Book Review

The Princeton Companion to Mathematics

Editor's Note: To review this unusually wide-ranging volume, the *Notices* invited five distinguished mathematicians who are both experts in their fields and broadly knowledgeable about mathematics in general. Their reports are presented in alphabetical order.

—Andy Magid

The Princeton Companion to Mathematics
Timothy Gowers, editor
June Barrow-Green and Imre Leader, associate editors
Princeton University Press, 2008
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Bryan Birch

This is an enormously ambitious book, full of beautiful things; I would wish to keep it on my bedside table, but that could only be possible by relays, since of course it is far too large. Timothy Gowers and his associate editors have aimed to give an account of as much of mathematics as can reasonably be made accessible; in particular, students at school should be helped to understand what mathematics is about, intending graduate students should be helped to decide what topics to research in, and established mathematicians should be helped to understand what their colleagues are doing.

Many of the articles have been written by the editors themselves, but most have been written by an enormous team of collaborators recruited by the editors. In his Introduction, Gowers stresses the importance of accessibility and pays tribute to his authors' willingness to revise their articles: if he didn't understand it, he asked them to change it—he must have been enormously tactful! Almost all of the articles I have sampled have been excellent, not just accessible but enjoyable to read, and varied voices add to the liveliness of the book. The book does not attempt to show the reader the frontiers of knowledge (in the nature of things, few of them are accessible), but there are signposts all over the place. In my opinion, the book is magnificently successful, but aspiring research students should be warned that this is a companion, not an encyclopaedia, and some important topics for research are not even touched on.

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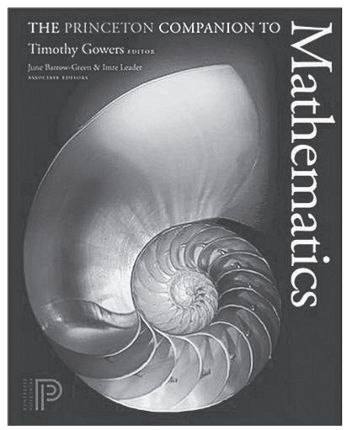

The plan of the book is a classic "arch" form in eight parts. The heart is Part IV, containing accounts of twenty-six "branches" of mathematics in an arbitrary but sensible linear order. This is where one learns what one's colleagues are doing! The first branch ("Algebraic numbers" by Barry Mazur) happened to be the one I know most about. He uses a light touch and stops short of p-adic numbers (in Part III), let alone class field theory (which has an article of its own in Part V). His simple examples convey the essence of the subject beautifully, but I would have liked him to have gone a little deeper; of course I enjoyed the article enormously, it is good to read stuff one knows well, written by as good a writer as Barry. When I went on to the second branch ("Analytic number theory" by Andrew Granville), the writing was still excellent and I continued to enjoy myself; but whereas the part of analytic number theory with which I am familiar is mainly concerned with polynomial equations, Andrew Granville's article is centred on the properties of primes, and there is nothing in it about Waring's problem (which gets mentioned later on, particularly in the biographical Part VI). More seriously, transcendence theory and Diophantine approximation seem to have gone missing too; even if one divides mathematics into twenty-six branches and keeps it accessible, the branches remain huge and gaps start opening! "Computational number theory", "Algebraic geometry", "Arithmetic geometry", "Algebraic topology"—a particularly nice chapter—but I was starting to find it tougher. I jumped ahead to IV.25, "Probabilistic models of critical phenomena", of which I know very little but have enjoyed what I know, and Gordon Slade didn't disappoint me.

What about the rest of the book? Part I is the editor's introduction to what mathematics is about and to the language of mathematics. It is addressed to a beginning student, but its simple examples illustrate sophisticated points that we can all learn from. Part II explains where modern mathematics has come from, in particular containing substantial historical articles. Part III at first sight appears something of a ragbag, containing bits and pieces which need to go somewhere because they are important but don't fit conveniently into any of the branches. Arranged in alphabetical order for easy reference, it is surprisingly satisfactory to consult. The bias of the book is toward the questions and results of mathematics, and that is how it has been split into branches in Part IV; but concepts and methods are important too, and don't split the same way. Accordingly, important procedures and techniques have their description in Part III, sometimes in extensive articles. I note that cohomology gets short shrift; it is a valuable and pervasive technique, but may be hard to write about attractively.

