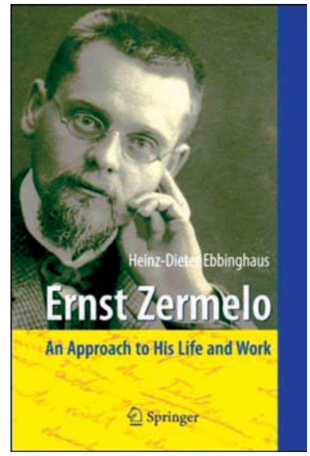
Ernst Zermelo: An Approach to His Life and Work

Reviewed by Gregory H. Moore



Ernst Zermelo: An Approach to His Life and Work
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Ernst Zermelo is familiar to mathematicians as the creator of the controversial Axiom of Choice in 1904 and the theorem, based on the Axiom of Choice, that every set can be well ordered. Many will be aware that in 1908 he axiomatized set theory—in a form later modified by Abraham Fraenkel (1922) and then by Zermelo himself (1930). Some will know of Zermelo's conflict with Ludwig Boltzmann over the Poincaré Recurrence Theorem and its role in understanding the Second Law of Thermodynamics.

Fewer will know of the following: Zermelo's work in the calculus of variations at the start of his career; his pioneering work in game theory before that of Emile Borel and John von Neumann; his conflict with Thoralf Skolem over whether axiomatic set theory should be formulated in first-order or second-order logic (and the resulting conflict over the Löwenheim–Skolem Theorem and the existence of a countable model of set theory); his contributions to infinitary logic; or finally his conflict with Gödel over the latter's Incompleteness Theorems and Zermelo's attempts to circumvent them.

The book under review is the first full-length biography of Zermelo. It is based on a great deal of archival research, on the cooperation of Zermelo's widow Gertrud (who outlived him by half a century), and on work by earlier historians of mathematics, including the reviewer. The best parts of the book are the previously unpublished letters. Yet relatively few of them from Zermelo appear in the book, even when they were available, such as his letters to Hilbert. Thanks largely to his widow, this book contains more photographs of Zermelo than any other source.

The biography begins with Zermelo's ancestors, then turns to his youth spent in Berlin, where his father died shortly before Zermelo finished high school and where Zermelo already experienced the poor health that would plague him through much of his life. Next the book discusses his university studies, which focused on mathematics, physics, and philosophy (a combination that was later shared by Paul Bernays and Kurt Gödel). Like many German students of the time, he studied at several universities—in his case Berlin, Freiburg, and Halle. At Halle he attended lectures by Georg Cantor on elliptic functions and number theory, but Cantor's work had no particular influence on him at the time. His interest in set theory was stimulated later, circa 1900, by David Hilbert.

Zermelo's 1894 doctoral dissertation under H. A. Schwarz was on the calculus of variations. Schwarz, who had been Weierstrass's student at Berlin, succeeded him there in 1892, and Zermelo was Schwarz's first doctoral student. After his dissertation, Zermelo worked for three years as assistant to Max Planck at the Berlin Institute for Theoretical Physics. It was during this period that Zermelo's polemical dispute with Boltzmann over the Second Law of Thermodynamics occurred.

In 1897 Zermelo wrote to Felix Klein at Göttingen, expressing the wish to come there to prepare a *Habilitation* in theoretical physics or mechanics. Although Klein's reply is not extant, apparently he encouraged Zermelo, for Zermelo

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came, and his 1899 *Habilitationsschrift* at Göttingen was in theoretical hydrodynamics. (In the German academic system a *Habilitationsschrift*, which was like a second doctoral dissertation, was required before one was allowed to give a lecture course at a university.) At Göttingen, Zermelo then returned to researching the calculus of variations, on which he had many conversations with Constantin Carathéodory, who became a lifelong friend. In fact, they undertook a joint book on the calculus of variations, which, as Hermann Minkowski wrote to Wilhelm Wien in 1906, "promises to become the best in this field" (p. 33). Unfortunately the book was never finished.

