Book Review

What Then? Plato's Ghost: The Modernist Transformation of Mathematics

Reviewed by Yuri Manin

Plato's Ghost: The Modernist Transformation of Mathematics
Jeremy Gray
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The new book of the renowned historian of mathematics Jeremy Gray can be succinctly described as a rich and thorough study of Western ethnomathematics in the period of five decades or so (1880–1930) preceding and following World War I. What distinguishes such a project from more conventional Bourbaki–style enterprises is the stress on extra–mathematical background: social and political structures of society, economic practices, forms of professional self–organization, and culture.

In fact, culture (represented by visual arts, music, philosophy) dominates the discourse at several key points, and culture furnished the basic metaphor for the book. The time span in question is characterized as the period of modernist transformation in mathematics, and the first quotation in the Introduction is taken from Guillaume Apollinaire's "The Beginning of Cubism" (1912).

The scope of the book is already ambitious. Chronologically, it starts well before the arrival of "modernism": There are brief essays on Monge, Poncelet, and projective geometry, followed by a discussion of Lobachevsky, Bolyai, and non-Euclidean geometry, including a description of Gauss's role in its formation and the subsequent contributions of Riemann and Poincaré that led to a new vision of what geometry is and what it should be. Similar essays on algebra, analysis, "British Algebra and Logic", and philosophy follow, all of this interspersed with information about professional organizations, teaching, journals, and quotations from contemporary articles and letters. (I cannot resist the temptation to reproduce a delightfully funny sentence from a letter by C. G. J. Jacobi to A. von Humboldt, although in the book it appears only considerably later, on page 268:

> If Gauss says he has proved something, it seems very probable to me; if Cauchy says so, it is about as likely as not; if Dirichlet says so, it is certain.)

Chapter 2, containing this wealth of information, is called "Before Modernism", and ends with a brief synopsis "Consensus in 1880".

The arrival of mathematical modernism is addressed in Chapter 3. Again, the discussion is consecutively focused on geometry (in particular, reemerging connections to physics and Schwarzschild's paper of 1900); then analysis, algebra, and logic. With hindsight, the central modernizing figure appears in the person of Georg Cantor. Especially interesting for me were pages discussing religious overtones of Cantor's transcendental flights in the infinity of infinities and the related subsection on "Catholic Modernism". (As I have written elsewhere, I discern a subtle mockery in the famous and often quoted sentence of Hilbert

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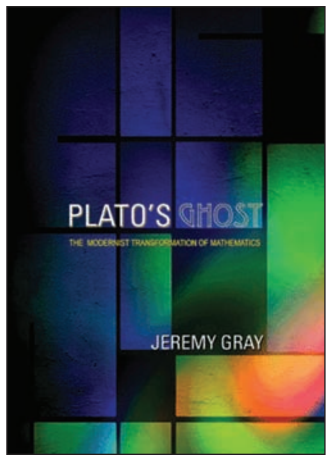
about "Cantor's Paradise", usually perceived only as unconditional support for set theory).

The subsequent chapters are called "Modernism Avowed", "Faces of Mathematics", "Mathematics, Language and Psychology", and "After the War" and are bursting with information, insight, and scholarship. As far as I know, nobody before Jeremy Gray discussed in this context "Popularizing Mathematics around 1900" (section 5.4), or "Languages Natural and Artificial" (section 6.1).

Constrained by the restrictions of space, I will end this brief survey of the detailed contents of this remarkable book and turn to the discussion of several of its grand themes, sometimes underlying the exposition and sometimes explicitly invoked in it.



Mathematics is a product of civilization: it is discovered or created (see discussion below) by a very limited number of human beings in each generation, then encoded in a mixture of words, formulas, and pictures (nowadays software and hardware might be added to this list), and in this way transmitted to the next generation, which continues this process. To estimate the number of persons creating mathematics these days, one can refer to the attendance of International Congresses of Mathematicians that are held every four years: recently it was about three to five thousand. Undoubtedly, this should be viewed as a great explosion in comparison with previous centuries. In Newton's time the respective number probably did not exceed a few dozen.