In contrast to Part III, Part V ("Theorems and problems") is more by way of a beauty contest. I am gratified to have a share in fourth place (albeit for alphabetical reasons); the B-S-D conjecture has been stated clearly and simply, although only in the weak form; again, this is a companion, not an encyclopaedia! Millennium problems aside, this part contains some lovely plums; for a number theorist like me the sections from V.27 onward are a delight. (Gowers has treated number theory very generously in this companion!) At the very end, Osserman's article on the Weil conjectures gets it exactly right.

The book concludes with a part containing the lives and works of great mathematicians, a part entitled "Influence of mathematics", and finally "Final perspectives". "Influence of mathematics" begins with various applications of mathematics, including particularly authoritative articles by Daubechies on wavelets, by Frank Kelly on traffic, by Sudan on coding, and by Cocks on cryptography; these are followed by articles on music and on art. "Final perspectives" contains five essays which are intended to provoke thought and certainly do so, and at the very end there are letters of advice to a young mathematician from Atiyah, Bollobás, Connes, Dusa Macduff, and Peter Sarnak.

To sum up, the book is really excellent. I know of no book that will give a young student a better idea of what mathematics is about. I am certain that this is the only single book that is likely to tell me what my colleagues are doing. I am less sure that an intending graduate student should take this book as a guide to his choice of research topic, since different fields of similar importance may not be served equally in this volume—some may even be missing; but he or she should certainly read the final part. A final carping comment: the publishers should consider issuing a library edition in at least three volumes; I fear that the spine of the present heavy volume may break from overuse.

Simon Donaldson

Two extracts from the preface convey a good overall picture of what the book is about:

"[The companion] simply aims to present for the reader a large and representative sample of the ideas that mathematicians are grappling with at the beginning of the twenty-first century."

"The companion is not an encyclopaedia: …the book is like a human companion, complete with gaps in its knowledge and views on some topics that may not be universally shared."

The book has eight parts which contain rather different kinds of material aimed at different likely readership. Roughly, the material could be divided into three classes:

(1) material for a general audience, similar to a *Scientific American* article or popular mathematics lecture;

(2) material at an undergraduate mathematics level;

(3) expository articles aimed at professional mathematicians, somewhat in the style of the AMS *Bulletin* articles.

There are contributions from many different authors, but the book is permeated by Gowers' distinct vision and he has written a large amount of the material.

Working outward, the heart of the book is Part IV, which consists of expositions of twenty-six "Branches of mathematics" of about fifteen pages each. For example we have "Algebraic numbers" (Mazur), "Differential topology" (Taubes), "Partial differential equations" (Klainerman), "Extremal and probabilistic combinatorics" (Alon and Krivelevich). These are roughly at the level (2)–(3) above. The most successful of these articles are excellent, giving overview and insight that would

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be hard to find elsewhere. Two of the articles I particularly liked are "Partial differential equations" (tackling such a large subject in such a short space) and "Representation theory" (Gronowski), which moves from the elementary theory up to advanced topics but conveys an overall unity. The first two articles, "Algebraic numbers" and "Analytic number theory" (Granville), are very informative and interesting, at least for this reader, and illustrate a contrast in styles. The first explains many of the basic ideas in the subject (such as ideal class groups and unique factorisation) with outline proofs, while the second concentrates more on stating results and interesting open questions. Both approaches work well and are probably the right ones for the subjects. Each of these articles in Part IV finishes with a short list of further reading.

