WHAT IS...
Stanley Depth?

M. R. Pournaki, S. A. Seyed Fakhari, M. Tousi, and S. Yassemi

History and Background

Richard P. Stanley is well known for his fundamental and important contributions to combinatorics and its relationship to algebra and geometry, in particular in the theory of simplicial complexes. Two kinds of simplicial complexes play central roles in combinatorics: partitionable complexes and Cohen-Macaulay complexes. Stanley posed a central conjecture relating these two notions: Are all Cohen-Macaulay simplicial complexes partitionable? In a 1982 *Inventiones Mathematicae* paper [4], Stanley defined what is now called the Stanley depth of a graded module over a graded commutative ring. Stanley depth is a geometric invariant of a module that, by a conjecture of Stanley, relates to an algebraic invariant of the module, called simply the *depth*. It is shown in [2] that this conjecture implies his conjecture about partitionable Cohen-Macaulay simplicial complexes. Our aim here is to introduce the notion of the Stanley depth.

Let $\mathbb{K}$ be a field and $S = \mathbb{K}[x_1, \ldots, x_n]$ the $\mathbb{K}$-algebra of polynomials over $\mathbb{K}$ in n indeterminates $x_1, \ldots, x_n$. We may write $\mathbf{x} = \{x_1, \ldots, x_n\}$ and denote S by $\mathbb{K}[\mathbf{x}]$ for convenience. A *monomial* in S is a product $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \ldots x_n^{a_n}$ for a vector $\mathbf{a} = (a_1, \ldots, a_n) \in \omega^n$ of nonnegative integers. The set of all monomials in S, denoted by $\mathrm{Mon}(S)$, forms a $\mathbb{K}$-basis of S.

A *monomial ideal* I of S is an ideal generated by monomials in S, so that there exists a subset $A \subseteq \mathrm{Mon}(S)$ with $I = \langle \mathbf{x}^{\mathbf{a}} \mid \mathbf{x}^{\mathbf{a}} \in A\rangle$. The Hilbert basis theorem now implies that there exists a finite subset $B \subseteq A$ such that $I = \langle \mathbf{x}^{\mathbf{b}} \mid \mathbf{x}^{\mathbf{b}} \in B\rangle$. Therefore, every monomial ideal I of S may be written in the form $I = \langle \mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_\ell}\rangle$, where $\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_\ell} \in \mathrm{Mon}(S)$. Monomial ideals are the most manageable among all types of ideals due to their simple structure.

Clearly, every monomial ideal I in S is a $\mathbb{K}$-subspace of S. The monomials in I form a $\mathbb{K}$-basis of I. Let us characterize monomials that lie in a given monomial ideal. In order to do this, fix a monomial ideal $I = \langle \mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_\ell}\rangle$ in S. Then a monomial $\mathbf{x}^{\mathbf{b}}$ lies in I if and only if there exists t with $1 \le t \le \ell$ such that $\mathbf{x}^{\mathbf{a}_t} \mid \mathbf{x}^{\mathbf{b}}$. This is equivalent to $\mathbf{x}^{\mathbf{b}} = \mathbf{x}^{\mathbf{a}_t}\mathbf{x}^{\mathbf{c}}$ or $\mathbf{b} = \mathbf{a}_t + \mathbf{c}$, for some $\mathbf{c} \in \omega^n$. Thus, the set $\bigcup_{t=1}^{\ell}(\mathbf{a}_t + \omega^n)$, which is a set of lattice points in ω^n, consists of the exponents of all monomials in I. For example, the set of all monomials in the ideal $I = \langle x_1x_2^2, x_1^2x_2\rangle$ of $S = \mathbb{K}[x_1, x_2]$ is $\{\mathbf{x}^{\mathbf{b}} \mid \mathbf{b} \in (1,2) + \omega^2\} \cup \{\mathbf{x}^{\mathbf{b}} \mid \mathbf{b} \in (2,1) + \omega^2\}$. The exponents of these monomials have been shown in Figure 1 by the blue points.

