

The Quest for Universal Spaces in Dimension Theory

Stephen Leon Lipscomb

For metric spaces, the quest for universal spaces in dimension theory spanned more than a century of mathematical research. The history breaks naturally into two periods—the *classical* (separable metric) and the *modern* (not necessarily separable metric).

To unify these two periods, we shall motivate the construction of the space J_A that did for the modern period what the unit interval I did for the classical period. The approach is chronological and dates from 1883 to 2009.

Classical Period

Cantor's 1883 Construction

The 1970s introduction of the space J_A was partly motivated by a construction of a topological copy of unit interval I . Indeed, the mapping $C \rightarrow I$ from Cantor's set C , known as *classical adjacent-endpoint identification*, yields, as a quotient space, a topological copy of the unit interval. (It is most likely that it was Cantor who introduced classical adjacent-endpoint identification—see Cantor [1883] and [1884], the English translation of Cantor [1884] in Edgar [1993], or Lipscomb [2009, pages 7 and 8].)

Recall that Cantor's set C is the limit set obtained by starting with the unit interval I , removing the middle-third open segment $(1/3, 2/3)$, and then recursively removing the “middle thirds” of the remaining segments ad infinitum. Moreover, notice that the first “middle third” $(1/3, 2/3)$ has

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endpoints

$$1/3 = \frac{0}{3^1} + \frac{2}{3^2} + \frac{2}{3^3} + \dots \equiv 0222 \dots \equiv 0\bar{2},$$

$$2/3 = \frac{2}{3^1} + \frac{0}{3^2} + \frac{0}{3^3} + \dots \equiv 2000 \dots \equiv 2\bar{0}.$$

It is also true that the endpoints of *all* “removed segments” correspond one-one with sequences in $\{0, 2\}$ that have the form

$$a_1 \dots a_t u v v v \dots = a_1 \dots a_t u \bar{v} \quad (u \neq v).$$

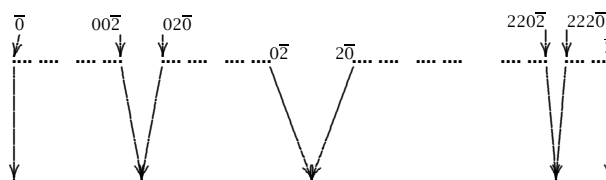
This one-one correspondence extends to a bijection between all sequences in $\{0, 2\}$ and all points in C according to “ $x \in C$ if and only if

$$x = \sum_{i=1}^{\infty} \frac{a_i}{3^i} \text{ for some } a_1 a_2 \dots \in N(\{0, 2\}),$$

where $N(\{0, 2\})$ is an instance of a *Baire space*”. (For any nonempty discrete space A , the countably infinite topological product $N(A) = A \times A \times \dots$ is called a Baire space.)

The one-one correspondence $C \rightarrow N(\{0, 2\})$ given above is a homeomorphism; in the context of viewing C as an attractor of an iterated function system, the inverse mapping $N(\{0, 2\}) \rightarrow C$ is the *address map* and $N(\{0, 2\})$ the *code space*.

So using the sequential representations (addresses) of the endpoints in Cantor's set C , we may illustrate classical identification of adjacent endpoints $C \rightarrow I$ in the following graphic.

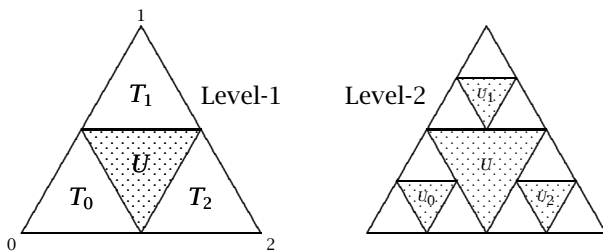


For our purposes, the key facts concerning C are detailed in the following theorem. For a proof that C is a universal space see Kuratowski [1966, page 285].