One of the finest parts of the biography concerns Hilbert's repeated attempts to help Zermelo's career. In 1901 Zermelo, who as a *Privatdozent* (lecturer) at Göttingen had to subsist solely on fees from students attending his lectures, applied for a grant intended to help such young lecturers. His application was supported by Hilbert, who wrote:

> Dr. Zermelo is a gifted scholar with a sharp judgment and a quick intellectual grasp. He shows a lively interest and open understanding of the questions of our [mathematical] science and, moreover, has comprehensive theoretical knowledge in the domain of mathematical physics. I am in continual scientific exchange with him. (p. 34)

Thanks to Hilbert's support, Zermelo's application was successful.

Unfortunately, Hilbert's support did not get Zermelo a position in 1903 at the University of Breslau. That university asked Hilbert to rate candidates for the position there (among whom Zermelo was not listed). Hilbert replied to Breslau in May 1903 by rating them, and then added:

> Now, concerning further names, I immediately start with the one whom I consider the real candidate for the Breslau Faculty, namely Zermelo.
>
> Zermelo is a modern mathematician who combines versatility with depth in a rare way. He is an expert in the calculus of variations.... I regard the calculus of variations as a branch of mathematics which will belong to the most important ones in the future....
>
> You must not presume that I intend to praise Zermelo into leaving [Göttingen]. Before Minkowski came here and before Blumenthal matured, Zermelo was my main mathematical company. I have learned a lot from him, e.g., the Weierstrassian calculus of variations, and so I would miss him here most of all. (pp. 35–36)

Zermelo, as he wrote three decades later (in a passage first published in 1980 by the reviewer), was much influenced by Hilbert in starting to do research on set theory:

> Thirty years ago, when I was a *Privatdozent* in Göttingen, I began, under the influence of D. Hilbert, to whom I owe more than anyone else for my scientific development, to concern myself with questions about the foundations of mathematics, especially with the fundamental problems of Cantorian *set theory*, which only came to my full consciousness in the productive cooperation among Göttingen mathematicians. (p. 28)

Since 1981 it has been known that Zermelo discovered Russell's Paradox before Bertrand Russell did and that Zermelo's version of the paradox was discussed by mathematicians at Göttingen, including Hilbert, before Russell published his paradox in 1903.

But Zermelo's first immortal and yet controversial achievement in set theory was his proof, in a letter to Hilbert of September 1904, that every set can be well ordered. A month earlier, at the International Congress of Mathematicians in Heidelberg, Julius König had given a lecture purporting to disprove Cantor's Continuum Hypothesis and to show that the set of all real numbers cannot be well ordered. König's argument relied essentially on a "theorem" in Felix Bernstein's doctoral dissertation. Cantor, Hilbert, and Schoenflies contributed to the official discussion following the lecture.

What happened next has been the subject of dispute. Ebbinghaus tends to rely on the dubious account of Gerhard Kowalewski, written almost a half-century after the event. The reviewer published a letter from Felix Hausdorff to Hilbert of September 24, 1904, in Leipzig, where Hausdorff had checked in the library soon after his return home: "After the Continuum Problem had plagued me at Wengen almost like a monomania, naturally I looked first here at Bernstein's dissertation" [[6], 108]. Hausdorff informed Hilbert that the error lay where it had been suspected—in Bernstein's argument that if κ is an infinite cardinal with $\kappa < \lambda$ and if B is a set with cardinality λ, then *every* subset A of B (where A has cardinality κ) lies in an initial segment of λ. This is false when κ is $\aleph_0$ and λ is $\aleph_\omega$—precisely the case that König needed for his argument. The seminal concept hidden here, and waiting for Hausdorff to discover it, was cofinality.

