Life on the Mathematical Frontier: Legendary Figures and Their Adventures

Roger Cooke

How many fairy tales circulate as "universally known truths"!

B. L. van der Waerden, Preface to *Science Awakening*

A mathematical legend begins as an anecdote about a mathematician or a theorem, reported by someone at second hand. After many repetitions, usually with some variations, it solidifies into a legend. Thus a legend is characterized by two properties: 1) it is not attested by any primary documents or anyone claiming to have been an eyewitness; 2) it becomes widely known and cited. As the examples below will illustrate, a mathematical legend may be cited in more than one context to add color to a mathematical achievement. It often arises as a way of explaining how results were achieved when there is no paper trail showing the way, just as the legend of the Trojan horse, according to some classicists, explains how the Greeks were able to take Troy after the death of the great warrior Achilles when they could not do so with his help.

Any legend, mathematical or historical, may "migrate" from one person to another, just as the legends of the American frontiersmen Daniel Boone (1734–1820) and David Crockett (1786–1836) have become commingled through hagiography. A famous legend about Boone relates that he killed a wildcat as a boy, and according to another, he later killed a bear and carved a memorial of his feat into a nearby tree. (In the version I heard as a child, he showed his ignorance of spelling by writing "D. Boon cilled a bar this yre 1775.") The popular song that accompanied the Disney-promoted legend of Crockett said he "killed him a b'ar, when he was only three." It is more than coincidental that the same actor, Fess Parker, portrayed the two men as essentially identical in two popular American television series during the 1950s and 1960s.

The durability of a mathematical legend depends more on its resonance with the mathematical community than on its plausibility. If it illuminates some feature of a mathematician or a theory, it deserves a place in the history of the subject. For example, did Archimedes suddenly leap from his bath shouting "Eureka!" when he thought of the idea of measuring specific mass using the displacement of water? As the legend tells it, Archimedes had been ordered to do a forensic investigation for the King of Syracuse, who suspected that some of the gold he had provided to make a crown had been replaced with an equal weight of silver. It's an interesting story and seems plausible at first sight. But is there any eyewitness testimony to it? The oldest source for this story is the introduction to Book IX of *De architectura* by the first-century Roman architect Vitruvius, who lived two full centuries after Archimedes. Vitruvius cites no earlier source, and so we are left to conjecture whether the story is true and how it might have arisen if it isn't. What is valuable in the story is the picture of the sudden flash of inspiration that mathematicians sometimes experience. Whether true or not, this story will continue to be told

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because it amuses people and because it expresses some folklore concerning a legendary figure.

How much should one be willing to bet on the historical truth of this story? It seems doubtful that Archimedes could have weighed gold and silver and measured the amount of water they displace so precisely as to solve the problem he had been challenged with. The difference in the amount of water displaced must have been very small. Moreover, Vitruvius claims that Archimedes performed a quantitative analysis of the crown, determining the exact amount of silver that had been used to replace the gold. That would have been unnecessary if his only task had been to determine whether the gold had been alloyed with silver, which is probably all the king would have wished to know. And again, although the physical principle involved in the analysis is correct, the practical application of it appears to be much more challenging.

As is the case with other stories told of Archimedes by Vitruvius' contemporary Plutarch, such as the story that he designed a claw to be attached to a crane that could snatch Roman ships out of the water during the siege of Syracuse (*Life of Marcellus*, Ch. 15, § 3), there seem to be some practical difficulties. Although engineers working with both the BBC and the *Discovery Channel* have constructed machines that are capable of upending a Roman ship from shore, it is difficult to believe the ships could not have maneuvered out of reach. It has been conjectured that the claw was used at night, when the sailors could not see it approaching. But perhaps the main purpose of the claw was to keep the ships from attempting to land at all. In that case, it would indeed have been a practical defensive weapon that needed only to be demonstrated once, not regularly used. The same explanation would apply to Archimedes' supposed use of burning mirrors to set the Roman ships on fire. Revolved conic sections were well enough understood in his day to allow the design of such mirrors. Modern experimenters have found it difficult to set wood on fire in this way, but it is not necessary to set a ship on fire to make it unbearably hot for its human occupants. Thus, these two stories merit a somewhat higher degree of confidence than the story of the crown. If indeed these stories are only legends and not true reports, one can imagine how they may have arisen, since Archimedes wrote treatises on floating bodies, the lever, and the parabola.

Anecdotes reported by people who do not claim to have been eyewitnesses need to be used with caution when reconstructing serious history, precisely because mathematics is a logical subject, but mathematicians do not always discover their results in the seemingly natural logical order. There is a temptation to let logic be one's guide and conjecture an origin for mathematical ideas that is largely or entirely wrong. For example, Brouwer's proof of his fixed-point theorem was not constructive. From the point of view of his intuitionism, what he had proved is that *either* every continuous mapping of a closed ball into itself has a fixed point *or* the axioms of analysis are inconsistent. (There are now constructive proofs of this theorem.) I have heard rumors that it was this nonconstructive nature of the theorem that set Brouwer on the path to intuitionism. Although it seems psychologically natural that he should have developed it in this way, there is good documentary evidence pointing to a different conclusion. Brouwer was writing about the foundations of mathematics already in his 1907 doctoral dissertation, whereas his topological results were obtained between 1909 and 1913. Thus it was not doubts about topology that led him to intuitionism. However, his intuitionism did cause him to worry about the validity of his topological results. As van der Waerden said [8]:

> Even though his most important research contributions were in topology, Brouwer never gave courses on topology, but always on—and only on—the foundations of intuitionism. It seemed that he was no longer convinced of his results in topology because they were not correct from the point of view of intuitionism, and he judged everything he had done before, his greatest output, false according to his philosophy.