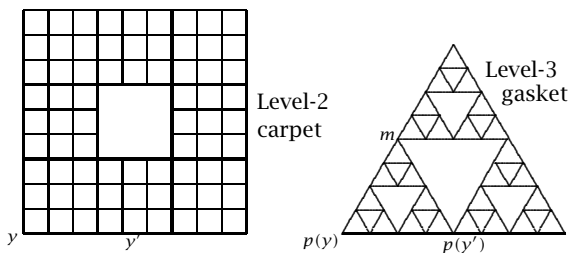
Theorem 1 (Dimension zero). *Cantor's set C is a fractal that is also universal for the class of zero-dimensional separable metric spaces.*¹

Classical Imbedding Theorems

The symbiotic beginnings (circa 1900) of topology and dimension theory were documented by Tony Crilly [1999]. In 1915, two years after L. E. J. Brouwer [1913] constructed his precise and topologically invariant definition of dimension, Waław Sierpiński [1915] introduced his classical fractal Sierpiński's triangle (or gasket). He conveyed his inductive construction in two illustrations; a partial rendition is given below. (Note that like Cantor's construction of his set C above, Sierpiński recursively cut holes in a manifold with boundary.)



And in the following year, 1916, Sierpiński [1916] introduced his *carpet*, a limit set obtained by starting with a square, dividing it into nine congruent subsquares, removing the middle subsquare, and then, on each of the remaining eight subsquares, repeating this process ad infinitum.



Sierpiński's carpet is another example of a classical fractal that is also a *universal space*.

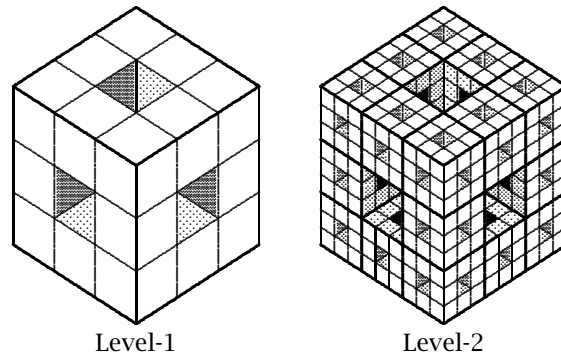
Theorem 2 (Planar and dimension one). *Sierpiński's carpet is a fractal that is also universal for the class of one-dimensional compact subspaces of the plane.*

¹For a given class \mathcal{C} of topological spaces, $U \in \mathcal{C}$ is universal for \mathcal{C} if each member of \mathcal{C} is homeomorphic to a subspace of U . All spaces in this article are metrizable.

In passing, we note that Sierpiński's carpet and triangle are quite distinct; e.g., no subspace of his triangle is homeomorphic to any plane figure that has five or more line segments meeting at a common point (Peitgen, Jürgens, and Saupe [1992, page 30]).

By 1926 Menger extended Sierpiński's construction of the carpet by recursively cutting holes in the unit cube I^3 . Menger's construction is commonly known as Menger's *sponge*. Like the carpet, the sponge turned out to be a fractal that is also a universal space (Peitgen, Jürgens, and Saupe [1992, page 108], Menger [1926], and the illustration below).

Theorem 3 (Dimension one). *The fractal introduced by Menger (Menger's sponge) is universal for the class of one-dimensional compact metric spaces.*



Menger also formulated a statement for the classical n -dimensional case: any compact metric space of dimension less than or equal to n is homeomorphic to a subspace of I^{2n+1} —the topological product of $2n + 1$ copies of the unit interval I . And then Menger's student G. Nöbeling [1931] proved the Classical Theorem.

Theorem 4 (Classical Theorem 1931). *The set of points in I^{2n+1} , at most n of whose coordinates are rational, is universal for the class of n -dimensional separable metric spaces.*

For $n = 0$ the Classical Theorem yields another dimension-zero universal space.

Theorem 5 (Dimension zero). *The subspace of irrational points in the unit interval is universal for the class of zero-dimensional separable metric spaces.*

It also follows from the Classical Theorem that the class of finite-dimensional separable-metric spaces may be viewed as the class of subsets of finite-dimensional Euclidean spaces. And viewing each I^{2n+1} as a subspace of the countably infinite product I^∞ , Nöbeling's classical 1931 theorem dovetails nicely with Urysohn's universal space theorem (Urysohn [1925]).

Theorem 6 (Urysohn 1925). *The countably infinite product I^∞ is universal for the class of separable metric spaces.*

Dimension Theory (1940s–1960s)

Following its emergence during the early 1900s, topological dimension theory evolved into an elegant body of mathematics within the context of separable (weight $\leq \aleph_0$) metric spaces. By the 1940s, when this now classical theory was well established, an extension to more general spaces seemed improbable (Hurewicz and Wallman 1941). Nevertheless, by the mid-1960s a surprisingly new and natural theory for general (weight $\geq \aleph_0$) metric spaces was rapidly maturing.