Kowalewski [[4], 202] had claimed that Zermelo found the error the day after König's talk. Grattan-Guinness [[3], 334] asserted that Zermelo had not found the error. Ebbinghaus (p. 52) discovered a letter of October 27, 1904, from Zermelo to Max Dehn, which showed that Zermelo was one of those to suspect the error lay in Bernstein's "theorem", but was able to verify this only when, after the holidays, he returned to Göttingen and could visit the library. Consequently, Grattan-Guinness was mistaken. However, the clearest light on what happened is shed by a letter (not mentioned by Ebbinghaus) from Otto Blumenthal to Emile Borel on December 1, 1904. Blumenthal informed Borel that König himself was the first to realize that his proof was not valid, followed (independently of each other) by Cantor, Bernstein, and Zermelo [[1], 74].

In 1907 Zermelo had been teaching as a *Privatdozent* at Göttingen for eight years, and his grant for doing so was finished. He either had to find a salaried position somewhere or abandon academic life altogether. So Hilbert turned to the Ministry of Cultural Affairs in Berlin (responsible for all universities in Germany) and made a case for establishing a lectureship in mathematical logic at Göttingen and for Zermelo to hold this lectureship. Hilbert persuaded Friedrich Althoff, the all-powerful director of the First Educational Department at the ministry, to establish this lectureship—the first in mathematical logic in Germany or, to the best of my knowledge, anywhere else. However, lecture courses in mathematical logic had previously been given in Germany by Ernst Schröder and Gottlob Frege, and in England by Russell.

Zermelo held his first course in mathematical logic in 1908, but almost all that we know of this course comes from lecture notes taken by Kurt Grelling. Zermelo's own notes for the course were in Gabelsberger, a form of shorthand that Bernays and Gödel also used frequently. Almost no one knows Gabelsberger now, since it was superseded in Germany by a different form of shorthand. Further, Ebbinghaus does not inform the reader that Zermelo's lecture notes are in Gabelsberger and hence inaccessible even to Germans. By contrast, the Gödel Editorial Project trained someone in Gabelsberger in order to transcribe unpublished manuscripts by Gödel into German. Unfortunately, the manuscripts of Zermelo and of Bernays have not been transcribed.

Early in 1905 Zermelo fell seriously ill with an inflammation of the lungs, and had to abandon his teaching for several months. A year later, he again had an illness in his lungs, finally diagnosed as tuberculosis. Once again, he had to stop teaching. In 1906 Minkowski, who was asked to recommend candidates for a position at the University of Würzburg, gave a strong recommendation to Zermelo, whom he described as "a mathematician of the highest qualities, of the broadest knowledge, of quick and penetrating grasp, of rare critical gift" (p. 106). But Minkowski also commented on his personality:

> Above all, his conspicuous lack of good luck stems from his outer appearance, his nervous haste which shows in his speaking and conduct. Only very recently it is giving way to a more calm, serene nature. Because of the clarity of his intellect he is a first-class teacher for the more sophisticated students, for whom it is important to penetrate the depths of science. They, like all the younger lecturers here with whom he is a close friend, appreciate him extraordinarily. However, he is not a teacher for beginners.... (p. 107)

Zermelo did not obtain the professorship at Würzburg in 1906, nor again in 1909. In 1910, when Bernstein was being considered at Göttingen for an extraordinary professorship in actuarial mathematics, the mathematics department urged the Minister to offer an extraordinary professorship there to Zermelo as well. In the end it was not offered because of news from Switzerland—in 1910 Zermelo obtained a full professorship at the University of Zurich. At that time, Zermelo's friend Erhard Schmidt was still professor at Zurich and was on the committee to select his replacement. Schmidt vigorously supported Zermelo's application, which included a very strong recommendation from Hilbert.

Zermelo held this professorship at Zurich only until 1916. It was ended at the request of the university because of his tuberculosis. He had undergone surgery for the disease and had missed several terms of teaching due to ill health. However Fraenkel, who began working on axiomatic set theory around 1919 and whose relations with Zermelo were strained, published in his autobiography a claim that Zermelo lost his professorship because, when a guest at a hotel in Germany he wrote under his nationality "Thank God, not Swiss!", a comment unfortunately read in the hotel register by the head of Zurich's education department. Ebbinghaus made thorough use of documents from Zermelo's official file at the University of Zurich to show that his illness, by making him unable to teach for several semesters, was the reason for his involuntary retirement. He was given a generous pension.

