An Invitation to Cauchy-Riemann and Sub-Riemannian Geometries

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A certain amount of complex analysis, in both one and several variables, combines with some differential geometry to form the basis of research in diverse parts of mathematics. We first describe that foundation and then invite the reader to follow specific geometric paths based upon it. We discuss such topics as the failure of the Riemann mapping theorem in several variables, the geometry of real hypersurfaces in complex Euclidean space, the Heisenberg group, sub-Riemannian manifolds and their metric space structure, the Hopf fibration, sub-Riemannian geodesics on the three-sphere, CR mappings invariant under a finite group, and finite type conditions. All of these topics fall within the branch of mathematics described by the title, but of course they form only a small part of it. We hope that the chosen topics are representative and appealing. The connections among them provide a fertile ground for future research. We modestly hope that this article inspires others to develop these connections further. To this end, our reference list includes a diverse collection of books and accessible articles.

Given Riemann's contributions to geometry, it is not surprising that his name occurs twice in the title. It is more surprising that the uses of his name here come from different parts of mathematics. We find it delightful that contemporary mathematics is forging connections that even Riemann could not have anticipated.

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Figure 1. Georg Friedrich Bernhard Riemann.

There Is No Riemann Mapping Theorem in Higher Dimensions

In an attempt to understand complex analysis in several variables, especially questions revolving around the failure of the Riemann mapping theorem, many mathematicians have focused attention on geometric properties of the boundaries of domains and how those geometric properties influence the complex analysis on the domain. Such investigations have led to the fields of CR geometry and sub-Riemannian geometry.

We begin with the Riemann mapping theorem. Let Ω be an open, connected, and simply connected proper subset of $\mathbb{C}$. Then there is a bijective holomorphic mapping f from Ω to the unit disc $\mathbb{D}$. The inverse mapping is also holomorphic. We say that Ω and $\mathbb{D}$ are *biholomorphically equivalent*. The entire plane must of course be excluded, because a bounded holomorphic function on $\mathbb{C}$ reduces to a constant. This theorem has applications throughout both pure and applied mathematics. In many of these applications, such as uniform fluid flow, the upper half plane substitutes for the unit disc. The explicit linear fractional transformation f (the *Cayley transformation*) defined by

$$z = f(w) = \frac{i - w}{i + w} \tag{1}$$

maps the upper half plane biholomorphically to the unit disc.

We mention in passing one of the most fundamental geometric generalizations of the Riemann mapping theorem, although we will not head in that direction. The *uniformization theorem* states that the universal cover of a compact Riemann surface must be one of three objects: the unit disc $\mathbb{D}$, the complex plane $\mathbb{C}$, or the Riemann sphere $\overline{\mathbb{C}}$. As a reference for analysis in one complex variable, we need only mention the classical text by Ahlfors [Ahl].

Let us now consider complex Euclidean space $\mathbb{C}^n$ of arbitrary dimension. We equip $\mathbb{C}^n$ with the usual Hermitian inner product given by $\langle z, w\rangle = \sum z_j \overline{w}_j$ and the corresponding norm given by $|z| = \langle z, z\rangle^{1/2}$. The unitary group $\mathcal{U}(n)$ consists of linear mappings preserving the inner product. The unit ball B_n is the set of z for which $|z| < 1$.

In dimension at least two the unit ball is not biholomorphically equivalent to a half space. There are many possible proofs. One approach takes into account the geometry of the boundaries. When the boundaries are smooth manifolds (as they are in this case), the *Levi form* plays a crucial role. The Levi form for a real hypersurface in $\mathbb{C}^n$ (for $n \geq 2$) is the complex variable analogue of the second fundamental form for a real hypersurface in $\mathbb{R}^n$. We define and discuss the Levi form on page 211.

For now we simply note that the Levi form is a Hermitian form defined on part of the tangent spaces of the boundary. There is no Levi form in one dimension, because the tangent space is one real dimension and the part on which the Levi form acts does not exist. For a half plane in dimension at least 2 the Levi form is identically zero. For the unit sphere it is positive definite. As a result the boundary geometries are so different that no biholomorphic mapping can exist. By contrast, in one dimension, different boundary curves are indistinguishable from this point of view.

A polydisc is a Cartesian product of discs. For example, the subset of $\mathbb{C}^2$ defined by $|z_1| < 1$ and $|z_2| < 1$ is a polydisc. It is also not biholomorphically equivalent to B_2. In fact, for $n \geq 2$ there is no biholomorphic map from any polydisc to any ball. One classical proof of this statement involves showing that the groups of biholomorphic automorphisms of the ball and of the polydisc have different dimensions.

A second proof (see [D1, page 14]) proceeds in the following manner. First consider a biholomorphic (or even proper) image Ω of the Cartesian product of bounded domains; one shows that $b\Omega$ must contain complex analytic sets. Then an easy computation, given in Lemma 6, shows that the definiteness of the Levi form precludes the existence of positive-dimensional complex analytic subsets of $b\Omega$. This computation helps unify several of the ideas in this article. For example, higher order commutators of complex vector fields can determine obstructions to the existence of complex analytic sets in a real hypersurface. Such obstructions lead to geometric finite type conditions for subelliptic estimates. In two complex dimensions a subelliptic estimate (see page 217) holds if and only if the bracket-generating property (see page 212) holds.

Consider a domain Ω with smooth boundary $b\Omega$. In order that geometric results about $b\Omega$ be meaningful for complex analysis on Ω, one requires results concerning the boundary smoothness of biholomorphic maps. Two early references in this area are [Bel] and [Fef]; we do not attempt to list any of the many newer references. In general, smoothly bounded domains, topologically equivalent to the unit ball B_n, are seldom biholomorphically equivalent to it. On the other hand, Fridman [Fri] has established a wonderful approximate Riemann mapping theorem: given any topological ball and a positive ϵ, there is a biholomorphic mapping to a domain whose boundary is always within distance ϵ of the unit sphere.

The Unit Sphere and the Heisenberg Group

The quintessential principle connecting CR geometry and sub-Riemannian geometry arises from the correspondence between the unit sphere bB_n and the Heisenberg group, which bounds the unbounded Siegel upper half space H_n. We will obtain a useful identification of bH_n with $\mathbb{C}^{n-1} \times \mathbb{R}$, on which we will impose a nonabelian group law arising from the biholomorphic automorphisms of H_n. The equivalence between H_n and B_n generalizes the one-dimensional Cayley transformation.

The Siegel upper half space H_n is the set of w in $\mathbb{C}^n$ such that

$$\operatorname{Im}(w_n) > \sum_{j=1}^{n-1} |w_j|^2. \tag{2}$$

The following elementary but useful identity is the key to discovering the biholomorphic map:

$$\operatorname{Im}(\zeta) = \left|\frac{i + \zeta}{2}\right|^2 - \left|\frac{i - \zeta}{2}\right|^2. \tag{3}$$

With $\zeta = w_n$ we plug (3) into (2) and rewrite the result to obtain

$$(4) \qquad \left|\frac{i+w_n}{2}\right|^2 > \sum_{j=1}^{n-1} |w_j|^2 + \left|\frac{i-w_n}{2}\right|^2 .$$

After dividing by $|\frac{i+w_n}{2}|^2$ and changing notation, inequality (4) becomes

$$1 > \sum_{j=1}^{n} |z_j|^2.$$

Here $z_j = \frac{2w_j}{i+w_n}$ for $j < n$ and $z_n = \frac{i-w_n}{i+w_n}$. The reader should check that this transformation $f : w \mapsto z$ is biholomorphic from H_n to B_n and also compare it with the transformation in (1).