The extension of the classical theory was initiated by Stone [1948], who recognized a symbiosis between open coverings and metric spaces. This symbiosis was further developed (in the context of general topology) by Bing [1951], Nagata [1950], and Smirnov [1951] in their metrization theorems. And on that foundation, Katětov [1952] and Morita [1954] created a significant and elegant dimension theory for general metric spaces (see Nagata [1967]).

Modern Period Star Spaces

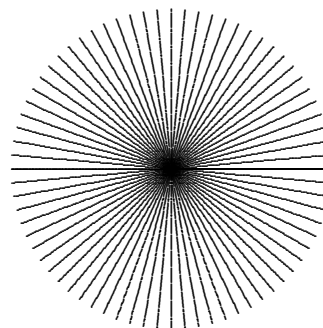
As detailed in the previous sections, the unit interval as a base space dominated universal space theorems in dimension theory up through 1931 when a subspace of I^{2n+1} was shown to be universal for the class of n -dimensional separable-metric spaces.

For n -dimensional weight $|A| \geq \aleph_0$ metric spaces, the unit interval continued to be central through the 1960s when it was used to construct the star space $S(A)$, which is known in the literature as a *hedgehog* with $|A|$ *prickles*, each prickle being a copy of the unit interval.

More precisely, a (Kowalsky) *star space* is a metric space $S(A) = (\bigcup_a I_a, d_S)$ where the set $\bigcup_a I_a$ is the *star-shaped set* obtained by identifying the zeros of a disjoint union of $|A| \geq \aleph_0$ unit intervals I_a (the a th arm), and the metric d_S is given by

$$d_S(x, y) = \begin{cases} |x - y| & x \text{ and } y \text{ in same arm,} \\ |x + y| & x \text{ and } y \text{ in distinct arms.} \end{cases}$$

A star space with a finite number 72 of prickles (or arms) is illustrated above right.



Historically, the one-dimensional star spaces appeared as the base space in the countably infinite product space $S(A)^\infty$, which is *infinite dimensional*.

Theorem 7 (Kowalsky 1957). *A topological space X is metrizable if and only if it can be imbedded in a countable product $P = S(A)^\infty$ of star spaces for some infinite set A .*

Theorem 8 (Nagata 1963). *A metric space X has (covering) dimension $\leq n$, if and only if it can be imbedded in the subset K_n of a countable product $P = S(A)^\infty$ of star spaces for some infinite set A , where K_n is the set of points in P at most n of whose nonvanishing coordinates are rational.*

By 1966, Nagata [1967], contrasting his universal spaces (subspaces of *infinite-dimensional* spaces) with the classical universal spaces (subspaces of *finite-dimensional* Euclidean cubes), stated:

Comparing the general imbedding theorem with the classical one for separable metric spaces, we notice that $P(A)$ has infinite dimension while every n -dimensional separable metric space is imbedded in the $(2n + 1)$ -dimensional Euclidean cube I^{2n+1} . This leads us to the following problem: Improve the general imbedding theorem finding another universal n -dimensional space instead of $P(A)$.

Nagata's statement drew attention to the fact that Nöbeling's [1931] Classical Imbedding Theorem rests on the *one-dimensional* unit interval I as the *base space* in " I^{2n+1} ". In other words, to construct the desired general imbedding theorem (analogous to Nöbeling's), one needs a *one-dimensional weight* $\alpha \geq \aleph_0$ *metric space* X whose k th product X^k , for some finite k , contains a weight $\alpha \geq \aleph_0$ universal space.

1971, Adjacent Endpoints in $N(A)$

So prior to the 1970s, it was Nagata's research and quotation above that served as motivation for seeking an analogue of the unit interval.

Also prior to the 1970s, there were *four* well-known results that indicated how to construct such an analogue:

- (a) Cantor's construction $C \rightarrow I$ of I
- (b) C is a topological copy of $N(\{0, 2\})$
- (c) $N(A)$ is a generalization of $N(\{0, 2\})$
- (d) Morita's Theorem (see below where "dim" denotes *the covering dimension*).

Theorem 9 (Morita 1955). *Let X be a metric space. Then $\dim X \leq n$ if and only if there exists a subspace S of $N(A)$ for suitable A and a closed continuous surjection $f : S \rightarrow X$ such that each fiber $f^{-1}(x)$ contains at most $n + 1$ points.*

For an example of how Morita's Theorem relates to statements (a), (b), and (c), let $A = \{0, 2\}$, $S = N(A)$, and $f : N(A) \rightarrow I$ classical adjacent-endpoint identification. Then since all fibers $f^{-1}(x)$ of f have size less than or equal to two, the dimension of the unit interval I is less than or equal to one.