One remarkable feature of mathematical knowledge is this: we learn more and more about the same objects that ancient mathematicians already started to see with their mental eyes: integers and prime numbers, real numbers, polynomial equations in one or many variables, space and various space forms… To illustrate this statement, I will mention just one string of events. Some time around 300 BC, the Hellenistic scholar Pappus discovered a remarkable and, in the Euclidean context, quite unusual theorem (see the modernized statement and the picture on page 463 of Gray's book, and beware of a small misprint: $B'A$ must be BA'). Pappus's Theorem does not conform to the spirit of Euclidean geometry, because lengths, angles, and rigid plane motions are quite irrelevant here: only points, lines, and incidence relations "a point x lies on a line X" play a role. In the nineteenth century it was understood that Pappus's Theorem characterizes projective geometry of the real plane. When an abstract projective plane—as a set of points P, with a set of subsets $l \subset P$ called lines—was introduced at the beginning of the twentieth century, it turned out that Pappus's Theorem, now taken as an axiom, is a necessary and sufficient condition for the possibility of introducing projective coordinates in P with values in an abstract field (at least, if P is infinite). Several more decades elapsed, and it was discovered that this latter statement can be vastly generalized in the theory of models of formal languages: "Zariski geometries" of Zilber and Hrushovski describe when abstract sets with subsets satisfying certain combinatorial requirements can be conceived as sets of solutions of arbitrary polynomial equations over a field.

A *sine qua non* condition of such permanency, continuity of preoccupation, is the creation and maintenance of a very safe, reliable, intergenerational flow of information. Linguistic resources of mathematics by which this information is encoded and transferred are arguably much more variable, fickle, subject to sweeping winds of history, than the content of the information itself.

The whole bulk of discoveries of one generation, or one of its active but geographically restricted subgroups, might in a few decades become almost incomprehensible to another generation, unless new expressive means, with their new intuitions and hygienic rules, come to the rescue. The old arguments are translated into a new language, rewritten, purged of perceived obscurities and errors, in the process of mastering old results and moving to new discoveries.

My professional youth was spent in this exhilarating process of assimilation of glorious Italian algebraic geometry and recasting it using schemes, coherent sheaves, and homological algebra of the emerging "new age" algebraic geometry, which in two to three decades was to become one of the powerful vehicles not only in pure mathematics but also in theoretical physics and quantum field theory. Perhaps for this personal reason, I became perceptive of similar events that took place in other times and locations.

From this perspective, "the modernist transformation of mathematics", which is the subtitle of Jeremy Gray's book, was one such periodically occurring refurbishing of basic vocabulary, grammar, and, yes, aesthetic requirements for mathematical thought and mathematical texts. It took about thirty years, and its outcomes dominated the next thirty years: all in all, just about the span of three generations. The fact that during the lives of these generations two world wars swept Europe (or, according to some accounts, one Thirty Years' War 1914–1945) influenced mathematics and the wider culture in many ways.

The Bourbaki group formed itself in the 1930s in a conscious attempt to fill the void left by the "lost generation": not in the somewhat egocentric sense of the phrase coined by Gertrude Stein, who referred to bohemian American émigrés in Paris, but a real and painful void. About 40 percent of French mathematicians died in the first world war.

Although my argument posits "the modernist transformation" as one of the series of structurally similar transformations that occurred in other times and places, both the choice of the time span and the use of the adjective "modernist" for it seem to me very felicitous. After formulating what he sees as characteristic of modernist aesthetics, creative psychology, the relationship with society, and self–perception, shared by, say, painting and mathematics of that period, Jeremy Gray himself warns that this looks like a case of "convergent evolution" (page 8) rather than an effect of "common genes".

In his own sober words: "I am not sure there is more than a resonance that links mathematical modernism to the arrival of modernity, modern capitalism, and its horrific opponent, the Nazi state. We are far from knowing much about the societies we inhabit."

In a book largely dedicated to the history and philosophy of mathematics, Plato's name might have been invoked in many contexts, but the first that comes to mind, that of "Mathematical Platonism", is relegated to the very end of the discussion, section 7.3, pages 440–452. Besides this section 7.3, Jeremy Gray, according to the Index, mentions Plato or Platonism on twelve more pages scattered throughout the text. By contrast, more than fifty entries in the Index invoke Poincaré and only slightly fewer, Bertrand Russell.

Why, then, is this monograph called "Plato's Ghost" ? The explanation is on the very first page, which quotes in its entirety Yeats's "What Then?", a poem of four five-line stanzas about an accomplished life of an accomplished human being:

> Everything he wrote was read,
> After certain years he won
> Sufficient money for his need;
> Friends that have been friends indeed;
> *"What then?" sang Plato's ghost, "What then?"*

The most direct reading of these stanzas suggests to me a simple and universal human emotion, *existential anxiety,* as beautifully expressed by Yeats as by many others before and after him.

Jeremy Gray himself hears in Plato's voice denial of a claim to perfection (page 14). This interpretation is readily accommodated in a chronicle of (para)philosophical controversies around mathematics. But, significantly, the word "anxious" appears already in the first paragraph of the Introduction, and "anxiety" pops up on page 4, where it is first introduced as "a well established theme in writing about modernism". Later this theme is developed in the section 4.8, "Anxiety", already fully in the context of the mathematics of the first half of the twentieth century, against the background of the "crisis of foundations".

Is it conceivable that Gray's whole discourse on modernism in mathematics is informed by existential anxiety and that the last section, 7.5, "The Work Is Done", is as self–referential as it sounds?

