Solving Mathematical Problems: A Personal Perspective

Reviewed by Loren Larson

Solving Mathematical Problems: A Personal Perspective
Terence Tao
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In 1980 Paul Halmos concluded his *Monthly* article "The Heart of Mathematics" [1], with this thought:

> I do believe that problems are the heart of mathematics, and I hope that as teachers, in the classroom, in seminars, and in the books and articles we write, we will emphasize them more and more, and that we will train our students to be better problem posers and problem solvers than we are.

Now, thirty years later, I think he would be pleased with what has transpired. Problem solving is central in current mathematics education, from kindergarten through middle school, from high school through college and beyond. Learning mathematics means *doing* mathematics, a mix of practice *exercises* to develop skills and *problems* for deeper understanding. The proportion of each depends on the context, anything from 100/0 for crash courses where the problems will come later (in more than one sense?) to 0/100 for Moore-method courses where mastering techniques will come later. Both are necessary, but from a pedagogical point of view, appropriately chosen problems under the right conditions are more fun and offer more satisfaction, at least for mathematically inclined students. To paraphrase Albrecht Dürer, "love and delight are better teachers than compulsion."

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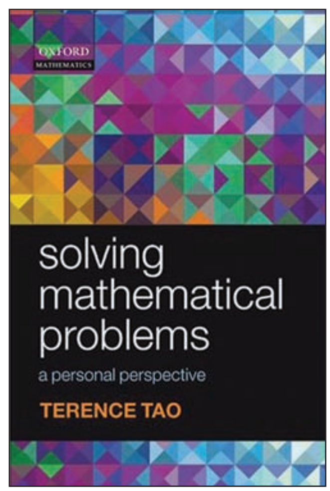

One means of promoting problem solving is by organizing math contests, and there are now a variety of contests and plenty of opportunities for participation. These contests are supported by a vast literature of preparatory supplements: anthologies, compilations, how-to-solve books, special topics, problem-solving strategies, online classes, forums, summer math camps, and videos (e.g., see `http://www.artofproblemsolving.com`). Contests have a proven record for fostering interest in mathematics, but they aren't for everyone. Some students are turned off by competition, or have mathematical interests that aren't particularly amenable to a contest format, or feel that contests favor speed over power (not that they can't coexist). But there are other ways of promoting problem solving and personal involvement, such as problem seminars, independent research, and interdisciplinary projects.

Finding suitable problems is a time-intensive task. Problem books can provide ideas and starting points, for example nontrivial tweaks or special cases of published results. But this approach can lead to stagnation and irrelevance. A better source for keeping ideas meaningful and up-to-date is the research community—through personal communication, networking, newsgroups, problem sections of professional journals, and journal articles. Problems for high-level math competitions not only need to be original, but they should also be presented with a certain flair. Composers of such problems have to indulge their feel for language, their artistic temperament, and even their sense of humor. It's in the best interest of our community to encourage this kind of effort, as getting students interested in current research through problems is an important means of renewal. Paul Erdős was a master at introducing rich mathematical ideas by way of simply stated problems. Putnam problems have stayed fresh over the years because of contributions from researchers and inveterate problem enthusiasts. A book by Paul and Judith Sally [4] is helpful in showing how a single mathematical concept can sometimes be adapted and framed into an interesting problem at virtually every level of mathematical expertise.

Terence Tao, the author of the book under review, is one of math's luminaries, a winner of the Fields Medal in 2006. If you aren't familiar with his prodigious accomplishments and life as a child prodigy, by all means check out his amazing story. This book was written when he was just fifteen years old, but even by this age he was well-qualified for the task: he had already competed in three International Mathematical Olympiads, winning a bronze medal, a silver medal, and then a gold medal just days after his thirteenth birthday.