Part III of the book is made up of shorter articles (one or two pages) about ninety-nine "Mathematical concepts". The level and style varies greatly. A sample of three alphabetically adjacent articles conveys the contrasting approaches:

> "The Euler and Navier-Stokes equations" (Fefferman). A statement of the equations and a discussion of the long-time existence problem, the distinction between weak and strong solutions, and some modern results. Culminates in the insight, "We need to understand why a tiny viscosity dissipates a lot of energy."
>
> "Expanders" (Wigderson). The definition (a graph with n-vertices is a c expander if for every $m \leq n/2$ and every set S of m vertices there are least cn edges between S and its complement). Description of constructions and why expanders are important, including surprising (and recent) applications to estimating averages over large sets.
>
> "The exponential and logarithm functions" (Gowers). This is pitched at a much more elementary level: the problem of defining 2^a when a is integral, rational or irrational. The exponential function defined by power series or the limit of $(1 + x/n)^n$, with outline proofs. Logarithms and extension to complex variables.

Again, these are all excellent in their different ways. The first two will be concise, insightful references at the level (2)–(3) above, and the third nicely summarises standard material around about the beginning undergraduate/senior high school level in a way that might be more digestible than when buried in a textbook. There are comprehensive cross-references between the different articles. In Part III there are few references to other sources, and more would be useful (for example the original research articles on "Expanders" discussed above, and sources where the reader might find more about the Ricci flow).

So much for Parts III and IV. The "Introduction", Part I (76 pages, written by Gowers), could stand alone as a general description of modern mathematics. The discussion includes, among much else, "What is mathematics about?"; "Some fundamental mathematical definitions"; different modes of thought characterising algebra, geometry and analysis; formal and informal language used in proofs; "What do you find in a mathematical paper?". This could be extremely valuable for an undergraduate contemplating a career in mathematical research. Part II consists of seven substantial historical articles. Part V is made up of short articles, somewhat like Part III but focused on particular results and problems (again with a wide range, from "The fundamental theorem of arithmetic" to "The Poincaré conjecture"). Part VI is historical again, brief biographies of ninety-six mathematicians, and Part VIII ("Final perspectives") consists of a variety of essays on general, sometimes more philosophical, topics.

Part VII ("The influence of mathematics") deserves special mention. As explained in the preface, the central focus of the book is on *pure mathematics* but with a sympathy to applications. Part VII addresses applications in more detail, and, while the coverage has to be very selective, the articles are particularly interesting; perhaps the most informative for the professional pure mathematician. Here "applications" should be interpreted broadly: the articles include "Wavelets", "Medical statistics", "Mathematics and music".

It is easy to complain about what is not covered in the book, although such criticism is largely deflected by the not-an-encyclopaedia quote above. There is very little on differential geometry. I was hoping to find a broad discussion of the influence of cohomology in various guises (surely one of the main developments of the twentieth century), but was disappointed. It would have been interesting and topical to see more on quantum field theory, as a notable idea "that mathematicians are grappling with at the beginning of the twenty-first century" (although there is some coverage of this under the headings "Mirror symmetry" and "Vertex operator algebras"). In general I would often have been happier with a little more formality in the definitions, etc., but this would make the book more of an encyclopaedia, more standard and less distinctive.

Overall this book is an enormous achievement for which the authors deserve to be thanked. It contains a wealth of material, much of a kind one would not find elsewhere, and can be enjoyed by readers with many different backgrounds.

Gil Kalai

Praise

This book is an unusually rich description of the many facets of mathematics as a science, as an art, as a powerful tool, and as a human activity.

The human face of mathematics comes to play not only in the general history chapters and the little chapters on individual mathematicians, but often also in chapters devoted to areas of mathematics and to concepts, problems, and results. Take, for example, this nice quotation: "Borcherds was struck by the formal similarity between V^1 and the chirals algebras of CFT's" (the story of the proof of the moonshine conjecture from the chapter on vertex operators, p. 549.) You do not have to fully understand these objects in order to get a good sense of the discovery's first moments, and the joy it produced. Another example of a defining moment in mathematics described with a personal touch: "Gerhard Frey realized that such a curve might be so unusual that it may contradict the Shimura-Tanayama-Weil conjectures" (From the chapter, "Fermat's last theorem", p. 692). Oh yeah!