Of course it is, as the present author willingly testifies, relying upon his own experience, and as Jeremy Gray himself hints in the last lines of page 4.

But before bidding farewell to the reader, let's brace ourselves and try to face the Ghost question "What then?", which I will interpret in the most prosaic way, as a question about the direction in which self–consciousness of mathematicians moved during the last few decades, separated by more than a half–century from the 1930s and 1940s where the detailed analysis of Gray stops.

So far as I can judge, "Platonism" of working mathematicians is based on a feeling that important mathematical facts are *discoveries* rather than *inventions.*

The Bering Strait was named after Vitus Bering who discovered it, whereas the diesel engine was named after Rudolf Diesel who invented the engine. What about Galois groups? If you feel that they were discovered by Évariste Galois, rather than invented by him, you are in a sense a Platonist.

I will call such an attitude *emotional Platonism* in order to stress that (in my view) it is intellectually indefensible, but not to the least degree invalidated by this fact, since our emotions happily resist rational arguments.

Being such an emotional Platonist myself, I do not want to say that *all* mathematics is a discovery of a Platonic world, whatever that could mean. Certainly, the history of mathematics is also marked by inventions of marvelous intellectual telescopes and efficient vehicles, allowing people to travel from one discovery to another. Moreover, there are mathematicians whose oeuvre deserves comparison to the *Odyssey* rather than to Columbus's travelogue, insofar as mathematics is exteriorized in texts forming a part of the much vaster general culture of the written word.

Elaborating the latter metaphor, one can represent the history of the "modernist transformation of mathematics" simply as a tale about the birth and development of a certain style of thought, expression, and teaching, starting with Georg Cantor and culminating with Bourbaki. Much will have to be left out of such a tale, but it could serve as a good starting point for guessing "What then?"

This style is formed around a system of pretty explicit prescriptions: how a mathematician is supposed to introduce his or her object of study ("what

I am talking about"), how his or her results should be stated ("what I am saying"), and finally how one should write arguments convincing her/himself and potential readers that the stated results are correct ("proofs").

Briefly, the object of study (group, space with measure, topological space…) is introduced by a *definition*, presenting it as a Bourbaki–style *structure*, which in turn is a collection of Cantorian sets satisfying certain relationships stated in terms of the basic relation "being an element of" and standard logical means. The results are stated as *theorems*, statements of the type "if a structure has a certain property P, then it has another property Q as well" etc. Finally, proofs are texts written in a mixture of natural language, formulas, and sometimes pictures (although the latter are considered "bad taste" in the Bourbaki aesthetics). Such a text can be considered as a valid proof only if in principle it can be *replaced by a sequence of statements* such that each term of this sequence *is valid either by definition, or can be obtained by an elementary logical step from previously obtained statements.*

There are several more or less hidden or explicit sources of self–referentiality in this picture, of which I will mention two.

One is that the notion of mathematical reasoning invoked in the previous paragraphs can itself be rigidified to become a mathematical structure. This was of course the main discovery of the formalist program; as soon as it had crystallized, Gödel's and Tarski's theorems formalizing the "liar's paradox" or Cantor's diagonal argument became inevitable.

Another source of self–referentiality is less formal. Namely, a "proof" presented in a mathematical paper must convince the reader not just that the stated theorem is true, but that it by itself is a PROOF, the incarnation of an ideal object residing in the Platonic territory of formalized mathematics. Severe rules of hygiene are imposed upon mathematical exposition in order to ensure this. The reader must be alerted even to the occurrence of an "empty proof" (empty in the set–theoretical sense), as Edmund Landau used to practice with his inimitable "*Beweis:* Klar. (*Proof:* Clear.)".

The imposition of the hygienic restrictions was a part of the reaction to the perceived crisis of foundations, but in the real life of mathematics of the twentieth century it played another, and probably unexpected, role: that of unification of many diverse fields of mathematical studies. Algebraists, analysts, geometers, probabilists, number theorists, logicians could now speak in the same language, even when speaking about different structures.

Logicists stubbornly struggled in the losing battle against this loss of their dominance as Keepers of Foundations. Nicolas Bourbaki did not really understand the structures emerging in new mathematical logic and botched the respective chapters of his Treatise. As a result, the basic new discoveries in logic, the Turing–Church Thesis and the theories of computability and of complexity, suffered because their ties to the mainstream mathematics were loosened. Fortunately, such researchers as A. N. Kolmogorov helped tighten these ties.

What we see now in the flow of mathematical research can be termed loosely as a "post-modern" period, but the term can be taken literally only if we adopt Jeremy Gray's concept of "modernism". Contemporary mathematics bears no traces or connotations of Lyotard's description of the post-modern condition, whose main characteristic is the loss of credibility of all grand narratives. The grand narrative of mathematics is steadily developing, reaching new depths and new sophistication.