In 1921 Zermelo left Switzerland permanently and settled at Freiburg in Germany, where he remained until his death in 1953. In Freiburg

Zermelo became friends with Reinhold Baer and Arnold Scholz (both algebraists), who were successively assistants there. In 1926 the two professors of mathematics at Freiburg, Lothar Heffter and Alfred Loewy, applied to the Ministry of Education to make Zermelo a full honorary professor. This permitted Zermelo to teach (which he did for seven years without salary). During this period his mathematical research and publications resumed, both in the foundations of mathematics and in applied mathematics. Then, early in 1935, he was dismissed from his position because he refused to give the Hitler salute. In 1946 he wrote to the rector of the university, asking to be rehabilitated: "As an *honorary professor* I gave regular lectures in pure and applied mathematics for a number of years until I was forced under the Hitler government by political intrigues to give up this activity. Circumstances having now changed,…I therefore request the University…to look favorably on my reappointment as an honorary professor" (p. 251). His request was granted.

Even after his dismissal, Zermelo maintained some scientific contacts. In 1941 Scholz organized a colloquium at Göttingen for Zermelo's seventieth birthday, and among the speakers was Bartel van der Waerden. Zermelo gave three talks at the colloquium, all on applied mathematics.

As the review has mentioned, this biography has many positive features. However, there are certain matters where a more critical approach is necessary. We leave to one side various minor errors, due to its author writing in English, which is not his mother tongue: "inconstructive character of the axiom of choice" (p. vii) rather than "nonconstructive character", "*n*th derivation" (p. 12) for "*n*th derivative", etc. Nor do we wish to emphasize his misleading claim (p. 40) that Frege took a logicist position on the foundations of geometry, i.e., that geometrical objects are built on the basis of logic alone (p. 40). This was not Frege's view at all, since he was very much a traditionalist in regard to geometry (unlike his treatment of the real numbers, where he was a logicist). Likewise, we pass over Ebbinghaus's failure (pp. 136–8), when discussing at length the Axiom of Replacement and its invention by Fraenkel and Skolem, to point out that this axiom had actually been published years before them by Dmitri Mirimanoff in 1917 [[5], 262].

But on one matter Ebbinghaus leaves a serious misimpression that must be corrected: namely Zermelo's 1929 dispute with Skolem on the foundations of set theory and the role of first-order (or second-order) logic in set theory. Ebbinghaus implies that Skolem was right, and Zermelo wrong (and wrongheaded), in this dispute, since first-order logic (with its Completeness Theorem) is now the dominant form of mathematical logic. Yet that Completeness Theorem (every valid first-order sentence is provable) was published only after this dispute. Even now, many mathematicians do not understand what is meant by "first-order logic" and, consequently, the import of the Löwenheim-Skolem Theorem and the Compactness Theorem for this logic. Because the chief question at issue between Zermelo and Skolem was the latter's formulation of set theory within first-order logic, we must be clear (as almost all those at the time were not) about what was involved: the difference between first-order and second-order logic.

The essential difference between these two logics (relative to axiomatizing set theory) is in how to interpret a quantifier ranging over the subsets of a given set within a given model A. In second-order logic, the expression "for every subset of a given set x" is interpreted in the way natural to mathematicians who are not logicians, i.e., that expression means what it says: "every subset" means "every subset", whether or not a given subset happens to be in A. But in first-order logic, "for every subset of a given set x" in a model A means only those subsets of x that are members of A. Thus, within second-order logic, "the set S of all subsets of $\mathbb{N}$", where $\mathbb{N}$ is the set of all natural numbers, is uncountable, whether looked at from inside a model A or from outside A. By contrast, in first-order logic, "the set S of all subsets of $\mathbb{N}$" can be uncountable if looked at from inside the model A but countable if looked at from outside the model A. From outside, the apparent uncountability of S is seen to be an artifact caused by the lack of a certain bijection inside A, a bijection between S and $\mathbb{N}$ that is present outside A. This is the so-called "Skolem paradox", which was precisely the point at issue between Zermelo and Skolem.