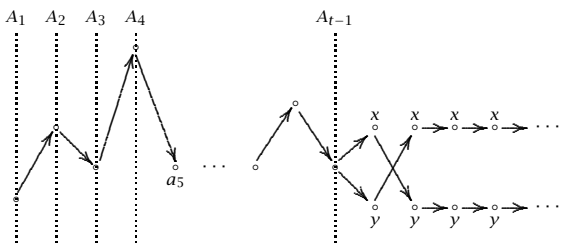
Moreover, straightforward arguments using Morita's Theorem yield the fact that $N(A)$ itself is a universal space.

Theorem 10 (Dimension zero). *For each infinite set A , the Baire Space $N(A)$ is universal for the class of zero-dimensional weight $|A|$ metric spaces.*

So comparing Theorem 10 with Theorem 1, we see that the role of Baire spaces $N(A)$ in the modern period parallels the role of Cantor's set in the classical period. From this observation and (a), (b), and (c) it seemed natural to this author to introduce *modern adjacent-endpoint identification*.

The extension was accomplished as follows: A point $a = a_1 a_2 \dots$ in $N(A)$ is an *endpoint of $N(A)$* if there exists an index k such that $a_k = a_{k+1} = \dots$. Distinct endpoints $a \neq b$ are *adjacent endpoints* when there exists $x \neq y$ in A such that $a = a_1 a_2 \dots a_{t-1} x y y y \dots$ and $b = a_1 a_2 \dots a_{t-1} y x x x \dots$.

The concept of *adjacent endpoints in $N(A)$* is graphically illustrated below where each $A_i = A$.



1971, $J(A)$ Quotient Space²

With adjacent endpoints in $N(A)$ well defined, the next obvious step was to define J_A as the *identify adjacent endpoints in $N(A)$* quotient space. So the natural mapping $p : N(A) \rightarrow J_A$ for the modern period is an extension of the natural mapping $C \rightarrow I$ of the classical period.

Moreover, key topological properties of $C \rightarrow I$ extend to $N(A) \rightarrow J_A$: both mappings are closed and continuous surjections with fibers of size less than or equal to two, and in both cases Morita's Theorem applies, showing that both I and J_A are one-dimensional.

As a bonus, by defining a point $x \in J_A$ as *rational* whenever $p^{-1}(x)$ has size two, we see that J_A partitions into *rationals* and *irrationals* as counterparts to those in the unit interval. In fact, in the J_2 case there is a homeomorphism $J_2 \rightarrow I$ that maps the J_2 -rationals onto the rational reals contained in the open interval $(0, 1)$.

1970s, J_A Universal Space Theorems

By 1975 the space J_A played a role in the modern period that paralleled the role played by the unit interval I in the classical period (see Lipscomb [1975]).

Theorem 11 (Modern Theorem 1975). *Let A be an infinite set and let $n \geq 0$. Then the set of points in J_A^{n+1} at most n of whose coordinates are rational is universal for the class of n -dimensional weight $|A|$ metric spaces.*

The index " $n + 1$ " in the Modern Theorem is the best possible because Borsuk [1975], using homology, proved that the 2-sphere S^2 cannot be imbedded in the product of two one-dimensional spaces.

For $n = 0$, Theorem 11 yields the following result, which should be compared to Theorem 5.

Theorem 12 (Dimension zero). *Let A be an infinite set. Then the subspace of irrational points in J_A is universal for the class of zero-dimensional weight $|A|$ metric spaces.*

And for the analogue of Urysohn's [1925] classical universal space theorem (Theorem 6 above), we provide its modern counterpart.

Theorem 13 (Lipscomb 1976). *Let A be an infinite set. Then the countably infinite product J_A^∞ is universal for the class of weight $|A|$ metric spaces.*

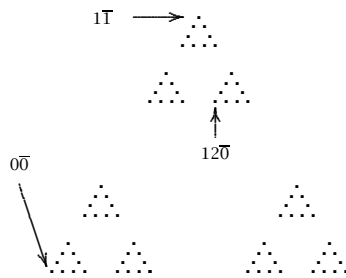
²Because J_A was conceived as a generalization of the unit interval I , it seemed natural to select a notation that serves as a mnemonic of the extension—select the letter that follows the letter I , namely the letter J .