In the narrowest sense, the book is aimed at showing high school students how to solve Math-Olympiad-like problems. Almost all the problems are taken from published collections of problem sets for mathematics competitions. In the Preface to this second edition, the mature Tao at age thirty comments that if he were to write a problems book now it would be very different. But he resisted the temptation to tamper with it except for a few organizational changes, noting that "his younger self was almost certainly more attuned to the world of the high-school problem solver." Indeed, one of the most charming features of the book is the exuberant exposition, which the elder Tao points out "[sometimes] has a certain innocence, or even naiveté." Here are some delightful excerpts taken from chapter previews (and if you've worked with precocious young people, you'll recognize the voice):

> **Algebra:** Algebra is the basic foundation of a large part of applied mathematics. Problems of mechanics, economics, chemistry, electronics, optimization, and so on are answered by algebra and differential calculus, which is an advanced form of algebra. In fact, algebra is so important that most of its secrets have been discovered—so it can be safely put into a high school curriculum. However, a few gems can still be found here and there.

> **Number Theory:** Unlike algebra, which has as its backbone the laws of manipulating equations, number theory seems to derive its results from a source unknown. Take, for example, Lagrange's Theorem, … . Number theory is a fundamental cornerstone which supports a sizeable chunk of mathematics.

> **Analysis:** Analysis is the study of functions and their properties. …Functional equations form a sort of "pocket mathematics", where instead of the three dozen or so axioms and countless thousands of theorems, one has only a handful of "axioms" (i.e., data) to use and there is a clear direction in which to go. And yet, it still has surprises.

> **Geometry:** The true beauty of geometry is in how a non-obvious-looking fact can be shown to be undeniably true by repeated application of obvious facts. As an example: the midpoints of the four sides of a quadrilateral always make up a parallelogram. These facts—they have a certain something about them.

Statements like these have to be put into context. Remember that his target reader is someone like himself at age nine, intensely eager and capable and focused on learning as much as possible about Olympiad-level math problems. A certain panache and dash, even some exaggeration and eccentricity, is expected at this age. I was all ears, a teenager again, fired-up and flattered to be treated as an equal. His intentions for the book are laid out in the Preface:

> I will try to demonstrate some tricks of the trade when problem solving. Two of the main weapons—experience and knowledge—are not easy to put into a book: they have to be acquired over time. But there are many simpler tricks that take less time to learn. There are ways of looking at a problem that make it easier to find a feasible attack plan. There are systematic ways of reducing a problem into successively easier subproblems.

Math "how-to" books usually focus on specific applications of methods given by George Pólya [3], and this book is no exception—the intention is to motivate the solution. But the manner in which Tao does it is the book's most distinguishing feature. For not only does Tao motivate the solution but he also walks through his entire thinking process, including rejected ideas and false starts. He seems to be thinking aloud, explaining as he goes—each journey an apparently effortless flow of ideas. It's a slim book, about 100 pages, but the problems are well chosen, only twenty-six of them, with about the same number of instructive exercises, without solutions. The solutions to the problems are almost incidental to Tao's observations and reformulations along the way. He shows us how to think like mathematicians. For example, the actual solution to Problem 2.2 (Is there a power of 2 whose digits could be rearranged and made into another power of 2?) takes only six *lines*, but it is preceded by five *pages* of discussion.

The defining characteristic of Tao's perspective on problem solving is based on Pólya's dictum, nicely stated by Halmos [1]:

> Make it easier. In slightly greater detail: if you cannot solve a problem, then there is an easier problem that you cannot solve, and your first job is to find it! Make it sharp. By that I mean: do not insist immediately on asking the natural question ("what is…?", "when is…?", "how much is…?"), but focus first on an easy (but nontrivial) yes-or-no question ("is it…?").

Essentially, this approach is analogous to the *Bolzano-Weierstrass Method* for catching a lion in the desert [2]: Repeatedly bisect the area with alternating vertical and horizontal lines, choosing the half that contains the lion. At each stage, build a fence. The lion is ultimately enclosed by a fence of arbitrarily small perimeter.

