Book Review

The Millenium Prize Problems

Reviewed by Robert C. Gunning

Problems are the lifeblood of mathematics. A good deal of time and effort is spent on solving problems while learning almost any part of mathematics, from addition in grade school through calculus and differential equations in college and on to the wide range of graduate courses introducing prospective mathematicians to current work. One of the hardest parts of supervising doctoral theses is finding or, even better, leading students to find on their own reasonable problems to attack: problems that are not too easy, so lead to nothing interesting, nor too difficult, so lead to nothing at all; problems that require the right amount of background, not too little so as to encourage triviality, nor too much so that there is no time and energy left to tackle the problem itself; problems that lie in reasonable areas, not those areas of no current interest, so that the solutions might be a waste of time, nor those areas that are so overwhelmed by current attention that it is difficult to discover what is already known in the time available. That the final reports of mathematical meetings and conferences frequently include lists of problems related to the topic of the meeting is quite common, and is greatly appreciated.

A number of mathematicians have proposed on their own lists of problems in the areas of mathematics in which they were particularly interested; and some, such as Paul Erdős for instance, have even proposed prizes for their solutions. From time to time particularly daring and thoughtful mathematicians even have proposed lists of problems of interest throughout all of mathematics.[1] The great example, of course, is the list of twenty-three problems that David Hilbert proposed in his speech at the International Congress of Mathematicians held in Paris in 1900 and in somewhat extended forms in some subsequent publications.[2] The extent to which that list of problems is known and has influenced the direction of mathematical research is truly amazing. In part that is no doubt due to the stature of the proposer, one of the giants of mathematics of that or any period; but it is also due to the proposer's wide knowledge of mathematics and insight into the importance both of problems in general and of the particular problems that he posed, an insight substantiated by the amount of effort that has been put into solving these problems since they were first proposed. The Symposium in Mathematics devoted to further problems inspired by Hilbert's list and sponsored by the American Mathematical Society in 1974 attests to the interest of the original list three-quarters of a century later. The *Proceedings* of that symposium included a range of problems in twenty-seven different areas of mathematics, proposed by a number of mathematicians and edited by Jean Dieudonné, in homage to the Hilbert list and in an effort to compose another list of important and interesting problems that might serve to guide mathematical research for another three-quarters of a century. Like most of the other general lists of problems, however, this list has

Robert C. Gunning is professor of mathematics at Princeton University. His email address is gunning@math.princeton.edu.

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[1]*A few examples are S. Ulam, "A collection of mathematical problems",* Interscience, *1960; V. Arnold, M. Atiyah, P. Lax, and B. Mazur, editors,* Mathematical Frontiers and Perspectives, *IMU/AMS 2000.*

[2]Göttinger Nachrichten, *1900, pp. 253-297;* Archiv der Mathematik und Physik, *vol. 1 (1901), pp. 44-63 and 213-37.*

vanished into the depths of libraries across the world with very few ripples to mark its passing.

The incredible growth of mathematics since the beginning of the twentieth century and the astonishingly wide range of current mathematical research really preclude anything as comprehensive or brash as Hilbert's list today. The problem list in the volume under review here is of a somewhat different nature, not so much a prevision or prescription for mathematics of the future as a collection of some well-known problems that have been circulating throughout the world of mathematics for a number of years, indeed from later in the nineteenth century through the middle of the twentieth century, that have been the focus of a great deal of mathematical research for a long time, and that doubtless will continue to attract the efforts of numerous mathematicians to tackle them without any further inducements. However, this list stands a much better chance of having a lasting existence than almost any other post-Hilbert list, not just from the quality of the problems selected, but also from the liberality of the awards for their solutions. The idea of prizes for solutions of mathematical problems is an old one; an excellent history by Jeremy Gray of the role of prizes in mathematics, from ancient Greece through the Fields Prize in the middle of the twentieth century, is included in the book and is well worth reading on its own. Landon T. Clay, through the Clay Mathematical Institute that he and his wife, Lavinia D. Clay, founded and endowed, has been an outstanding modern patron of mathematical research; sponsoring this volume of problems and the prizes for their solutions is one of his many contributions to the development and flourishing of mathematics, beyond the research conferences, research awards, research fellowships, senior scholarships, liftoff fellowships, and other mathematical activities that the Clay Foundation supports, with the advice of a distinguished and remarkably sensible scientific advisory board. This list, with the excitement and publicity aroused by the munificence of the prizes, may have been intended to serve a purpose other than those of most other lists: that of increasing the awareness of and interest in mathematical research among nonmathematicians and of inspiring budding prospective mathematicians and scientists to pursue challenging careers involving mathematics and its applications.

The seven prize problem areas (the Birch and Swinnerton-Dyer conjecture, the Hodge conjecture, the existence and smoothness of solutions of the Navier-Stokes equation, the Poincaré conjecture, the P versus NP problem, the Riemann hypothesis, quantum Yang-Mills theory) that the scientific advisory board selected are sufficiently well known that it is not necessary to include any more detailed discussion in this review. The individual articles briefly describing the background and significance of the problems are excellent and written by current major figures in mathematics. The photographs of the authors of the individual essays, and of other mathematicians who have worked on these problems and closely related areas, are a part of a very handsome publication that should also inspire many nonprofessional mathematicians to have a look, and perhaps be inspired to take modern mathematical research more seriously.