The automorphism group of H_n

We alluded earlier to the biholomorphic automorphism groups of the ball and the polydisc. For any complex manifold the automorphism group is a fundamental algebraic invariant. For example, the automorphism group of the upper half plane in $\mathbb{C}$ is $\mathrm{PSL}(2,\mathbb{R})$. Its elements act on the (one-point compactification of the) boundary of the upper half plane as linear fractional transformations $x \mapsto \frac{ax+b}{cx+d}$, for $ad - bc = 1$. Any such transformation can be represented as the composition of dilations $x \mapsto rx$, translations $x \mapsto x + x_0$, and possibly the inversion $x \mapsto \frac{1}{x}$. Automorphisms fixing the point at infinity do not require the inversion.

We now describe the analogous situation on bH_n, defined by equality in (2). The last variable in (2) plays a different role, and hence for $w \in \mathbb{C}^n$ we write $w = (w', w_n)$. Two natural families of biholomorphic self-maps of H_n are the *dilations* $\delta_r : H_n \to H_n$, for $r > 0$, given by $\delta_r(w', w_n) = (rw', r^2 w_n)$, and the *rotations* $R_A : H_n \to H_n$, for $A \in \mathcal{U}(n-1)$, given by $R_A(w', w_n) = (A(w'), w_n)$. To introduce an analogue of translation we consider the biholomorphisms $\tau_p : H_n \to H_n$, for $p = (\zeta, t) \in \mathbb{C}^{n-1} \times \mathbb{R}$, given by

$$(5) \quad \tau_p(w', w_n) = (w' + \zeta, w_n + t + 2i\langle w', \zeta\rangle + i|\zeta|^2).$$

All of the preceding maps extend to self-maps of $bH_n = \{(w', w_n) : w_n = |w'|^2\}$. The action of the family $\{\tau_p : p = (\zeta, t) \in \mathbb{C}^{n-1} \times \mathbb{R}\}$ on $H_n \cup bH_n$ is faithful and the action on bH_n is simply transitive. By this method we equip $\mathbb{C}^{n-1} \times \mathbb{R}$ with a group law $(p, q) \mapsto p \cdot q$, characterized by the identity $\tau_p \circ \tau_q = \tau_{p \cdot q}$. The resulting space is the *Heisenberg group*. Often in the literature $\mathbb{C}^{n-1}$ is replaced by $\mathbb{R}^{2n-2}$ and the group law is expressed using real variables. Finally we observe that the group of biholomorphic automorphisms of H_n which fix the point at infinity is generated by dilations, rotations, and translations. See Stein's textbook in harmonic analysis [Ste, Chapters XII and XIII] for the complete story.

CR Structures

We next discuss the geometry of a general real hypersurface M in $\mathbb{C}^n$. The beautiful interplay between real and complex geometry dominates the discussion. See [BER], [D1], [DT], [Jac], [Tre] and their references for more about CR geometry.

We wish to consider complex vector fields and therefore start by considering the complexified tangent bundle

$$(6) \qquad \mathbb{C}T(\mathbb{C}^n) = T(\mathbb{C}^n) \otimes \mathbb{C}.$$

As usual in complex analysis we define the complex partial derivative operators

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)$$

and

$$\frac{\partial}{\partial \overline{z_j}} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

Using these operators we decompose the exterior derivative d as

$$(7) \quad df = \partial f + \overline{\partial} f = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_j + \sum_{k=1}^{n} \frac{\partial f}{\partial \overline{z_k}} d\overline{z_k}.$$

Since $d^2 = 0$, it follows that $\partial^2 = \overline{\partial}^2 = \partial\overline{\partial} + \overline{\partial}\partial = 0$.

A section of the bundle (6) is a complex vector field. Each such vector field is a combination of both z and $\overline{z}$ derivatives with smooth coefficients:

$$L = \sum_{j=1}^{n} a_j(z)\frac{\partial}{\partial z_j} + \sum_{j=1}^{n} b_j(z)\frac{\partial}{\partial \overline{z}_j}.$$

We obtain two naturally defined integrable subbundles of $\mathbb{C}T(\mathbb{C}^n)$. Let $T^{10}(\mathbb{C}^n)$ denote the bundle whose sections are vector fields L of the form

$$(8) \qquad L = \sum_{j=1}^{n} a_j(z)\frac{\partial}{\partial z_j},$$

where the a_j are smooth complex-valued functions. We note that this bundle is integrable in the sense of Frobenius: if K, L are sections of $T^{10}(\mathbb{C}^n)$, then so is their commutator, or Lie bracket, given by $[K, L] = KL - LK$. The complex conjugate bundle, denoted $T^{01}(\mathbb{C}^n)$, is also integrable. Since the zero section is the only section of both bundles, we have $T^{10}(\mathbb{C}^n) \cap T^{01}(\mathbb{C}^n) = 0$. We obtain a splitting

$$\mathbb{C}T(\mathbb{C}^n) = T^{10}(\mathbb{C}^n) \oplus T^{01}(\mathbb{C}^n).$$

This splitting of the tangent bundle plays a crucial role in all aspects of complex geometry. CR geometry studies the extent to which this splitting holds on real manifolds.

Let M be a smooth real hypersurface of $\mathbb{C}^n$, or more generally, of a complex manifold. The tangent spaces of M then inherit some of the ambient complex structure. We write $\mathbb{C}TM$ for $T(M) \otimes \mathbb{C}$. We define $T^{10}(M) = T^{10}(\mathbb{C}^n) \cap \mathbb{C}TM$, and denote its complex conjugate bundle by $T^{01}(M)$. Again we have

$$(9) \qquad T^{10}(M) \cap T^{01}(M) = 0.$$

Let M be an abstract real manifold M with complexified tangent bundle $\mathbb{C}TM$. We say that a subbundle $T^{10}(M)$ of $\mathbb{C}TM$ defines a *CR structure* on M if it is integrable and its intersection with its conjugate bundle satisfies (9). We then call M a *CR manifold*. Its *horizontal bundle* is the direct sum

$$\mathcal{H}(M) = T^{10}(M) \oplus T^{01}(M). \tag{10}$$

We say that M is of *hypersurface type* if the fibers of $\mathcal{H}(M)$ have codimension one in $\mathbb{C}TM$.

In general, the *CR codimension* of M is the codimension of $\mathcal{H}(M)$ in $\mathbb{C}TM$. Real submanifolds of $\mathbb{C}^n$ of arbitrary dimension define CR manifolds of various CR codimensions. Consider a k-dimensional real subspace of $\mathbb{C}^n$. As a real submanifold it has codimension $2n - k$. On the other hand, its CR codimension can be any integer ℓ between 0 and k such that $k - \ell$ is even. The subspace has maximal CR codimension (equal to k) if it is totally real, and it has minimal CR codimension (equal to zero) if it is complex. Any complex manifold can be regarded as a CR manifold of CR codimension zero. Such manifolds are precisely the integrable almost complex manifolds.

