WHAT IS... the Complex Dual to the Real Sphere?

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The observation of this note is connected with some modern considerations in integral geometry. At the same time it returns us back to the era of great projective geometry of Poncelet-Plücker and to the understanding that some phenomena of real geometry need a language of complex geometry. This era started with Poncelet's discovery that circles can be defined as ellipses passing through two universal imaginary points at infinity—the cyclic points. We consider here a canonical object dual to the real sphere $S = S^n$ that was not considered earlier, probably because it is complex.

Spherical and hyperbolic geometry are real forms of the same complex geometry, but in many respects hyperbolic geometry is richer than spherical geometry. In hyperbolic geometry, horospheres ("spheres of infinite radius") play an important role, but they have no analogues in spherical geometry. Our initial point is that it makes perfect sense to consider complex horospheres on the real sphere.

Let us start from the hyperbolic picture. We realize hyperbolic space as the hyperboloid $H = H^n \subset \mathbb{R}^{n+1}$,

$$x_1^2 - x_2^2 - \cdots - x_{n+1}^2 = 1,$$

relative to the action of the pseudoorthogonal group $O(1;n)$. The dual object is the cone $\hat{H}$,

$$\xi_1^2 - \xi_2^2 - \cdots - \xi_{n+1}^2 = 0, \xi \neq 0,$$

without the vertex, where the group $O(1;n)$ also acts transitively. Points $\xi \in \hat{H}$ parameterize the horospheres, which are intersections of H by the (isotropic) hyperplanes $\xi \cdot x = 1$. Here the dot-product corresponds to the same quadratic form.

For the real sphere $S = S^n$

$$x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1,$$

we consider its complexification $\mathbb{C}S$,

$$z_1^2 + z_2^2 + \cdots + z_{n+1}^2 = 1, z = x + iy \in \mathbb{C}^{n+1}$$

and complex horospheres $E(\zeta)$—intersections of $\mathbb{C}S$ by the hyperplanes

$$\zeta \cdot z = \zeta_1 z_1 + \cdots + \zeta_{n+1} z_{n+1} = 1, \zeta \cdot \zeta = 0, \zeta \neq 0.$$

So complex horospheres are parameterized by points of the complex cone $C \subset \mathbb{C}^{n+1}$ without the vertex. The crucial moment in such constructions comes when one selects from all horospheres some that have a special relation with the real sphere. We suggest considering horospheres $E(\zeta)$ that do not intersect the real sphere S and interpreting the manifold $\hat{S} \subset C$ of their parameters ζ as the dual object for the sphere S. Direct computation shows that the domain $\hat{S}$ on the cone C is defined by the condition

$$\xi_1^2 + \cdots + \xi_{n+1}^2 < 1, \xi = \Re\zeta.$$

This domain is invariant relative to the orthogonal group $O(n+1)$ (but, of course, is inhomogeneous).

To support this interpretation we will state one analytic fact. Analytic dualities as consequences of geometric dualities are important components of such considerations (the Radon transform and projective duality is the classic example). Let $Hyp(S)$ be the space of hyperfunctions on $S \subset \mathbb{C}S$—functionals on the space $\mathcal{O}(S)$ of holomorphic functions on $\mathbb{C}S$ in some neighborhoods of S—and let $\mathcal{O}(\hat{S})$ be the space of holomorphic functions in $\hat{S}$.

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Theorem 1. *There is an $O(n)$-isomorphism between $Hyp(S)$ and $\mathcal{O}(\hat{S})$.*

The operators, which establish the isomorphism in both directions, are explicit. If $f \in Hyp(S)$ and $\zeta \in \hat{S}$, then the evaluation of this functional on the function $\varphi_\zeta = 1/(1-\zeta \cdot z)$, which is holomorphic in a neighborhood of S, gives $\hat{f}(\zeta) \in \mathcal{O}(\hat{S})$. To construct the inverse operator we need an analogue of the Cauchy-Fantappie integral formula on $\mathbb{C}S$, which makes it possible to extend functionals from the functions of the form φ_ζ to all holomorphic functions in neighborhoods of S.

