## About the Cover

# Representations of $U(n)$

The cover illustrates the Gelfand-Tsetlin scheme associated to one of the irreducible representations of $U(3)$, and was suggested by the article in this issue with authors William Klink and Tuong Ton-That.

An irreducible finite-dimensional complex representation $\pi$ of $U(3)$ decomposes into a direct sum of eigenspaces on which the group $D$ of diagonal elements act by a character:

$$\pi \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} : \; v \longmapsto t_1^{m_1} t_2^{m_2} t_3^{m_3} v .$$

The integral vectors $(m_i)$ are called the **weights** of the representation. The dimension of the space on which $D$ acts by a given weight is called its **multiplicity**, and in spite of many years of investigation the problem of computing this multiplicity in general has not found a clearly optimal solution.

The $3 \times 3$ permutation matrices normalize the diagonal group, so the set of weights is invariant with respect to permutation. Under the assumption of irreducibility, the weights are the lattice points in the convex hull of the permutations of a unique **dominant** weight with $m_1 \geq m_2 \geq m_3$.

Something similar is valid for all $U(n)$. In particular, an irreducible representation of $U(2)$ with dominant weight $(m_1, m_2)$ is the direct sum of one-dimensional spaces with weights

$$(m_1, m_2),\; (m_1 - 1, m_2 + 1),\; \ldots,\; (m_2, m_1) .$$

Hermann Weyl described the restriction of an irreducible representation of $U(n)$ to $U(n-1)$, which turns out to be relatively simple. Around 1950, Gelfand and Tsetlin pointed out that if Weyl's result were applied recursively it would allow an interesting geometric interpretation of weights. For $U(3)$ their observation is that the one-dimensional eigenspaces of the diagonal group are parametrized by diagrams:

$$\begin{matrix} m_1^{(3)} & & m_2^{(3)} & & m_3^{(3)} \\ & m_1^{(2)} & & m_2^{(2)} & \\ & & m_1^{(1)} & & \end{matrix}$$

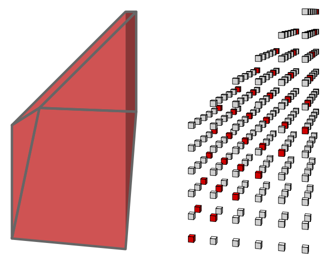
where $m_i^{(3)} = m_i$, and the interlace conditions

$$m_1^{(3)} \geq m_1^{(2)} \geq m_2^{(3)} \geq m_2^{(2)} \geq m_3^{(3)}$$
$$m_1^{(2)} \geq m_1^{(1)} \geq m_2^{(2)}$$

are satisfied. The $m_i^{(3)}$ are fixed for a given representation, so the parametrization is by three independent integer variables $x = m_1^{(2)}$, $y = m_2^{(2)}$, $z = m_1^{(1)}$ satisfying the six linear inequalities

$$m_2 \leq x \leq m_1$$
$$m_3 \leq y \leq m_2$$
$$y \leq z$$
$$z \leq x .$$

This is a rectangular cylinder sliced by top and bottom planes $x - z \leq 0$, $z - y \geq 0$:



A decomposition into one-dimensional eigenspaces thus corresponds to lattice points inside the **Gelfand-Tsetlin polytope**, and these are what the cover shows for dominant weight $(5, 0, -5)$.

The map from points $(m_i^{(j)})$ to weights $(n_i)$ is according to the formulas

$$n_1 + n_2 + n_3 = m_1^{(3)} + m_2^{(3)} + m_3^{(3)}$$
$$n_1 + n_2 = m_1^{(2)} + m_2^{(2)}$$
$$n_1 = m_1^{(1)}$$

The cover shows in red the lines of $m$ corresponding to a given weight, for the particular weights $(n, 0, -n)$. The symmetry of the set of weights with respect to $\mathfrak{S}_3$ is skewed because of the choice of coordinate system but nonetheless apparent.

The literature on these matters is huge, but I have found especially useful the recent M.I.T. thesis of Étienne Rassart, which can be found on the Internet. It shows that the geometry of Gelfand-Tsetlin polytopes is extremely useful in analyzing weight multiplicities.

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