We focus on CR manifolds of hypersurface type. On such manifolds there is a nonvanishing differential one-form η, defined up to a multiple, annihilating $\mathcal{H}(M)$. We may assume that η is purely imaginary. Each of the summands in (9) is integrable, but something new happens; their sum is not integrable. The Levi form measures the failure of integrability of the sum and leads to both CR geometry and sub-Riemannian geometry.

Definition 1. Let M be a CR manifold of hypersurface type. The Levi form λ is the Hermitian form on $T^{10}(M)$ defined by

$$\lambda(L, \overline{K}) = \langle \eta, [L, \overline{K}] \rangle.$$

Any vector field T for which $\langle \eta, T \rangle \neq 0$ has a component in the *missing direction*. The Levi form $\lambda(L, \overline{L}\rangle$ gives the component of $[L, \overline{L}]$ in the missing direction. In the following definition we identify λ with a Hermitian linear transformation of $T^{10}(M)$.

Definition 2. A CR manifold of hypersurface type is *pseudoconvex* if all nonzero eigenvalues of λ have the same sign. It is called *strongly pseudoconvex* if λ is definite, that is, all eigenvalues have the same nonzero sign.

Observe that there is an ambiguity of sign in the definition of the Levi form. Even for a real hypersurface in real Euclidean space the sign of the second fundamental form is defined only modulo the choice of normal. For a compact hypersurface in $\mathbb{R}^n$ consider a point p farthest from the origin. Since the sphere centered at the origin osculates M to order two at p, the second fundamental form at p agrees with that of the sphere, thus determining its sign. The story is the same for real hypersurfaces in $\mathbb{C}^n$ and the Levi form. On the other hand, there is no natural way to resolve the ambiguity of sign for an abstract CR manifold.

Next we express the Levi form on a hypersurface in terms of partial derivatives. In a neighborhood of a given point we suppose that M is the zero set of r, where r is smooth and $dr \neq 0$ where $r = 0$. A complex vector field L is tangent to M if and only if $L(r) = \langle dr, L \rangle = 0$ on M. For the one-form η we may use $\frac{1}{2}(\partial - \overline{\partial})(r)$. In this case $d\eta = -\partial\overline{\partial}r$.

The Cartan formula for the exterior derivative of a 1-form η states that

$$\langle d\eta, A \wedge B \rangle = A\langle \eta, B \rangle - B\langle \eta, A \rangle - \langle \eta, [A, B] \rangle. \tag{11}$$

The commutator term ensures that $d\eta$ is linear over the module of smooth functions. Let L and K be local sections of $T^{10}(M)$. Then $\langle \eta, L \rangle = \langle \eta, \overline{K} \rangle = 0$. By the Cartan formula,

$$\lambda(L, \overline{K}) = \langle \eta, [L, \overline{K}] \rangle = \langle -d\eta, L \wedge \overline{K} \rangle = \langle \partial\overline{\partial}r, L \wedge \overline{K} \rangle. \tag{12}$$

We have interpreted the Levi form as the restriction of the complex Hessian of r to sections of $T^{10}(M)$. It is therefore analogous to the real Hessian, which governs Euclidean convexity. Euclidean convexity implies pseudoconvexity, but the converse fails. See [Hör] and [Kra] for lengthy discussion, especially for the meaning of pseudoconvexity on a domain whose boundary is not smooth.

Example 3. The zero set of $r(z) = \mathrm{Im}(z_n)$ is a half-space Σ. Its horizontal space $\mathcal{H}_z(\Sigma)$ decomposes into holomorphic and conjugate holomorphic subspaces, spanned respectively by the vector fields $L_j = \partial/\partial z_j$, $j = 1, \ldots, n-1$, and their conjugates. In this case, the complex Hessian of the defining function is identically equal to zero, and hence the Levi form also vanishes identically.

Example 4. The zero set of $r(z) = \sum_{j=1}^{n} |z_j|^2 - 1$ is the sphere $\mathbb{S}^{2n-1}$. The horizontal space $\mathcal{H}_z(\mathbb{S}^{2n-1})$ decomposes into the holomorphic subspace $T_z^{10}(\mathbb{S}^{2n-1}) = \mathrm{span}\{L_1, \ldots, L_{n-1}\}$ and its conjugate $T_z^{01}(\mathbb{S}^{2n-1})$, where

$$L_j = \overline{z_j}\frac{\partial}{\partial z_n} - \overline{z_n}\frac{\partial}{\partial z_j}. \tag{13}$$

The annihilating one-form can be taken to be

$$\eta = \frac{1}{2}\sum_{j=1}^{n} \overline{z_j}dz_j - z_j d\overline{z_j}. \tag{14}$$

The Levi form, after dividing out a nonzero factor, satisfies $\lambda(L_j, \overline{L_k}) = \delta_{jk} + \overline{z}_j z_k$. Hence the unit sphere is strongly pseudoconvex. The Heisenberg group is also strongly pseudoconvex. Its Levi form, computed similarly, is $\lambda(L_j, \overline{L_k}) = \delta_{jk}$.

Remark 5 (Hans Lewy equation)**.** In 1957 Hans Lewy produced his famous example of an *unsolvable* first-order linear PDE. His equation is $\overline{L}u = f$, where L is the type $(1,0)$ vector field tangent to $\mathbb{S}^3$ given in (13). See [Tre] for an elegant and readable account of developments in PDE and CR geometry based on this operator.

Positive definiteness of the second fundamental form for a real hypersurface M in $\mathbb{R}^n$ is a curvature condition: it precludes the existence of straight line segments in M. The next lemma provides a subtle analogue for the Levi form.

Lemma 6. *Let M be a strongly pseudoconvex real hypersurface in $\mathbb{C}^n$. Then M contains no complex analytic sets of positive dimension.*

Proof. We prove the contrapositive statement: if M contains such sets, then M cannot be strongly pseudoconvex. Assume M contains a complex analytic set V of positive dimension. Let p be a nonsingular point of V. We can then find a one-dimensional nonsingular holomorphic curve $t \to z(t) \in \mathbb{C}^n$ with $z(0) = p$, with tangent vector $z'(0) \neq 0$, and with $z(t) \subset M$ for $|t| < 1$. For each local defining function r for M near p we have $r(z(t)) = 0$ for $|t|$ small.

Taking $\frac{\partial}{\partial t}$ of the identity $r(z(t)) = 0$ and evaluating at $t = 0$ gives

$$\langle \partial r(p), z'(0) \rangle = 0. \tag{15}$$

Taking the Laplacian $\frac{\partial^2}{\partial t \partial \bar{t}}$ of the same identity and evaluating at $t = 0$ gives

$$\langle \partial\bar{\partial} r, z'(0) \wedge \overline{z'(0)} \rangle = 0. \tag{16}$$

Equation (15) says that $z'(0) \in T_p^{10}(M)$ and equation (16) says that $z'(0)$ is in the null space of the Levi form. Since $z'(0) \neq 0$, M is not strongly pseudoconvex at p. □

Sub-Riemannian Geometry

It has been evident so far that the stratification of the complexified tangent spaces of a CR manifold given by the horizontal bundle together with the missing direction is at the heart of CR geometry. In this section we will consider real manifolds equipped with a similar horizontal subbundle of the (uncomplexified) tangent bundle. In both cases there are horizontal directions which are infinitesimally accessible and missing directions. When the Levi form is nonzero we can recover the missing direction by commutators. The ability to recover missing directions via commutators of horizontal vector fields is the defining property of a sub-Riemannian manifold. This nonintegrability condition translates to a connectivity condition whereby such manifolds are equipped with a singular metric. See Theorem 9.