All regular functions and distributions are contained in $Hyp(S)$. Let μ be the invariant form of maximal degree on S: $\mu \wedge d(z \cdot z) = dz$. Then for any function $\psi(z), z \in S$, we consider the hyperfunction-functional $(f[\psi], \phi) = \int_S \psi(z)\phi(z)\mu, \phi \in \mathcal{O}(S)$. We identify $\hat{f}[\psi](\zeta) = \hat{\psi}(\zeta)$. If ψ is holomorphic in some neighborhood of S, then in the integral defining $\hat{\psi}(\zeta)$ we can by deforming S extend $\hat{\psi}(\zeta)$ holomorphically outside of the domain $\hat{S}$. If ψ is holomorphic on $\mathbb{C}S$ then $\hat{\psi}$ holomorphically extends to the whole of C.

Theorem 2. *There is an $O(n, \mathbb{C})$-isomorphism between $\mathcal{O}(\mathbb{C}S)$ and $\mathcal{O}(C)$ that identifies the spaces of polynomials on these manifolds.*

This isomorphism is surprising since complex homogeneous manifolds $\mathbb{C}S$ and C are not isomorphic as homogeneous manifolds, nor are they isomorphic as complex ones. There are some intermediate isomorphisms for spaces of holomorphic functions on horospherically convex domains $D \subset \mathbb{C}S$ (their complements are unions of horospheres). This situation is similar to the complex linear convexity of Martineau. It is essential that the sphere S is horospherically convex compact. It would be interesting to investigate horospherically convex compacts inside S as an example of the influence of complex geometry on real geometry.

In the isomorphism of Theorem 2, homogeneous polynomials on C correspond to spherical polynomials on S. Spherical polynomials are eigenfunctions of the Laplace-Beltrami operator on the sphere. Similarly, we can consider spherical functions on the hyperbolic space H. In the latter case there is the Poisson integral reconstructing spherical functions through their boundary values (we transfer to the bounded model in the intersection of H by the hyperplane $x_1 = 1$).

Is there an analogue of the Poisson integral for spherical polynomials? Of course, S has no real boundary, but we can consider the complex boundary of $\mathbb{C}S$, which we will identify with the projectivization B of the cone C. We extend spherical polynomials $f(x)$ on $\mathbb{C}^{n+1}$ and take restrictions $\hat{f}$ to the cone C. They are homogeneous polynomials on C—sections of line bundles on B. We interpret $\hat{f}$ as boundary values of f. The operator $f \to \hat{f}$ is compatible with the isomorphism in Theorem 2 above. Let C_z be the intersection of C by the hyperplane $\zeta \cdot z = 1$ and ω be a holomorphic $(n-1)$-form such that $d(\zeta \cdot z) \wedge \omega = \mu$. Let $\gamma \subset C_z$ be any cycle homological to the sphere S^{n-1}.

Theorem 3. *We have*

$$\int_\gamma \hat{f}\omega = c(m,n)f(z), m = \deg \hat{f}.$$

In this formula we reconstruct the extensions of spherical polynomials on the whole space. We do not give the explicit value of the constant c. To make this formula similar to the Poisson formula on H we need to use the homogeneity of $\hat{f}$ to replace the integration in C_z by the integration in a fixed section of C. Doing so will add to the integrand a factor—a Poisson kernel. The new essential moment comes when we integrate not on the whole complex boundary but on any cycle there. Let us mention that the connections between spherical polynomials on S and homogeneous polynomials on the complex cone C were discovered by Maxwell although he considered a different isomorphism.

There are interesting complex constructions connecting with the hyperbolic geometry as well. Here is one example. Let H_+ be one sheet of the hyperboloid H ($x_1 > 0$). Let us consider its complex neighborhood $Crown(H) = \{z = x + iy \in \mathbb{C}^{n+1}, z \cdot z = 1, x_1^2 - x_2^2 - \cdots - x_{n+1}^2, x_1 > 0\}$, which we will call the complex crown of H. It is biholomorphically equivalent to the future tube.

Theorem 4. *All spherical functions on H_+ admit holomorphic extensions on $Crown(H)$, and it is the maximal joint holomorphy domain for these functions.*

All these constructions can be generalized to arbitrary compact symmetric spaces.

Further Reading

[1] S. Gindikin, Complex horospherical transform on real sphere, *Geometric Analysis of PDE and Several Complex Variables*, Contemporary Mathematics, number 368, Amer. Math. Soc., 2005, pp. 227–232.

[2] ———, Horospherical Cauchy-Radon transform on compact symmetric spaces, *Mosk. Math. J.*, **6**(2) (2006), 299–305.

[3] ———, Harmonic analysis on symmetric manifolds from the point of view of complex analysis, *Japanese J. Math.*, **1** (2006), 87–105.