One trend that visibly changes the face of set theory as the foundational language of mathematics is the current popularity of categories (functors, enriched categories, polycategories…) as the dialect in which basic definitions of mathematical objects are formulated. Another related, but not identical, trend is the evolution of homotopy theory, which has gradually become a kind of new language for postmodern mathematics (for an initiated reader, I can mention "brave new rings").

Briefly, these two trends together revolutionize our collective vision of both semantics and syntax of the language(s) of mathematics.

First, basic new objects "category" (up to equivalence) and "homotopy type" are *not* Bourbaki structures: they are formed by a class of Bourbaki structures that are related by certain equivalence relations. Both "class" and "relation" in this statement can be represented set–theoretically by "indefinitely large" collections of sets.

Accepting this, a working mathematician not only sheds the primeval horror of large infinities but opens his/her imagination to a radically new vision even of common objects. For example, natural numbers "simply" count things (or cardinalities of finite sets), but the history of mathematics teaches us how late zero and negative integers were introduced and accepted. External references such as the notion of "debt" in trade were necessary in order to legitimate negatives. Nowadays integers are homotopy classes of pointed loops in a real plane with a deleted point: this is a germ of the idea of "brave new rings". Even more generally, "small sets" that in the Cantor/Bourbaki paradigms could be turned into topological spaces by imposing an additional structure now become secondary/derived objects: say π_0 of a homotopy type. The traditional view "continuous from discrete" gives way to the inverted paradigm: "discrete from continuous".

Second, the notion of formal language that crystallized in the 1930s as a purified written form of natural languages, after it became treated as a Bourbaki structure, could be vastly extended. Seemingly, there are other Bourbaki structures (or better, categories) that have "language–like properties", such as categories of graphs. The intuition behind considering, say, a directed graph as a potential linguistic entity is that of "flowcharts". Generally, computer science, that no–nonsense child of mathematical logic, will exert growing influence on our thinking about the languages by which we express our vision of mathematics.

Finally, I want to discuss briefly the collaboration of mathematicians and physicists, which became dormant during the several decades of "mathematical modernism" but renewed with new vigor after 1950s and 1960s. I will primarily stress the benefits of this collaboration for mathematics and will describe these benefits as the emergence of a vast research program that could be called "quantization of classical mathematics".

This program is historically related to the fact that when physicists started to see quantum phenomena as the basic natural causes underlying observable classical behavior of matter and fields, they had to gradually discover what kind of changes must be made in order to proceed from a known mathematical description of a classical system to a new, quantum description ("quantization"), and back ("classical approximation"). Some of the discovered prescriptions, such as "deformation quantization" involving Planck's constant as a small parameter, turned out to be much more universal mathematically than suggested just by their initial uses.

Another prescription stressing quantum observables as operators in (generally infinite–dimensional) Hilbert spaces led to "non–commutative geometry", one germ of which was the Heisenberg commutation relation, and later to "quantum groups".

Philosophical problems related to the changed role of observation and measurement led to discussions of "quantum logic" and later, in a more applied vein, "quantum computation".

In the 1940s a development started that produced some mathematical miracles. Richard Feynman developed path integrals, a notion that is highly intuitively appealing though mathematically vague, as well as the powerful machinery of Feynman diagrams, which are well understood mathematically but are motivated only by the idea that they somehow capture path integrals. It was a great success in quantum field theory and elementary particles, but nobody could foresee how, in mathematics, it would backfire.

This became reality after many physics papers of Witten and his collaborators, which mathematicians could interpret as rich and powerful heuristic tools to guess precise and striking new mathematical facts, such as "quantum invariants of knots", from the physical intuition related to, say, "topological quantum field theories" and the respective formalism of path integrals. These facts could afterwards be investigated and put on firm ground by mathematicians: we got "quantum topology", "quantum cohomology" (from string theory), and much more.

If you search in the arXiv, MathSciNet, and Google for the terms I put in quotation marks above, you will find oceans of information about this stuff.

At this point, Plato's apparition intervenes again and sings *"What then?"*

And I respond: *"For us, there is only the trying. The rest is not our business."*[1]

References

[1] Yuri Manin, Georg Cantor and his heritage, in *Mathematics as Metaphor*, Amer. Math. Soc., Providence, RI, 2007, pp. 45–54.

[2] Vladimir Tasić, *Mathematics and the Roots of Postmodern Thought*, Oxford University Press, 2001.

[3] Philippe Grotard, *Géométrie Arithmétique et Géométrie Quantique, de Gauss à Grothendieck et de Witten à Kontsevich*, manuscript dated May 22, 2002, 540 pp.

[1] *T. S. Eliot,* Four Quartets.