CR manifolds provide a natural class of examples, but the general theory covers a much wider class of spaces arising in PDE, control theory, geometric group theory, and many other settings. Example 8 puts a sub-Riemannian structure on spaces of kth order Taylor polynomials (jets). This example formalizes the well-known procedure for reducing a differential equation of high order to a first-order system. The references [Mon], [Str], [Gro], [Bel], [FS], [CCG], [CDPT], and [CC] provide many examples and discussion of other sub-Riemannian manifolds.

Differential geometric aspects

Let M be a smooth real manifold and let $\mathcal{H}(M)$ be a distribution (subbundle) in the tangent bundle $T(M)$. The classical Frobenius theorem deals with the case when $\mathcal{H}(M)$ is integrable, that is, closed under the Lie bracket. In this case M is foliated by $\mathcal{H}(M)$-integral submanifolds. We consider the opposite extreme, when $\mathcal{H}(M)$ is *completely nonintegrable*: the Lie bracket span of $\mathcal{H}(M)$, at any point $p \in M$, coincides with $T_p(M)$. In this case we say that the pair $(M, \mathcal{H}(M))$ satisfies the *bracket-generating property*, also known as *Hörmander's condition*. Put another way, for each $p \in M$ there exists an integer $s = s(p) < \infty$ so that the values at p of all s-fold iterated Lie brackets of vector fields valued in $\mathcal{H}(M)$ fill out the entire tangent space $T_p(M)$. We call $s(p)$ the *step* of M at p. We emphasize that $s(p)$ may depend on the point p. In the case when s is uniformly bounded on M, we call $\sup_{p\in M} s(p)$ the *step* of M.

We elaborate with some examples. If M is a CR manifold of hypersurface type, we define $\mathcal{H}(M)$ as in (10). The missing direction might or might not be obtained via iterated Lie brackets of horizontal vector fields. Example 3 shows that Levi flat hypersurfaces do not satisfy the bracket-generating property. The opposite extreme is the strongly pseudoconvex case. If L is a $(1,0)$ vector field and not zero at p, then the bracket $[L, \overline{L}]$ has a component in the missing direction. In this case the missing direction arises upon taking a single Lie bracket of horizontal vector fields, and the induced sub-Riemannian structure is of step two. To be step two we need only one such vector field; we recover the missing direction at p with a single Lie bracket whenever the Levi form is not zero at p. The simplest example of a higher step structure is the pseudoconvex hypersurface defined by $\mathrm{Re}(z_2) = |z_1|^{2m}$ in $\mathbb{C}^2$ near the origin. One requires an iterated commutator

$$[\ldots[L, \overline{L}], \ldots, \overline{L}]$$

with $2m$ total brackets to obtain the missing direction. In this case the origin is called a point of type $2m$, and one says that the vector field L is of type $2m$ there. Notice in this case that the step at most points is 2. See page 217 for related information.

Example 7. The Heisenberg group bH_n provides the canonical example of a sub-Riemannian manifold. The group law was discussed in the paragraph following (5). As usual, the left invariant vector fields define the Heisenberg Lie algebra. As a basis for the horizontal distribution $\mathcal{H}(bH_n)$ in $T(bH_n)$ we take the vector fields

(17)
$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, j = 1, \dots, n-1,$$

where $w_j = x_j + iy_j$ and $t = \mathrm{Re}(w_n)$. Set $T = \frac{\partial}{\partial t}$. Then $[X_j, Y_k] = -4T\delta_{jk}$ for $j, k = 1, \dots, n-1$. By putting $L_j = \frac{1}{2}(X_j - iY_j)$ and $\overline{L_j} = \frac{1}{2}(X_j + iY_j)$ and regarding them as sections of $\mathbb{C}T(bH_n)$, we return precisely to the CR setting.

Example 8. Jet spaces provide a geometric interpretation for Taylor polynomials, by viewing the equivalence classes of C^m functions modulo m-jets (mth order Taylor approximations) as points in an abstract space. We illustrate in the one-dimensional case. Consider C^m functions $f : \mathbb{R} \to \mathbb{R}$. The underlying space M is $\mathbb{R}^{m+2}$, with coordinates x (corresponding to the base point x_0 for the Taylor approximation of f) and $u_m, \dots, u_0$ (corresponding to $f^{(m)}(x_0), \dots, f(x_0)$). The sub-Riemannian structure is defined by a family of smooth 1-forms $\eta_j = du_j - u_{j+1}\,dx$, $j = 0, \dots, m-1$, corresponding to the differential equalities $df^{(j-1)}(x) = f^{(j)}(x)\,dx$. These forms, together with dx and du_m, frame the cotangent bundle $T^*(M)$. Introduce the dual frame $X, U_m, \dots, U_0$ for $T(M)$. Let $\mathcal{H}(M)$ be spanned by X and U_m. The Cartan formula (11) gives

$$\begin{aligned}\langle \eta_{j-1}, [U_k, X] \rangle &= -\langle d\eta_{j-1}, X \wedge U_k \rangle \\ &= \langle du_j \wedge dx, U_k \wedge X \rangle = \delta_{jk},\end{aligned}$$

which yields the Lie bracket identities $[U_j, X] = U_{j-1}$, $j = 1, \dots, m$. We observe that $\mathcal{H}(M)$ is a bracket-generating horizontal distribution of rank two inducing a sub-Riemannian structure on $M = \mathbb{R}^{m+2}$ of step $m+1$, called the *mth order jet space* (on the real line). The first-order jet space is isomorphic (as a Lie group) to the Heisenberg group bH_2. For additional details see [CDPT, Chapter 3.1] or [Mon, Chapter 6].

The fundamental theorem of sub-Riemannian geometry

Recall that a Riemannian metric on a smooth manifold M is a smoothly varying family of inner products defined on the tangent spaces. We say that a pair $(M, \mathcal{H}(M))$ satisfying the Hörmander condition is a *sub-Riemannian manifold* if it is equipped with a smoothly varying family of inner products $\langle \cdot, \cdot \rangle_p$ (the *sub-Riemannian metric*) defined on the horizontal tangent spaces $H_p(M)$. Such a metric permits us to define notions of length, volume, angle and other geometric concepts for objects taking values in the horizontal subbundle. For instance, a smooth curve $\gamma : (a, b) \to M$ is termed *horizontal* if $\gamma'(t) \in \mathcal{H}_{\gamma(t)}M$ for all t.

As in the Riemannian case, the sub-Riemannian metric induces a distance function $d : M \times M \to [0, \infty)$ (the so-called *Carnot-Carathéodory distance*): for $p, q \in M$, $d(p, q)$ is the infimum of

(18)
$$\int_a^b \langle \gamma'(t), \gamma'(t) \rangle^{1/2}\,dt$$

over all smooth horizontal curves $\gamma : [a, b] \to M$ with $\gamma(a) = p$ and $\gamma(b) = q$. The fundamental result is Theorem 9 below, the Chow–Rashevsky theorem. See the preface in [Mon] for the history leading to this theorem.