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Let $\mathbf{x}^{\mathbf{a}}$ be a monomial in $S = \mathbb{K}[\mathbf{x}]$ and fix it. The $\mathbb{K}$-subspace of S whose basis consists of all monomials $\mathbf{x}^{\mathbf{a}}\mathbf{u}$, where $\mathbf{u}$ is a monomial in $\mathbb{K}[\mathbf{z}]$, $\mathbf{z} \subseteq \mathbf{x}$, is called a *Stanley space* of dimension $|\mathbf{z}|$ and is denoted by $\mathbf{x}^{\mathbf{a}}\mathbb{K}[\mathbf{z}]$. Here $|\mathbf{z}|$ denotes the number of elements of $\mathbf{z}$. For example, if $S = \mathbb{K}[x_1, x_2]$, then $x_1^2x_2\mathbb{K}$, $x_1^3x_2\mathbb{K}[x_1]$, $x_1x_2^2\mathbb{K}[x_2]$ and $x_1^2x_2^2\mathbb{K}[x_1, x_2]$ are all Stanley spaces with dimensions 0, 1, 1 and 2, respectively. In Figure 2, the red, orange and blue points consist of the exponents of all monomials in $x_1^3x_2\mathbb{K}[x_1]$, $x_1x_2^2\mathbb{K}[x_2]$ and $x_1^2x_2^2\mathbb{K}[x_1, x_2]$,

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respectively. Also $x_1^2x_2\mathbb{K}$ consists of only one monomial, namely $x_1^2x_2$. We draw the exponent of this monomial by a violet point in Figure 2.

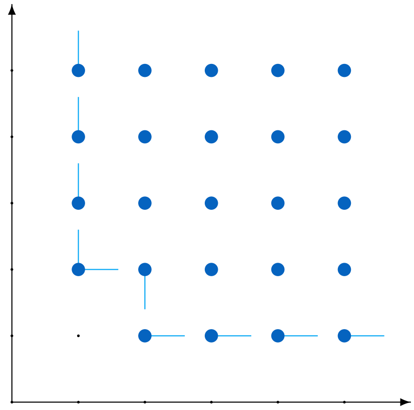


Figure 1. The blue points mark the exponents of all monomials belonging to the ideal $I = \langle x_1x_2^2, x_1^2x_2\rangle$ of $S = \mathbb{K}[x_1, x_2]$.

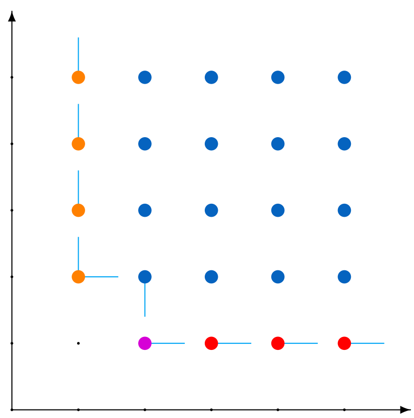


Figure 2. The violet, red, orange and blue points consist of the exponents of all monomials which belong to $x_1^2x_2\mathbb{K}$, $x_1^3x_2\mathbb{K}[x_1]$, $x_1x_2^2\mathbb{K}[x_2]$ and $x_1^2x_2^2\mathbb{K}[x_1, x_2]$, respectively.

Let I be a fixed monomial ideal of $S = \mathbb{K}[\mathbf{x}]$. A decomposition $\mathcal{D}$ of I as a finite direct sum of Stanley spaces is called a *Stanley decomposition* of I. It is known that there exists at least one Stanley decomposition of I. For example,

$$\mathcal{D}_1: \; I = x_1^2x_2\mathbb{K} \oplus x_1^3x_2\mathbb{K}[x_1] \oplus x_1x_2^2\mathbb{K}[x_2] \oplus x_1^2x_2^2\mathbb{K}[x_1, x_2]$$

is a Stanley decomposition for $I = \langle x_1x_2^2, x_1^2x_2\rangle$ in $S = \mathbb{K}[x_1, x_2]$. The two other Stanley decompositions of I may be written as follows:

$$\mathcal{D}_2: \; I = x_1^2x_2\mathbb{K} \oplus x_1^3x_2\mathbb{K} \oplus x_1^4x_2\mathbb{K}[x_1] \oplus x_1x_2^2\mathbb{K}[x_2] \oplus x_1^2x_2^2\mathbb{K}[x_1, x_2],$$

$$\mathcal{D}_3: \; I = x_1x_2^2\mathbb{K}[x_2] \oplus x_1^2x_2\mathbb{K}[x_1, x_2].$$