1975–1990s, J_{n+1} and Fractals

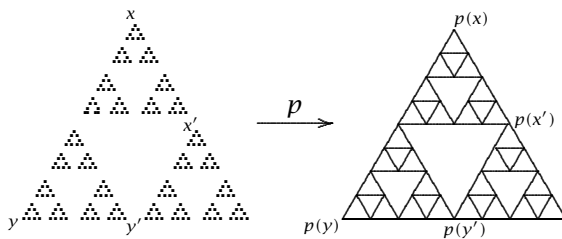
The term *fractal* was coined by Mandelbrot [1975] in the same year that the J_A universal space theorem (Theorem 11) was introduced (Lipscomb 1975).

Nevertheless, prior to 1975 pictures of $J_A = J_{n+1}$ for finite sets A of size $n + 1 = 3$ and $n + 1 = 4$ were obtained; i.e., graphical illustrations of approximations to J_3 and J_4 were obtained by simply thinking about the quotient mapping $N(A) \rightarrow J_A$.

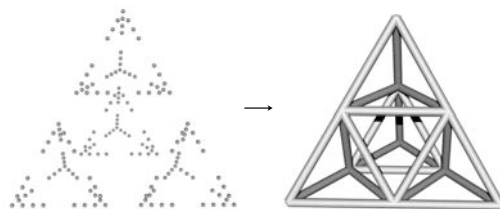
Indeed, for $A = \{0, 1, 2\}$ each finite list $a_1, \dots, a_k \in A$ and doubleton subset D of A define a Cantor subspace $\{a_1\} \times \{a_2\} \times \dots \times \{a_k\} \times D \times D \times \dots$ of $N(A)$. In the graphic below, lists “ a_1, a_2 ” of length two faithfully index the nine small triangles $T(a_1, a_2)$ whose three “edges”—each a linear arrangement of four dots—are faithfully indexed by the doubleton sets D . So, e.g., the list “1,1” indexes the small triangle T_{11} at the top, while the left side, right side, and bottom edges of T_{11} correspond to $\{1\} \times \{1\} \times D \times D \times \dots$, where D is $\{0, 1\}$, $\{1, 2\}$, and $\{0, 2\}$, respectively.



With due diligence, using the fact that each such edge (Cantor subset) maps onto a copy of the unit interval, one finds that J_3 is indeed a topological copy of Sierpiński’s triangle.

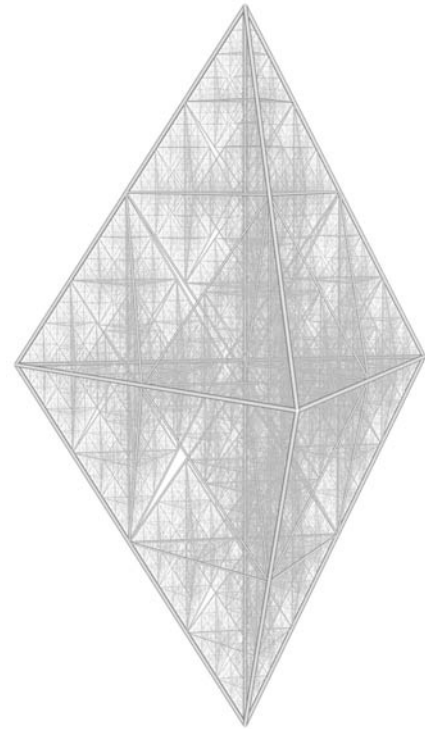


For the Baire space $N(\{0, 1, 2, 3\})$, as illustrated below, one may similarly deduce that J_4 is a topological copy of the *Sierpiński cheese* (the 3-space analogue of Sierpiński’s triangle).



By the mid-1990s the mathematics of viewing fractals as attractors of *finite* iterated function systems (IFSs) became rather well known—the ideas were introduced in 1981 (Hutchinson 1981) and then popularized in the late 1980s and early 1990s following the publication of Barnsley’s 1988 text.

Against this backdrop, it is not surprising that it was during the 1990s that J_{n+1} emerged as the attractor ω^n , called the “ n -web”, of an IFS \mathcal{F}_n whose $n + 1$ members are contractions by one-half toward the $n + 1$ vertices of an n -simplex.