Theorem 9. *Assume that Hörmander's condition is satisfied for the pair $(M, \mathcal{H}(M))$. Then all pairs of points in M are horizontally connected. Consequently, the Carnot-Carathéodory distance function on a sub-Riemannian manifold M endows M with the structure of a metric space.*

An *adapted Riemannian metric* on M is an extension of the sub-Riemannian metric to the full tangent bundle. If the step of M is finite, then the topologies defined by the sub-Riemannian metric and any adapted Riemannian metric coincide. However, the geometric and analytic properties of these two metrics differ substantially, unless the step is equal to one. For instance, if M has step at least two, then these two metrics are never bi-Lipschitz equivalent. (They are bi-Hölder equivalent with Hölder exponent $\frac{1}{s}$, where s is the step of M.)

Sub-Riemannian geodesics are smooth horizontal curves γ in M which locally minimize the length $\ell(\gamma)$ defined in (18). The horizontality condition is a family of nonlinear constraints which can be understood in the language of control theory. The path planning problem for wheeled motion (including such concrete subproblems as parallel parking an automobile) can be reinterpreted as the problem of finding geodesics joining prespecified points on certain sub-Riemannian manifolds modeled locally on the first Heisenberg group. Again, we reference [Mon] for more details.

Sub-Riemannian geodesics behave rather differently from their Riemannian counterparts. *Abnormal geodesics* are locally length minimizing curves which fail to satisfy the geodesic equations. Such curves cannot exist in Riemannian geometry. We refer to [Mon] for a thorough discussion of the geodesics in sub-Riemannian spaces and especially the issue of abnormal geodesics. We remark, however, that all geodesics in the sub-Riemannian geometries arising from CR structures on the boundaries of strictly pseudoconvex domains are necessarily normal. Additional information on the structure of sub-Riemannian geodesics can be found in [CCG] and the recent publication [CC].

A rich vein of current research activity in sub-Riemannian geometry that we do not address is the study of the Carnot-Carathéodory geometry of submanifolds of sub-Riemannian manifolds. For more information see [CDPT, Chapter 4].

Example 10. We equip the Heisenberg group bH_n with a Carnot-Carathéodory metric as follows. The left invariant vector fields from (17) define the horizontal distribution. We declare them to be an orthonormal basis. The geodesics for this metric have a beautiful geometric description. For simplicity, we restrict to the case $n = 2$. In view of the group structure it suffices to describe the geodesics from the identity element $o = (0,0,0)$ to an arbitrary point $p = (x_1, y_1, t_1)$ with $t_1 \le 0$. Each geodesic is the horizontal lift of a circular arc in the (x, y) plane with endpoints $(0,0)$ and (x_1, y_1) so that the area of the planar region bounded by the arc together with the line segment joining these two points is equal to $-4t_1$. The sub-Riemannian distance from o to p is just the Euclidean length of the projected arc. When $t_1 = 0$ the geodesic is just a ray in the $t = 0$ plane from o to p. When $(x_1, y_1) = (0,0)$ geodesics are not unique; the family of geodesics from o to $(0,0,t_1)$ is rotationally symmetric about the t-axis.

The Unit 3 Sphere

> *Just like the circle and the two-sphere, the three-sphere is very round. But there are some beautiful, classical aspects to its roundness that are not easy to guess from its lower-dimensional sisters.*
>
> William Thurston [Thu]

We illustrate the preceding circle of ideas by describing the sub-Riemannian geodesics in the three-sphere $\mathbb{S}^3$ equipped with the Carnot-Carathéodory metric arising from its canonical CR structure. These geodesics have recently been computed by several authors using different techniques. See [HR], [BR], and [CC, Chapter 10]. We loosely follow the presentation by Hurtado and Rosales [HR]. The geodesics admit a simple and elegant description in terms of the Hopf fibration.

$\mathbb{S}^3$ as a Lie group and the Hopf fibration

Hamilton's search for an "algebra of vectors in $\mathbb{R}^3$", parallel to the identification of plane vectors with complex numbers, is mathematical folklore. His search led him to develop the theory of the *quaternions*, a noncommutative algebraic structure for vectors in $\mathbb{R}^4$. An arbitrary quaternion takes the form $a + bi + cj + dk$, where $a, b, c, d \in \mathbb{R}$ and i, j, and k are indeterminates satisfying $i^2 = j^2 = k^2 = ijk = -1$. We will identify quaternions with pairs of complex numbers by the rule $p = z_1 + z_2 j \leftrightarrow (z_1, z_2) \in \mathbb{C}^2$. Thus we identify $\mathbb{C}$ with the subspace of the quaternions $\mathbb{H}$ obtained by setting the coefficients of j and k equal to zero. Note that the quaternionic conjugate $\overline{p}$ is equal to $\overline{z_1} - z_2 \cdot j$ in this presentation. We point out the elementary identities $z \cdot j = j \cdot \overline{z}$ and $z \cdot k = k \cdot \overline{z}$ for $z \in \mathbb{C}$.

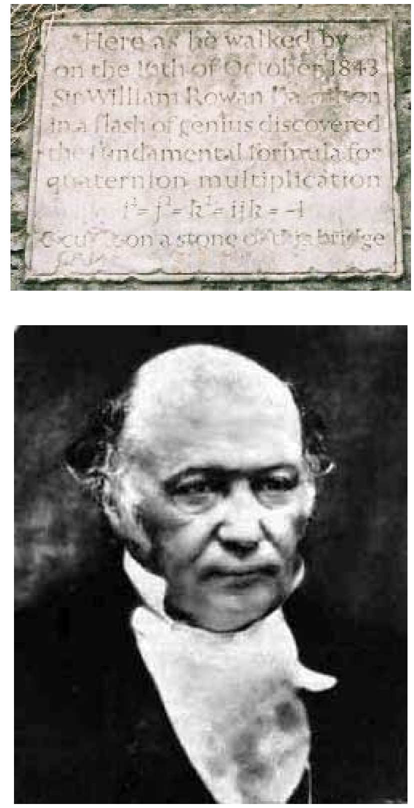

Figure 2. Sir William Rowan Hamilton and the plaque on Dublin's Brougham Bridge honoring Hamilton's invention of the quaternions. The Mathematics Department of the National University of Ireland at Maynooth commemorates the occasion with an annual walk to the site of this plaque on October 16, the anniversary of Hamilton's discovery.

We view $\mathbb{S}^3$ as the set of unit quaternions $p = z_1 + z_2 j$, where $|z_1|^2 + |z_2|^2 = 1$. With the preceding conventions in place, $\mathbb{S}^3$ is equipped with the Lie group law

$$p \cdot p' = (z_1 + z_2 j) \cdot (z_1' + z_2' j) = (z_1 z_1' - z_2 \overline{z_2'}) + (z_1 z_2' + \overline{z_1'} z_2) j.$$

It is well known that $\mathbb{S}^3$ is parallelizable: its tangent bundle admits a smoothly varying orthonormal frame. The quaternionic presentation provides an elegant description for such a frame: just consider the vector fields U, V, and W, where $U(p) = i \cdot p$, $V(p) = j \cdot p$, and $W(p) = k \cdot p$. These vector fields are linked by the bracket relations

(19)
$$[U, V] = -2W, \quad [W, U] = -2V, \quad [V, W] = -2U.$$

In particular, setting $\mathcal{H}(\mathbb{S}^3) = \mathrm{span}\{V, W\}$ defines a bracket-generating distribution on $\mathbb{S}^3$. If we set $p = z_1 + z_2 \cdot j$, then $V(p) = j \cdot (z_1 + z_2 j) =$

$-\overline{z_2} + \overline{z_1} j$, which coincides with the CR vector field L from (13). We have already observed that $\mathbb{S}^3$ is strongly pseudoconvex, hence step two. Formula (19) provides the same information. We define a sub-Riemannian structure by introducing a family of inner products on the horizontal bundle for which V and W are orthonormal. Our goal is to describe the geodesics of this sub-Riemannian structure.