For a given Stanley decomposition $\mathcal{D}$ of I, the minimum dimension of a Stanley space in $\mathcal{D}$ is called the *Stanley depth* of $\mathcal{D}$ and is denoted by $\operatorname{sdepth}(\mathcal{D})$. Therefore, in the above example we have $\operatorname{sdepth}(\mathcal{D}_1) = 0$, $\operatorname{sdepth}(\mathcal{D}_2) = 0$ and $\operatorname{sdepth}(\mathcal{D}_3) = 1$.

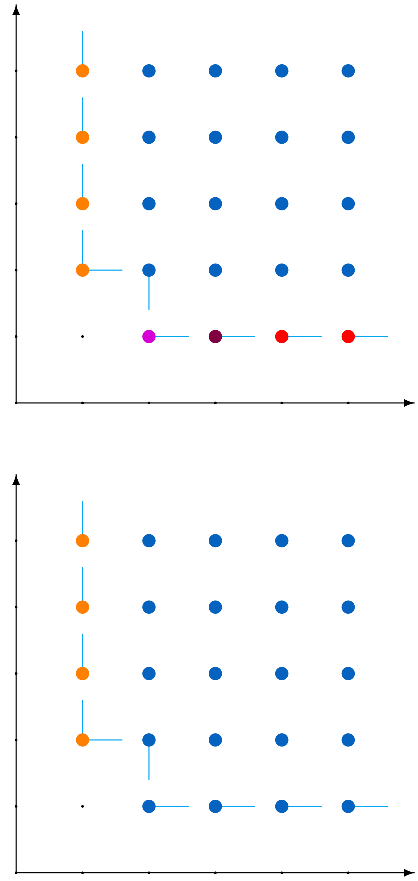


Figure 3. The upper figure demonstrates the Stanley decomposition $\mathcal{D}_2$, while the one on the bottom corresponds to $\mathcal{D}_3$.

Finally, the *Stanley depth* of I is defined to be

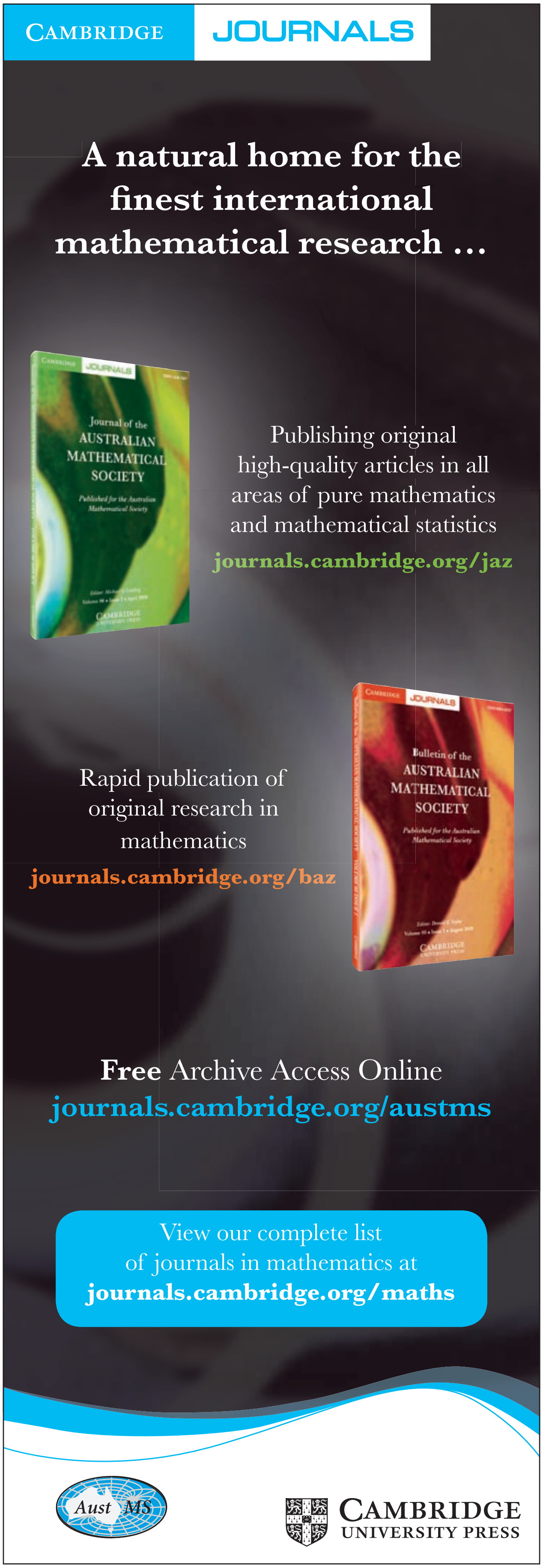
$$\operatorname{sdepth}(I) := \max\{\operatorname{sdepth}(\mathcal{D}) \mid \mathcal{D} \text{ is a Stanley decomposition of } I\}.$$