The different ways different authors chose to present a field, a concept, or a theorem shed light on different personal approaches toward mathematics.

While the self-mandate of the book is limited to pure mathematics, there is also a very strong spirit of applied mathematics. The oldest and strongest connection of mathematics is to physics and through it to other exact sciences and engineering; indeed, physics is strongly felt in most parts of the book. Next, in my view, comes statistics. The reader will get a strong taste of the importance of statistics and probability (and a taste for more). Optimization and algorithms are also amply represented in the book. It is not common to see as strong an emphasis on the philosophy of mathematics and mathematics in philosophy as appears in this book, and this is most welcomed.

The book is so rich and yet it is well done. A rare achievement indeed!

Critique

In a movie describing a human mission to the planet Mars, the proud families of three astronauts who had just landed there were about to talk to the astronauts. They greeted them and were told that because of the speed of light they would have to wait 90 seconds for a response. Once these 90 seconds passed and the movie paid its dues to science, the conversation continued back and forth without any further interruption. This is similar to a common problem (and a common unsuccessful solution) in trying to make mathematical presentation self-contained. In this book, the large introductory chapters to the mathematical endeavor as a whole are superb, but it seems that some chapters describing major areas of mathematics require a few introductory little chapters of a similar kind. The book is not and cannot really be self-contained.

Aside from that, reading a large encyclopedia-type book like *The Princeton Companion to Mathematics* (PCM) can be discouraging, recognizing how little your little corner of the woods in this huge forest is. Something to take comfort from is the fractal nature of science and of mathematics. A little discovery in a small corner, a concept or a theorem, can make a big difference for the large picture. Even more comfort comes from taking notice not just of the value of mathematics but of *mathematics as a value*, an idea that the PCM strongly champions.

The book, at 2.6 kilograms, is much too heavy. I hope contributors will put their chapters on their home pages and that a future edition will be divided into several volumes.

Advice

There are several pieces of advice in the book for young mathematicians, but an advice chapter for middle-aged and older mathematicians is notably missing. Béla Bollobás, in his nice advice, quotes G. H. Hardy who wrote that there is no permanent place in the world for ugly mathematics. When Hardy wrote this phrase the term he felt need to explain was "ugly" (and he elaborated on what beauty in mathematics means) but these days we probably have more trouble explaining the term "permanent". We cannot really accept Hardy's romantic saying as normative advice, nor can we really accept Bollobás's follow-up, that there is no place in the world for nonenthusiastic mathematicians. If you prove good lemmas and theorems or make other progress in exploring mathematics, the amount of enthusiasm for mathematics is up to you. Mathematics and mathematicians come in many kinds, as the book in front of us largely demonstrates, and there is a place in the world for all, though alas, perhaps not permanently.

Just Another Good Lemma

When Paul Erdős received the Wolf prize he said: "If I could get a good lemma—I wouldn't give it for a hundred medals." (Erdős was paraphrasing the Hungarian poet János Arany who wrote "If I could have a good sleep—I wouldn't give it for a hundred medals.") Gosta Mittag-Leffler seemed to have had another approach. Promoting mathematics (as Hardy said about him) more than any other mathematician of his time required him to avoid,

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at times, the temptation of proving lemmas. Contributing to mathematics and to the mathematical community also comes in many ways, and this book is a daring and successful attempt to enrich the infrastructure of mathematics. It offers the readers rich and useful sources on mathematics and mathematicians. The book is an achievement that Tim Gowers, June Barrow-Green, Imre Leader, and the many other contributors can be proud of, and that we all can take pleasure from.