The Lie group $\mathbb{S}^3$ is isomorphic to the matrix group $\mathcal{SU}(2)$ via the identification

$$z_1 + z_2 \cdot j \quad \longleftrightarrow \quad \begin{pmatrix} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{pmatrix}.$$

We can realize $\mathcal{SU}(2)$ as the double cover of the three-dimensional real rotation group $SO(3)$. The most elegant description, which dates back to Cayley, uses the quaternionic presentation: identify $\mathbb{R}^3$ with the space of purely imaginary quaternions via the basis $\{i, j, k\}$, and observe that the map $q \mapsto p \cdot q \cdot \overline{p}$ defines a rotation of $\mathbb{R}^3$. In the coordinates for the quaternions $\mathbb{H}$ described above, the map in question takes the form

$$\begin{aligned} p = \begin{pmatrix} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{pmatrix} &\in \mathcal{SU}(2) \mapsto R_p \\ := &\begin{pmatrix} |z_1|^2 - |z_2|^2 & -2\operatorname{Im}(\overline{z_1} z_2) & 2\operatorname{Re}(\overline{z_1} z_2) \\ 2\operatorname{Im}(z_1 z_2) & \operatorname{Re}(z_1^2 + z_2^2) & -\operatorname{Im}(z_1^2 + z_2^2) \\ -2\operatorname{Re}(z_1 z_2) & \operatorname{Im}(z_1^2 + z_2^2) & \operatorname{Re}(z_1^2 + z_2^2) \end{pmatrix} \\ &\in SO(3). \end{aligned}$$

This map is two-to-one, as it is invariant under the antipodal map $p \mapsto -p$ on $\mathbb{S}^3$.

Observe that the first row vector of R_p lies in $\mathbb{S}^2$ for every p. We obtain a map $\pi : \mathbb{S}^3 \to \mathbb{S}^2$. The circle group $\mathbb{S}^1$ acts on $\mathbb{S}^3$ via the diagonal action $e^{i\theta} \cdot (z_1, z_2) = (e^{i\theta} z_1, e^{i\theta} z_2)$, leaving the first row of the corresponding rotation matrix invariant. Thus we have exhibited $\mathbb{S}^3$ as an $\mathbb{S}^1$-bundle over $\mathbb{S}^2$, called the *Hopf bundle*. The induced fibration of $\mathbb{S}^3$ by (geometric) circles is the *Hopf fibration*.

The sub-Riemannian geodesics on $\mathbb{S}^3$

We begin by observing that the annihilating one-form η from (14) and the corresponding horizontal distribution $\mathcal{H}(\mathbb{S}^3)$ are invariant under the $\mathbb{S}^1$ action. This remark suggests that the structure of the sub-Riemannian geodesics should be related to the Hopf fibration. We now confirm this suggestion.

Extend the inner product to a Riemannian metric g on $\mathbb{S}^3$ by declaring U, V and W to be orthonormal. Let D be the Levi-Civitá connection for this metric g. Define a bundle isomorphism $J : \mathcal{H}(\mathbb{S}^3) \to \mathcal{H}(\mathbb{S}^3)$ by setting $J(V) = W$ and $J(W) = -V$. The data $(\mathbb{S}^3, \mathcal{H}(\mathbb{S}^3), \eta, J)$ defines a *pseudo-Hermitian structure* on $\mathbb{S}^3$, and the ensuing discussion can be framed in the language of pseudo-Hermitian geometry. See, for instance, [CHMY] and [DT].

The sub-Riemannian geodesics are critical points for the first variation of (horizontal) length by horizontal curves with fixed endpoints. Let $\gamma : I \to \mathbb{S}^3$ be a C^2 horizontal curve defined on a compact interval I. An *admissible variation* of γ is a C^2 map $G : I \times (-\epsilon_0, \epsilon_0) \to \mathbb{S}^3$ satisfying

(1) $G(s, 0) = \gamma(s)$ for all $s \in I$,
(2) $G(s, \epsilon) = G(s, 0)$ for all ϵ and all $s \in \partial I$,
(3) for each ϵ, $\gamma_\epsilon(s) := G(s, \epsilon)$ is horizontal.

Let X be the vector field along γ whose value at $\gamma(s)$ is equal to $(\partial G/\partial\epsilon)(s, \epsilon)|_{\epsilon=0}$. Admissibility of the variation can be characterized in terms of X: G is admissible if and only if X vanishes at the endpoints of γ and

$$\gamma'(\langle X, U\rangle) = 2\langle \pi_H X, J(\gamma')\rangle, \tag{20}$$

where π_H denotes projection into the horizontal space.

The classical formula for variation of length in $(\mathbb{S}^3, g)$, specialized to admissible variations, reads

$$\frac{\partial}{\partial\epsilon} \ell(\gamma_\epsilon)|_{\epsilon=0} = -\int_I \langle D_{\gamma'}\gamma', U\rangle.$$

Considering variations of the form $X = f J(\gamma')$ for C^1 functions $f : I \to \mathbb{R}$ with $f|_{\partial I} = 0$ and $\int_I f = 0$, and taking into account (20), one finds that the local minimizers for the sub-Riemannian length functional (18) are characterized by the Euler–Lagrange equation

$$D_{\gamma'}\gamma' + 2\kappa J(\gamma') = 0, \tag{21}$$

where κ is a real constant. This constant κ can be interpreted as a curvature of the geodesic γ. Geodesics of curvature zero are just great circles on $\mathbb{S}^3$. Geodesics of nonzero curvature are horizontal lifts of non-great circles on $\mathbb{S}^2$ through the Hopf fibration. Indeed, consider a smooth curve γ taking values in $\mathbb{S}^3$ and parameterized by arc length. The tangent space to $\mathbb{C}^2$ at $p = \gamma(t)$ splits as $T_p(\mathbb{C}^2) = T_p(\mathbb{S}^3) \oplus T_p(\mathbb{S}^3)^\perp$, which yields the following decomposition for the acceleration vector:

$$\gamma''(t) = D_{\gamma'(t)}\gamma'(t) - \gamma(t). \tag{22}$$

Then (21) reads $\gamma'' + 2\kappa J(\gamma') + \gamma = 0$. Viewing $\mathbb{S}^3$ as a subset of $\mathbb{C}^2$, this equation takes the form

$$\gamma'' + 2i\kappa\gamma' + \gamma = 0. \tag{23}$$

Equation (23) is a system of constant coefficient second-order ODE's whose explicit solution is easily obtained. Rather than presenting this solution, however, we focus on its geometric content.

Theorem 11 (Hurtado–Rosales)**.** *Let γ be a C^2 horizontal curve on $\mathbb{S}^3$ parameterized by arc length. Then the following are equivalent:*

(1) *γ is a geodesic of curvature κ in the sub-Riemannian metric on $\mathbb{S}^3$,*
(2) *$\langle\gamma'', J(\gamma')\rangle = -2\kappa$,*

(3) γ *is the horizontal lift under the Hopf fibration* $\mathbb{S}^3 \to \mathbb{S}^2$ *of a piece of a circle in* $\mathbb{S}^2$ *of constant geodesic curvature* κ.