In other words, first you have to survey the desert, that is, understand the problem and write down *everything* you can think of that might be relevant. This is especially important in problems at this level because the most obvious beginnings probably won't work. Then, with this list in hand, you start bisecting, eliminating ideas that appear to make things more difficult. Iterate on this process, systematically asking questions and reducing it into easier subproblems until the tricky parts are no longer tricky. "Make it easy" is the theme of the book, which also makes it fun to read. Expressed more flamboyantly by Tao:

> So that is it. We keep reducing the equation into simpler and simpler formulas, until it just collapses into nothing. A bit of a long haul, but sometimes it is the only way to resolve these very complicated questions: step by step reduction.

> Simplify repeatedly until the more unusable and unfriendly parts of the problem are exchanged with more natural, flexible, and cooperative methods.

> It is best to try elementary techniques first, as it may save a lot of dashing about in circles later.

> As long as one always tries to simplify and connect, chances are that the solution will soon fall into place. (Assuming, of course, that there is one—and most problems are not trying to pull your leg.)

Here's an example of how it plays out: One of the geometry problems asks you to prove that either the given triangle is isosceles or a particular angle is 60°. Tao quickly rules out coordinate geometry: "Hack-and-slash coordinate geometry is one long and boring way that is prone to abysmal complications and huge errors. Let us try that as a last resort." Realizing that the conclusion involves angles and the given data involves equal line segments, he needs results that relate length and angle. So he writes down relevant facts about right triangles, isosceles triangles, the law of cosines, the law of sines. There aren't a lot of right triangles in the figure so let's not create any yet. The equal line segments aren't part of an isosceles triangle that directly connects to the conclusion, so we'll pass on that for now. He continues: "The law of cosines usually complicates rather than simplifies, and it just creates more unknown lengths. This leaves only the law of sines as a feasible alternative." This leads to equations, then to further manipulations and reformulations and simplifications driven by the overall objective, and finally to the relevant angles.

Granted, this step-by-step reduction is particularly effective for Olympiad-like problems, and Tao acknowledges this in the original Preface:

> [Olympiad-like] mathematics problems are "sanitized" mathematics, where an elegant solution has already been found, the question is stripped of all superfluousness and posed in an interesting and (hopefully) thought-provoking way. If mathematics is likened to prospecting for gold, solving a good mathematics problem is akin to a "hide-and-seek" course in gold-prospecting: you are given a nugget to find, and you know what it looks like, that it is out there somewhere, that it is not too hard to reach, that its unearthing is within your capabilities, and you have conveniently been given the right equipment (i.e., data) to get it. It may be hidden in a cunning place,

but it will require ingenuity rather than digging to get it.

Contest problems provide a setting in which to acquire the habit of thinking mathematically. Here the psychological advantage makes it ideally suited for developing a way of thinking that will transfer to "unsanitized" mathematics, the habit of doing what you can and thinking ahead about how the complicated parts might be simplified.

The book assumes that the reader has a strong background in basic arithmetic and algebra (modular arithmetic, factorization formulas, elementary trig identities), standard Euclidean geometry, including elementary vector methods, and properties of polynomials and functions, as well as mathematical induction. If Tao were fifteen today his book would probably have included more topics from discrete math, such as the pigeonhole principle, elementary graph theory (Euler's formula $V - E + F = 2$, Euler and Hamiltonian circuits), and combinatorics (counting principles, recurrences). Nevertheless, the overall problem-solving approach would be the same, and it is largely this feature, along with the thorough, caring, and energizing delivery, that makes the book a noteworthy addition to the literature on problem solving and how to teach it.

References

[1] P. R. Halmos, The heart of mathematics, *The American Mathematical Monthly* **87** (7, Aug.-Sept.) (1980), 519–24.

[2] H. Pétard, A contribution to the mathematical theory of big game hunting, *The American Mathematical Monthly*, **45** (1938), 446–7.

[3] G. Pólya, *How to Solve It*, 2nd Edition, Princeton University Press, 1957.

[4] J. D. Sally and P. J. Sally Jr., *Roots to Research: A Vertical Development of Mathematical Problems*, AMS, 2007.