Richard Kenyon

What is this book? It is not immediately easy to say. It is not, what one might suspect at first glance, some sort of printed version of the Wikipedia of mathematics. Nor is it an encyclopedia of mathematics: you won't find tables of formulas or integrals or definitions of mathematical terms. It is not "complete" in this sense. Nor in fact is it necessarily a good resource for learning about a particular subject.

Part of what the book is, rather, is a description of what mathematics is, accessible to the public. I've been tempted to give it to my 15-year-old to read, as a way of explaining what I do. It also contains a history of mathematics: a concise chronological description of major mathematicians, theorems, definitions, and proofs.

Another part of what it is (and this is the part for me as a working mathematician) is a collection of facts/essays/ideas which every mathematician ought to know—at least in an ideal world. It is something to browse when I want to learn a little about those parts of math which…uh…are not part of my culture. Here is a collection of essays about various subjects of mathematics written by experts(!)—and here I mean my mathematical friends and colleagues—in a language which is not only understandable but downright accessible. I greatly enjoyed flipping through the book, browsing its articles, uncovering new gems of ideas. To be honest, the short articles are great but sometimes, well, frustratingly short. I wished some of them went into a little more depth! But maybe this is a sign that the book is serving its purpose: getting the reader interested enough in a subject to go out and get more information elsewhere.

Another delightful aspect of the short articles is that they were commissioned in a free-form way: authors were (apparently) not given strict guidelines about what to say. As a consequence, the article about, say, algebraic numbers, written by Barry Mazur, is not a dry list of definitions interspersed with lemmas and theorems, covering the basic facts in the subject. It is, rather, about what interests Barry Mazur as a mathematician, that is, a tale of tidbits, facts, and yes, definitions, which starts out from basic, motivating high-school algebra questions and leads up to (almost) serious but exciting questions of modern number theory. It is this personalized flavor that makes the whole program enjoyable, entertaining, and interesting.

My advice to you, reader, is to buy the book, open it to a random page, read, enjoy, and be enlightened.

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Angus Macintyre

The Preface begins with a notorious quotation from Bertrand Russell, giving a logicist "definition" of pure mathematics:

> Pure Mathematics is the class of all propositions of the form '*p* implies *q*', where *p* and *q* are propositions containing one or more variables, the same in the two propositions, and neither *p* nor *q* contains any constants except logical constants. And logical constants are all notions definable in terms of the following: Implication, the relation of a term to a class of which it is a member, the notion of *such that*, the notion of relation, and such further notions as may be involved in the general notion of propositions of the above form. In addition to these, mathematics *uses* a notion which is not a constituent of the propositions it considers, namely the notion of truth.

The *Princeton Companion to Mathematics* very neatly counters this by saying that it is about everything that Russell's definition leaves out. The aim is to present, in an attractive and accessible way, a large and representative sample of those ideas of modern, pure mathematics that most engage the mathematicians of our time. From the *Companion* I learned Eisenstein's "In the end, the best mathematical genius cannot discover alone what has been discovered by the collaboration of many outstanding minds." The *Companion* achieves its aims by such a collaboration, by skillful and unobtrusive editing. It makes possible a wide range of mathematical journeys, from short excursions for the untravelled to explorations that will reward the most experienced and accomplished of mathematicians.

Let us begin with origins and pioneers. While the historical sections do not have the pop drama of Bell's *Men of Mathematics* (a work known to have opened the eyes of many young people), they have their own authority of scholarship, and they enhance the longer articles. There are ninety-six

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short scientific biographies of deceased mathematicians, and one of Bourbaki. The ninety-six begin with Pythagoras, and end with Abraham Robinson. I was a little surprised to see among the ninety-six so many who had worked in logic (I counted eighteen, including of course some whose main claim to fame is elsewhere).

I wandered back and forth in the book. I recommend going early to the illuminating section "Final perspectives". I was particularly taken by the contributions of Tony Gardiner and Michael Harris. What they have in common, for me, is an emphasis on the specifics (or "motley") of mathematics, an evolving human activity unlike any other.