We recall that the *geodesic curvature* of a space curve γ contained in $\mathbb{S}^2$ is its curvature computed relative to the embedding in $\mathbb{S}^2$. For a curve γ parameterized at arbitrary speed, this curvature is given explicitly by the formula $\gamma \cdot (\gamma' \times \gamma'')/|\gamma'|^3$. For $|h| < 1$, the geodesic curvature of the circle $\gamma = \mathbb{S}^2 \cap \{z = h\}$ is $h/\sqrt{1-h^2}$.

We sketch the proof of Theorem 11. To see that (1) and (2) are equivalent, note that the triple $\{\gamma', J(\gamma'), U(\gamma)\}$ forms an orthonormal frame for $T\mathbb{S}^3$ along γ. The fact that γ is horizontal and arc length parameterized immediately gives that the components of $D_{\gamma'}\gamma'$ in the directions γ' and $U(\gamma)$ are equal to zero. Thus $D_{\gamma'}\gamma'$ is a multiple of $J(\gamma')$ and (21) can be rewritten $\langle D_{\gamma'}\gamma', J(\gamma')\rangle = -2\kappa$. Equation (22) shows that $\langle \gamma'', J(\gamma')\rangle = \langle D_{\gamma'}\gamma', J(\gamma')\rangle$, since $\gamma(t)$ is normal to the sphere and $J(\gamma'(t))$ is tangent to the sphere. To see the equivalence of (1) and (3), one calculates the binormal of the Hopf projection of γ, which turns out to be a constant. Thus this projection is a circle on $\mathbb{S}^2$. Another calculation shows that its geodesic curvature agrees with the curvature κ from (21).

The topological structure of these geodesics is quite interesting.

Proposition 12 (Hurtado–Rosales). *If* $h := \frac{\kappa}{\sqrt{1+\kappa^2}}$ *is a rational number, then* γ *is a closed curve, isotopic to a torus knot inside a group translate of the Clifford torus* T_h. *If* h *is irrational, then* γ *is diffeomorphic to* $\mathbb{R}$ *and is dense in some group translate of* T_h.

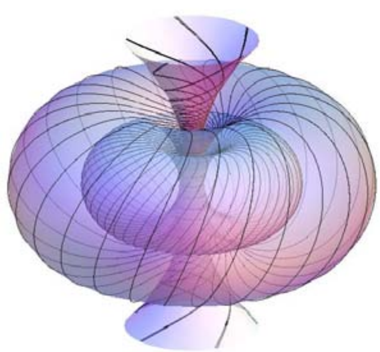

Figure 3. The Hopf fibration and sub-Riemannian geodesics.

Curves of the latter type are the well-known *skew lines* on the torus. The *Clifford tori* in $\mathbb{S}^3$ are tori of the form $T_\rho := \{(z_1, z_2) \in \mathbb{S}^3 : |z_1|^2 = (1+\rho)/2, |z_2|^2 = (1-\rho)/2\}$, where $|\rho| < 1$. Figure 3 illustrates the Hopf fibration, sub-Riemannian geodesics, and the Clifford tori.

Group-invariant Mappings between Spheres in Complex Spaces

The unit sphere $\mathbb{S}^3$ has dominated our discussion, because of its symmetry and its role in both CR and sub-Riemannian geometry. Earlier we observed an action of the circle on $\mathbb{S}^3$ defined by $(z_1, z_2) \to (e^{i\theta}z_1, e^{i\theta}z_2)$. In this section we study symmetries of $\mathbb{S}^3$ via actions of finite subgroups of $\mathcal{U}(2)$. The simplest case is when the group is cyclic; even in the cyclic case there is a remarkable depth and breadth of ideas. Given a positive integer p and an integer q with $1 \le q < p$, we choose a primitive p-th root of unity α, and consider the group action

$$(z_1, z_2) \to (\alpha z_1, \alpha^q z_2). \tag{24}$$

The quotient of the sphere by this group is called a *lens space*, written $L(p,q)$. Lens spaces, introduced by Tietze in 1908, are important in topology. The fundamental group of $L(p,q)$ is cyclic of order p for all q, and thus lens spaces with different p are not homotopy equivalent. Famous work of Alexander in 1919 showed that the lens spaces $L(5,1)$ and $L(5,2)$ are not homeomorphic even though they have isomorphic fundamental groups and the same homology. Necessary and sufficient conditions on p and q for being homeomorphic are known; similarly, necessary and sufficient conditions on p and q for being homotopy equivalent are also known. In Figure 4, the colors represent successive iterations of the group action (24). See `http://lukyanenko.net/math/heisenberg09/lens.html` for animation and additional details.

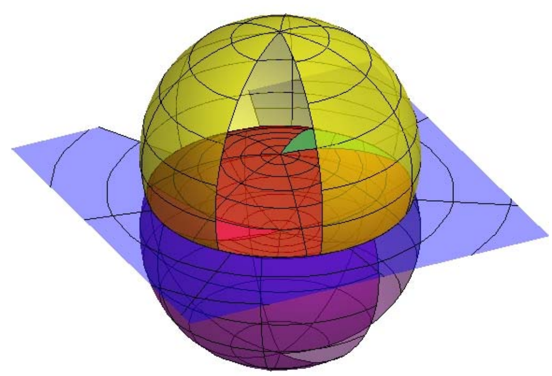

Figure 4. A provocative view of the lens space $L(5,2)$.

Lens spaces also arise in CR geometry. We naturally ask the following question. Given a finite subgroup Γ of $\mathcal{U}(n)$, is there an integer N and a nonconstant rational mapping, invariant under Γ, from $\mathbb{S}^{2n-1}$ to $\mathbb{S}^{2N-1}$? The answer is, in general, no. In fact, according to a theorem of Lichtblau [Lic], such a map can exist only when Γ is cyclic. Furthermore, (see [D1]), most representations of cyclic groups are also ruled out. We describe the results in the special case when $n = 2$. Roughly speaking, the only possibilities for nonconstant invariant rational maps to spheres are the special cases when the quotient space is $L(p,1)$ or $L(p,2)$. One amazing consequence is that there are homeomorphic lens spaces, for example $L(4,1)$ and $L(4,3)$, only one of which (in this case $L(4,1)$) admits a nonconstant rational CR mapping to a sphere.