The extremely limited expressive power of first-order logic can be made clearer as follows: First-order logic does not permit a theory of finite groups. That is, there is no first-order axiomatization whose models are all and only the finite groups. This is not accidental but essential since, by the Compactness Theorem, any collection of first-order sentences that has arbitrarily large finite models also has an infinite model. In this sense, the models of first-order logic cannot distinguish the finite from the infinite. Nor can first-order logic distinguish between different infinite cardinalities; for any first-order theory that has countably many primitive symbols (and these were the only first-order theories considered before 1936), if the theory has an infinite model, then it has a model of *every* infinite cardinality. The Löwenheim-Skolem Theorem is the special case that if a first-order theory has an infinite model, it has a model in the set of natural numbers. Thus, in first-order logic, the "set" of all real numbers has a model that is a subset of the natural numbers, and the set of all natural numbers also has uncountable models.

Moreover, the set of all first-order sentences true of the natural numbers has a countable model that is not isomorphic with the set of all natural numbers. All of these aspects of first-order logic are common knowledge to logicians, although usually unfamiliar to mathematicians who are not logicians.

Given all of the above, why would logicians formulate set theory within first-order logic? (The dominance of first-order logic only began in the 1950s when Alfred Tarski developed model theory.) In December 1938 at a conference in Zurich on the foundations of mathematics—a conference to which Zermelo regretted not being invited—Skolem returned again to the existence of countable models for set theory and to the Skolem paradox [[2], 37]. On this occasion he chose to emphasize the relativism, not only of set theory, but of mathematics as a whole. The discussion that followed Skolem's lecture revealed both interest in and ambivalence about the Löwenheim-Skolem Theorem. Bernays commented at length on this matter:

> The axiomatic restriction of the notion of set [to first-order logic] does not prevent one from obtaining all the usual theorems…of Cantorian set theory…. Nevertheless,…this way of making the notion of set (or that of predicate) precise has a consequence of another kind: the interpretation of the system is no longer necessarily unique…. It is to be observed that the impossibility of characterizing the finite with respect to the infinite comes from the restrictiveness of the [first-order] formalism. The impossibility of characterizing the countable with respect to the uncountable in a sense that is in some way unconditional—does this reveal, one might wonder, a certain inadequacy of the method under discussion here [first-order logic] for making axiomatizations precise? [Bernays in [2], 49–50]

Skolem objected vigorously to Bernays' suggestion and insisted that a first-order axiomatization is surely the most appropriate.

In 1958, at a colloquium held in Paris, Skolem reiterated his views on the relativism of fundamental mathematical notions and criticized Tarski's contributions:

> It is self-evident that the dubious character of the notion of set renders other notions dubious as well. For example, the semantic definition of mathematical truth proposed by A. Tarski and other logicians presupposes the general notion of set. [Skolem in [7], 13]

In the discussion following Skolem's lecture, Tarski responded to this criticism:

> [I] object to the desire shown by Mr. Skolem to reduce every theory to a denumerable model…. Because of a well-known generalization of the Löwenheim-Skolem Theorem, every formal system that has an infinite model has a model whose power is any transfinite cardinal given in advance. From this, one can just as well argue for excluding denumerable models from consideration in favor of uncountable models. [Tarski in [2], 17]

Skolem aimed to cripple set theory. But Tarski's view, allowing models of all infinite cardinalities within first-order logic, has dominated later developments and is a partial vindication of Zermelo's views. It is only partial, since today almost all set theorists formulate set theory within first-order logic with its rich theory of models.

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