hirzebruch's theorem says that

$$\varphi(M^{4m}) = K_m(p_1, p_2, \ldots, p_m)[M^{4m}].$$

Strict Multiplicativity. Like any genus, the signature satisfies $\sigma(M^n \times N^k) = \sigma(M^n)\sigma(N^k)$. It follows from a theorem of S. S. Chern, F. Hirzebruch, and J.-P. Serre that in fact a much stronger kind of multiplicativity holds. Let G be a compact connected Lie group, and let E be a principal G-bundle over a closed oriented manifold B. Let a smooth action of G on a closed oriented manifold V be given. Then one can form the associated bundle $E \times_G V$ over B with fiber V. Assuming that the orientation on $E \times_G V$ is compatible with the orientations of B and V, we have

$$\sigma(E \times_G V) = \sigma(B)\sigma(V),$$

which is often referred to as the *strict multiplicativity* of the signature. As an example, consider a complex vector bundle ξ over B of complex dimension k, and let $\mathbb{C}P(\xi)$ be the associated projective bundle. The fiber of $\mathbb{C}P(\xi)$ over a point $b \in B$ is the projective space $\mathbb{C}P(\xi_b) \cong \mathbb{C}P^{k-1}$, and strict multiplicativity implies

$$\sigma(\mathbb{C}P(\xi)) = \sigma(B)\sigma(\mathbb{C}P^{k-1}).$$

In particular, if k is even, $\sigma(\mathbb{C}P(\xi)) = 0$ for dimension reasons.

Elliptic Genera. A multiplicative genus φ is an *elliptic genus* if it vanishes on manifolds of the form $\mathbb{C}P(\xi)$, where ξ is an even-dimensional complex vector bundle over a closed oriented manifold B. The origin of the term "elliptic" is in the following theorem, which features an elliptic integral:

Theorem 1. *A genus φ is elliptic if and only if its logarithm $g(u)$ satisfies*

$$g(u) = \int_0^u \frac{dt}{\sqrt{1 - 2\delta t^2 + \varepsilon t^4}},$$

for some constants $\delta, \varepsilon \in \Lambda$.

Notice that for $\Lambda = \mathbb{C}$ and $\delta^2 \neq \varepsilon \neq 0$ (i.e., when the polynomial under the square root has four distinct roots), $g^{-1}(u)$ is the expansion at 0 of an odd elliptic function s. When $\delta^2 = \varepsilon$ or $\varepsilon = 0$, the elliptic genus is called degenerate. The two main examples are the signature ($\delta = \varepsilon = 1$) and the $\hat{A}$-genus ($\delta = -1/8, \varepsilon = 0$).

The projective space $\mathbb{C}P^{k-1}$ (k even) is an example of a *spin manifold*. A manifold V^n is a spin manifold if the structural group of its tangent bundle can be reduced to the group $\mathrm{Spin}(n)$, the two-fold cover of $\mathrm{SO}(n)$. Alternatively, V^n is a spin manifold if its tangent bundle can be trivialized over the 2-skeleton of any triangulation of V^n. The following theorem is equivalent to the *Rigidity Theorem* of R. Bott and C. Taubes:

Theorem 2. *Let G be a compact connected Lie group, let E be a principal G-bundle over a closed oriented manifold B, and let V be a closed spin manifold with a smooth G-action. Then for every elliptic genus φ, we have*

$$\varphi(E \times_G V) = \varphi(B)\varphi(V).$$

Modularity. Consider a non-degenerate elliptic genus φ over $\mathbb{C}$ with parameters $\delta, \varepsilon \in \mathbb{C}$. It is well-known that the Jacobi quartics

$$Y^2 = X^4 - 2\delta X^2 + \varepsilon$$

can be parametrized by points τ in the upper half-plane $H = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$. With this parametrization, δ and ε become level 2 modular forms for a certain subgroup $\Gamma_0(2)$ of the group of Möbius transformations of H. Since the values $\varphi(M^{4m})$ are polynomials in δ and ε, they are modular forms themselves and one can think of φ as an elliptic genus over the ring $\Lambda = M_*(\Gamma_0(2))$ of modular forms for $\Gamma_0(2)$.

Loop Spaces. Elliptic genera have a beautiful interpretation, due to E. Witten, in terms of elliptic operators on the *free loop space* $\mathcal{L}M$ of M, i.e., the infinite-dimensional manifold of smooth loops $S^1 \longrightarrow M$. Such operators play an important role in quantum field theory. The mathematical theory of such operators is still being developed, but conjectural extension of index theory to these operators has resulted in some remarkable insights. The Dirac operator on $\mathcal{L}M$ commutes with a natural circle action on $\mathcal{L}M$, and its index is an infinite-dimensional representation of S^1. Witten showed that the character of this representation can be naturally identified with the $M_*(\Gamma_0(2))$-valued elliptic genus of M.

Further Reading

Elliptic genera first appeared in [1]. The proceedings of the 1986 Princeton conference [3] contain, among many others, a paper by Witten that provides the physics interpretation of elliptic genera. The proof of the rigidity theorem is given in [4]. Finally, [2] is an elegant detailed introduction to the subject.

[1] S. Ochanine, S., Sur les genres multiplicatifs définis par des intégrales elliptiques, *Topology* **26** (1987), 143–151.

[2] F. Hirzebruch, Th. Berger, and R. Jung, *Manifolds and Modular Forms*, Vieweg, 1992.

[3] *Elliptic Curves and Modular Forms in Algebraic Topology*, P. S. Landweber, editor, Lecture Notes in Mathematics 1326, Springer-Verlag, 1986.

[4] R. Bott and C. H. Taubes, On the rigidity theorems of Witten, *J. Amer. Math. Soc.* **2** (1989), 137–186.