Chris Dupilka’s Level-6 representation of J_5 .

For example, as illustrated in the previous section, J_3 and J_4 are, respectively, copies of fractals known as the 2-web (Sierpiński’s triangle) and the 3-web (Sierpiński’s cheese).

Turning to J_5 , we also seek a picture—we desire to move the 4-web ω^4 from 4-space into 3-space. Any such isotopy should also preserve fractal dimension. The solution was obtained when Perry and Lipscomb [2003] constructed an isotopy that moves ω^4 from 4-space into 3-space with its fractal dimension preserved. (See Dupilka’s graphic of the 4-web $\omega^4 =_t J_5$ above.)

The existence or nonexistence of such isotopies for ω^5 in 5-space, ω^6 in 6-space, and ω^7 in 7-space are open problems.

2002, J_3 and Separable Metric Spaces

A quick review of the classical period shows that no universal space was derived from Sierpiński’s triangle J_3 .

This oversight changed in 2002. By modifying the decomposition approach used to prove the Modern Theorem, Ivanšić and Milutinović [2002] introduced the following theorem.

Theorem 14 (Ivanšić and Milutinović 2002). *The set of points in J_3^{n+1} at most n of whose coordinates are rational is universal for the class of n -dimensional separable metric spaces.*

1990s–2008, J_A as a Generalized Fractal

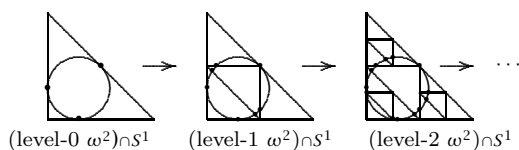
It turned out, as far as this author knows, that J_A for an infinite A may be counted among the first examples of a *generalized fractal*—an attractor of an *infinite* iterated function system.

The seeds of such a claim were planted in the early 1990s: For infinite A , Lipscomb and Perry [1992], and independently Milutinović [1992], produced imbeddings of J_A into Hilbert’s $l^2(A)$ space. Each imbedding involved an *infinite* IFS. In 1992, however, the IFS theory was limited to IFSs that were *finite*. In 1996, by modifying the topology of J_A , Perry [1996] constructed a subspace ω_c^A of the Tychonoff cube I^A that is *an attractor of an infinite* IFS. Perry also called attention to the open problem of showing that $\omega^A \subset l^2(A)$, a copy of J_A , is the attractor of an infinite IFS (of affine transformations of $l^2(A)$).

The open problem posed by Perry was solved by Miculescu and Mihail [2008]. They provided the mathematical context with an appropriate Hutchinson operator that had ω^A as its fixed point; i.e., ω^A is indeed the attractor of an infinite IFS.

An Application of J_5 A 3-Sphere Meets a 4-Web

Consider the leftmost graphic “(level-0 ω^2) $\cap S^1$ ” in the illustration below.



Simply put, the illustration shows how one may use the 2-web ω^2 to approximate a circle, which represents a 1-sphere S^1 .

The figure below similarly involves a 3-sphere S^3 and a 4-web ω^4 . The 4-web has five vertices consisting of the origin and the four standard orthonormal basis vectors in 4-space. The 3-sphere S^3 is also inside of 4-space and is represented as the solution to $\sum_1^4 (x_i - .25)^2 = .25^2$.



A 3-Sphere meets a 4-web.

In the 4-space case, however, the points of intersection are calculated inside of 4-space, and then faithfully moved into 3-space using the Perry and Lipscomb [2003] isotopy that “moves” the points of intersection into 3-space—the isotopy $H : \omega^4 \times I \rightarrow \mathbb{R}^4 \text{ rel } \omega^3$ is a homotopy such that, for $0 \leq t \leq 1$, each H_t is a homeomorphism that is the identity on $\omega^3 \subset \omega^4$, and each H_t is a linear transformation that preserves fractal dimension. In addition, $H_1 : \omega^4 \rightarrow \mathbb{R}^3 \subset \mathbb{R}^4$. In other words, the intersection (Level-7 ω^4) $\cap S^3$ in 4-space is faithfully moved into 3-space.

Comments

For more information see the author’s 2009 (Springer’s Monographs in Mathematics series) book *Fractals and Universal Spaces in Dimension Theory*. And for his excellent graphics—the J_5 graphic and the PovRay file that generated the 3-space view of (Level-7 ω^4) $\cap S^3$, I wish to acknowledge and thank Chris Dupilka.

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