Thus, in the above example, $\operatorname{sdepth}(I)$ is at least equal to one. One may easily show that, indeed, $\operatorname{sdepth}(I) = 1$.

If I^c denotes the $\mathbb{K}$-subspace of $S = \mathbb{K}[\mathbf{x}]$ generated by all monomials of S which do not belong to I, then we may define a Stanley decomposition as well as the Stanley depth of I^c. The Stanley depth of I^c is denoted by $\operatorname{sdepth}(S/I)$, instead of $\operatorname{sdepth}(I^c)$, since, as $\mathbb{K}$-subspaces, $S = I \oplus I^c$ and, hence, $S/I \cong I^c$. For example,

$$\mathcal{D}: \; I^c = x_1x_2\mathbb{K} \oplus \mathbb{K}[x_1] \oplus x_2\mathbb{K}[x_2]$$

is a Stanley decomposition of I^c for $I = \langle x_1x_2^2, x_1^2x_2\rangle$ in $S = \mathbb{K}[x_1, x_2]$. Here we have $\operatorname{sdepth}(\mathcal{D}) = 0$ and one may easily show that $\operatorname{sdepth}(S/I) = 0$.



Stanley [4] defined what is now called the *Stanley depth* of a $\mathbb{Z}^n$-graded module M over the ring S. Note that if M is of the form I or S/I, where I is a monomial ideal, then this latter notion coincides with the Stanley depth defined above. In general, no algorithm for the computation of the Stanley depth is known. However, in the particular case when M is of the form I/J, where $J \subseteq I$ are monomial ideals of S, such an algorithm was given by Herzog, Vladoiu, and Zheng (see [3]). They describe how to compute the Stanley depth in this case. Undoubtedly, this is one of the most important contributions to the theory so far. It provides the only method known to compute the Stanley depth and, for example, it was used to compute the Stanley depth for a complete intersection ideal. It is known that the Stanley depth of the monomial ideal $I = \langle x_1^{a_1}, \ldots, x_m^{a_m} \rangle$ in the ring $S = \mathbb{K}[x_1, \ldots, x_n]$ is equal to $n - m + \lceil m/2 \rceil$.

The depth of a module is one of its important algebraic invariants. For a $\mathbb{Z}^n$-graded module M over the ring S, the *depth* of M, denoted by $\operatorname{depth}(M)$, is the supremum of the lengths ℓ of all sequences $a_1, \ldots, a_\ell$ of $\mathbb{Z}$-homogenous elements of the ideal $\langle x_1, \ldots, x_n \rangle$, for which $M \neq \langle a_1, \ldots, a_\ell \rangle M$ and a_i is not a zero divisor of $M/\langle a_1, \ldots, a_{i-1} \rangle M$, for $i = 1, \ldots, \ell$. Stanley conjectured that $\operatorname{depth}(M) \leq \operatorname{sdepth}(M)$ holds for every nonzero finitely generated $\mathbb{Z}^n$-graded module M over S. Certain cases of the *Stanley conjecture* have been proven, for example, for monomial ideals generated by square free monomials of $S = \mathbb{K}[x_1, \ldots, x_n]$ where $n \leq 5$. The question is still largely open with a huge amount of active research behind it. For further reading we refer to [1], [2] and [3].

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