We do not give up. In order to get more groups involved, we can proceed in two ways. First, we can weaken the assumptions on the mapping. Smooth CR mappings in this context must be rational. Second, we could allow continuous functions satisfying the tangential Cauchy-Riemann equations in the sense of distributions. We then obtain spheres as targets of group-invariant maps for any fixed-point free subgroup of $\mathcal{U}(n)$. Such results require considerable machinery in function theory of several complex variables. If we weaken the assumption on the target by allowing hyperquadrics, then more elementary but fascinating ideas appear. Given a finite subgroup Γ of $\mathcal{U}(n)$, by [D1] we can always find an N and a nonconstant polynomial mapping $p : \mathbb{C}^n \to \mathbb{C}^N$ such that p is Γ-invariant, and p maps the unit sphere to a hyperquadric. The polynomial p arises from the following construction. First, one defines the Hermitian symmetric polynomial Φ_Γ as follows:

$$\Phi_\Gamma(z, \overline{w}) = 1 - \prod_{\gamma \in \Gamma}(1 - \langle \gamma z, w \rangle).$$

By elementary linear algebra, there exist holomorphic vector-valued Γ-invariant polynomial mappings g, h such that

$$\Phi_\Gamma(z, \overline{z}) = \sum_{j=1}^{N_+} |g_j(z)|^2 - \sum_{j=1}^{N_-} |h_j(z)|^2.$$

Then the polynomial map $p = (g, h)$ has the desired properties. Even for the cyclic subgroups $\Gamma(p,q)$ used to define lens spaces, the computation of $\Phi_{\Gamma(p,q)}$ is interesting. For $\Gamma(p,1)$ the result is $(|z_1|^2 + |z_2|^2)^p$. For $\Gamma(p,2)$ and $\Gamma(p,p-1)$ explicit formulas exist. For all q, the formula for $\Gamma(p,q)$ is a polynomial $f_{p,q}$ in the variables $|z_1|^2$ and $|z_2|^2$ with integer coefficients. In general these polynomials have many interesting properties; we give here a few examples. By [D2], for each q, the congruence

$$f_{p,q}(|z_1|^2, |z_2|^2) \cong |z_1|^{2p} + |z_2|^{2p} \pmod{p}$$

holds if and only if p is prime. A combinatorial interpretation of the integer coefficients appears in [LWW]. See [Mus] for an application to counting points on elliptic curves. See [D2] for many additional properties of these polynomials. See [BR] for a study of sub-Riemannian geometries on the lens spaces $L(p,q)$; these geometries are obtained by projecting the horizontal distribution from $\mathbb{S}^3$ to $L(p,q)$ via the natural quotient map.

Finite Type and Higher Brackets

Iterated commutators of vector fields provide a systematic method for dealing with higher order conditions, and their role in sub-Riemannian geometry has been evident in this article. Such conditions are also significant in partial differential equations. Given a finite set of vector fields $X_1, \ldots, X_k$ in a neighborhood of a point in $\mathbb{R}^d$, consider the operator $T = \sum_j X_j^2$. Such *sums of squares* of vector fields arise throughout partial differential equations and probability. When the X_j form a basis for the tangent space, T is elliptic, and hence hypoelliptic. Hypoellipticity means that solutions u to the equation $Tu = f$ are smooth when f is smooth. Subellipticity is almost as useful as ellipticity, because it also implies hypoellipticity. An elliptic operator of order two gains two derivatives in the Sobolev scale; a subelliptic operator gains fewer derivatives than its order, but the gain suffices to establish regularity results. The Hörmander condition, applied to the span of the X_j's, provides the necessary and sufficient condition for subellipticity of T.

On the other hand, subellipticity for systems is much more difficult. In two dimensions, subellipticity for the Cauchy-Riemann equations reduces to estimates for a scalar sum of squares operator. The following theorem precisely relates the gain in a subelliptic estimate with the step of the horizontal distribution on the boundary. We omit the technical definition of a subelliptic estimate with gain ϵ. Kohn established the first subelliptic estimate for weakly pseudoconvex domains in $\mathbb{C}^2$ under the condition of finite step, and Greiner established the necessity of this condition. The sharp result in Theorem 13 relies on a sophisticated theory of estimates developed by Rothschild and Stein [RS] based on the sub-Riemannian geometry of nilpotent Lie groups; their theory made it possible to relate the geometry and the estimates in a precise fashion.

Theorem 13. *Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^2$, and suppose $p \in b\Omega$. The following are equivalent:*

- *There is a subelliptic estimate at p with gain $\epsilon = \frac{1}{2m}$, but for no larger value of ϵ.*
- *For a $(1,0)$ vector field L on $b\Omega$ with $L(p) \neq 0$, we have type$(L,p) = 2m$.*
- *The bundle $\mathcal{H}(M) = T^{10}(M) \oplus T^{01}(M)$ has step $2m$ at p.*
- *The maximum order of contact at p of a complex analytic curve with $b\Omega$ equals $2m$.*

In higher dimensions things are more subtle; algebraic geometry enters the story. Consider the hypersurface M in $\mathbb{C}^3$ defined by

$$\mathrm{Re}(z_3) = |z_1^2 - z_2^3|^2.$$

The step at the origin is 4, and each $(1,0)$ vector field has type either 4 or 6 there. Thus each such vector field is of finite type at the origin. On the other hand, M contains the complex analytic variety V defined by $z_1^2 - z_2^3 = z_3 = 0$, which has a cusp at the origin. (See Figure 5.) At all nonsingular points of V, the step is 2, but there is a $(1,0)$ vector field (tangent to V) of infinite type. In particular the condition that each $(1,0)$ vector field be of finite type holds at the origin but fails at nearby points. This failure of openness has consequences in the study of subelliptic estimates. The condition for subellipticity must be an *open* finite-type condition. The answer turns out to be that there is a bound on the order of contact of (possibly singular) complex analytic one-dimensional varieties with M at p. See [Cat], [CD], [D1], [D3], [DK], and [Koh1].

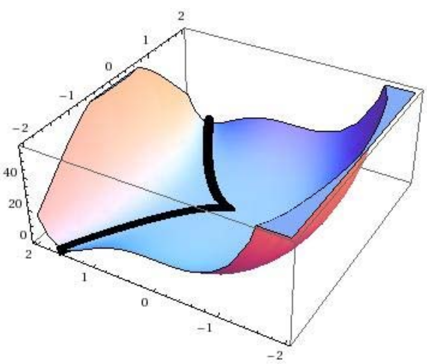

Figure 5. A singular variety in the hypersurface $M = \{z \in \mathbb{C}^3 : \mathrm{Re}(z_3) = |z_1^2 - z_2^3|^2\}$.

Concluding Remarks

Math 499, a required course for first-year graduate students at the University of Illinois, exposes students to a wide variety of mathematical points of view through engaging invitations to the research areas of the speakers. This note arose from discussing the background common to talks we have given in this course but includes material well beyond what we actually presented. We hope that we have provided sufficient insights and references to motivate further study.

Much of the material here is only tangentially related to differential geometry, especially if one appeals to the hilarious definition *"Differential geometry is the study of those properties invariant under change of notation"* sometimes attributed to Calabi. Regardless of the definition, differential geometry encompasses a diverse collection of subjects, including both CR geometry and sub-Riemannian geometry. While both these subjects have their roots in mapping theorems from complex analysis, they have diverged considerably over the past forty years. Perhaps this article is a small step toward their eventual convergence.

Acknowledgements

We would like to thank Harold Parks for encouraging us to write an article for the *Notices* and for his useful comments on a preliminary draft. We also acknowledge valuable suggestions from two referees. The Graduate Program at UIUC provided us the opportunity to speak in the Math 499 series. Luca Capogna, Joe Kohn, Chris Leininger, and Tom Nevins have invariably shared useful geometric insights with us over the years and thus indirectly aided the preparation of this article. The pictures of lens spaces, the Hopf fibration, and the sub-Riemannian geodesics on $\mathbb{S}^3$ were produced by Anton Lukyanenko using Mathematica; we gratefully recognize his contribution. We acknowledge the National Science Foundation for research funding under NSF Grants DMS-0500765 and DMS-0753978 (JPD) and Grants DMS-0555869 (JTT).

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