Gardiner's "The art of problem solving" is built around a delightful series of quotations (many familiar, but of enduring strength). He pursues the metaphor of exploration of a largely unexplored mental universe, where the great discoveries are rooted in detailed knowledge of "mathematics in the small". Mathematics is a craft, where serious insight is gained only through constant practice. He has a special interest in getting children started well in the craft, but much of what he says is relevant at all stages of our unending apprenticeship. There is a welcome scepticism about the jargon of theories of problem solving, and condemnation of "reforms" that reduce the emphasis on, and time for, serious elementary mathematics. In the penultimate subsection of "Final perspectives", Atiyah, Bollobás, Connes, McDuff, and Sarnak, each an inspiring master of our craft, have wise, and quite specific, advice for young mathematicians aiming to cross into the modern research world. Atiyah has a memorable phrase: "All the really creative aspects of mathematical research precede the proof stage."

Harris's title is "'Why mathematics?' You might ask", and it is what he says about mathematical ideas that I want to consider. His emphasis is on ideas and the experience of mathematics. Here are some of his phrases:

> "the basic unit of mathematics is the concept, not the theorem";
>
> "the purpose of a proof is to illuminate a concept";
>
> "Even the most ruthless funding agency is not yet so post-human as to require an answer to the question 'Why experience?'."

Note the extreme contrast to the Russell quotation cited earlier. I commend pages 973–975 for a light, convincing account of the objectivity of mathematical ideas (they can be stolen, or counted!) and an affirmation of the public intuition that underlies mathematics. The *Companion* itself amply confirms what Harris writes. There are no proofs, but many wonderful ideas (often linked to physics). That I, and I presume very many others, can go to a randomly chosen article and get the "drift" certainly confirms the public intuition. The deliberately flexible structure of the book (there is little point in reading it from cover to cover) conveys a sense of the deep mystery of the organism of mathematics. Those who regret the absence of proofs may turn to *Proofs from the Book.* Most of us will learn from both.

Parts I and II are the most skippable for professionals, but they are well done and indispensable for beginners.

Part III displays the motley of present-day concepts. That they come in alphabetical order did not affect my appreciation.

Part IV, "Branches of mathematics", leads to the most important and complex ideas. I read all the papers, and I can honestly say I enjoyed them all. Here one sees how new subjects have evolved and how others have been linked in unforeseen ways. Imagine how this book would have looked fifty years ago! There would have been little computational number theory, little cryptography, almost no arithmetic geometry, very little geometric group theory, a lot less dynamics, no fractals, no wavelets, no computational complexity, no mirror symmetry, no classification of finite simple groups, no proof of the Weil Conjectures, no Langlands Program, no vertex operator algebras, a very different and much more fragmented combinatorics, few links from stochastics to the rest of the mathematical world, no forcing in set theory, no modularity of elliptic curves over Q, no Ricci flow in the style of Hamilton and Perlman. Thus we must hope that the *Companion* will go through many revisions and be for later generations the rich resource it is for us.

I shared a couple of concerns with other reviewers. For an idea so pervasive in modern mathematics, cohomology gets rather little coverage, except for three pages in Totaro's beautiful paper. One can hope for much more in a revised edition. A quite different subject, of enormous difficulty, with a long history, and problems readily intelligible to all readers of the *Companion* seems to me neglected—namely transcendence theory.

The papers on PDE's satisfied me greatly. For those of an algebraic or logical bent, it has generally proved very demanding to extend one's understanding of PDE's much beyond the formal level of an undergraduate course. Reading the *Companion,* I now saw a much bigger picture. To quote Klainermann: "One looks in awe at how equations, such as the Laplace, heat, wave, Dirac, KdV, Maxwell, Yang-Mills, and Einstein equations, which were originally introduced in specific physical contexts, turned out to have very deep applications in areas such as geometry, topology, algebra and combinatorics."

Mutatis mutandis, such sentiments are appropriate reactions to most of the ideas in this book.