In most cases involving the modern era, there are enough documents to produce a clear picture of mathematical developments, and conjectures for which there is no eyewitness or documentary evidence are not needed. Even so, legends do arise. (Who has not heard the "explanation" of the absence of a Nobel Prize in mathematics?) The situation is different regarding ancient mathematics, however, especially in the period before Plato's students began to study geometry. Much of the prehistory involves allegations about the mysterious Pythagoreans, and sorting out what is reliable from what is not is a tricky task.

In this article, I will begin with some modern anecdotes that have become either legend or folklore, then work backward in time to take a more detailed look at Greek mathematics, especially the Pythagoreans, Plato, and Euclid. I hope at the very least that the reader finds my examples amusing, that being one of my goals. If readers also take away some new insight or mathematical aphorisms, expressing a sense of the worthiness of our calling, that would be even better. Still better would it be if readers become wary of inferring "the way it must have been" by looking at the mathematical achievements of the past through the lens of the present and learn to look for documentary or eyewitness confirmation before

repeating a legend. Best of all, I think, would be a reader who is inspired to undertake a systematic investigation of the many mathematical legends that are "out there", waiting to be studied. There are many examples of such anecdotes ([18], [9]), and a few examples of systematic investigation of them ([12], [10]). A definitive book on the subject remains to be written.

By way of preview, I note that both Plutarch (quoted in the section "The Pythagorean Theorem", below) and Vitruvius (again in the introduction to Book IX of *De architectura*) say that Pythagoras discovered the 3-4-5 right triangle. Vitruvius says he made a sacrifice of oxen after the discovery, but Plutarch says that sacrifice was occasioned by his discovery of the transformation of areas. This example shows how the same story can be attached to different discoveries—or perhaps that Pythagoras routinely sacrificed oxen.

Some Modern Legends

Even in the full light of day provided by modern journalism, legends arise quite easily, creating a large number of hits at websites where they can be reliably checked, such as `snopes.com`. One of those legends, carefully checked out by Snopes and uncharacteristically confirmed in most of its aspects, is a story about George Bernard Dantzig (1914–2005). According to legend, Dantzig arrived late for class one day in 1939, too late to hear that his professor Jerzy Neyman (1894–1981) had posted two unsolved problems on the board. Thinking they were homework, he went home and solved them, although it took a little longer than usual because they seemed to be rather difficult. All that is confirmed by Dantzig himself [1].

Such stories are the stuff of which legends are made, and this much of the legend is well attested. But the story does not end there. Dantzig told the story to Robert Schuller, the Lutheran minister best known for his broadcasts from the Crystal Cathedral. Schuller got Dantzig's permission to use the story in a book on what he called "possibility thinking". From there, it found its way into sermons all over the country, one of which was heard by Donald Knuth while visiting Indiana. In the sermons—fortunately or unfortunately, depending on one's view of legends—the story was embroidered with a few details that should have aroused suspicion in a careful listener or reader. It was said that the two problems were part of an examination, that Einstein had worked on them without success, and that Dantzig had immediately been offered a position at Stanford on the basis of this work. It appears that what Schuller thought he heard is not quite what Dantzig said.

The story continued to expand, reaching out from Dantzig to embrace at least one other eminent mathematician. A version essentially the same down to minor details was told to me by my undergraduate roommate in 1962, only in this version the student was said to be John Milnor. In this case, we can see how colorful details attach themselves to a core of facts, like barnacles on a ship. The mathematical result in this case was the Fáry-Milnor theorem, which Milnor proved as an undergraduate. Milnor did indeed hear about this problem in the classroom, from Al Tucker. But there is no indication in his own account [22] that he arrived late, or that he mistook the statement of the theorem for a homework problem. The colorful version of the story was widely believed among graduate students at Princeton at the time, and Dennis Sullivan once took the time to explain to me the technique that Milnor had used for solving the problem.

Unpublished Theorems

In the modern world, where so much mathematics gets into print, legends tend to involve the personal rather than the technical side of mathematics. There are many legends about the sayings and doings of famous modern mathematicians, almost none involving theorems or proofs passed along by an oral tradition. The closest approximation to an oral tradition comes from results tossed off as remarks because they are not regarded as serious or profound enough to merit formal statement as theorems. These may get passed around and repeated without citation, so that their origin may become obscure unless someone writes it up in a historical piece. An example is the following proof of the infinitude of the primes, reported in the reminiscences of Luzin's Moscow school of mathematics by L. A. Lyusternik ([21], p. 176).

According to Lyusternik, exotic proofs of the infinitude of primes were a routine challenge among Luzin's students, and many such proofs were found. But apparently no one thought of publishing them. The one that follows remained unpublished until Lyusternik wrote his reminiscences. The proof, which Lyusternik ascribed to Khinchin, is based on Euler's formula

$$\prod_{p\,\text{prime}} \left(1 - \frac{1}{p^2}\right)^{-1} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}\,.$$

If the set of primes were finite, the left-hand side of this formula would be a rational number, and hence π^2 would be rational.

Such proofs attract interest because they make unexpected connections. They often seem slightly whimsical, especially if (as in the case of Khinchin's proof) the principle invoked to prove the result is more complicated than the result itself. A good example is the remark of Loomis that the maximal ideal space of the Banach algebra of bounded continuous complex-valued functions on

a completely regular topological space X is the Stone-Čech compactification of X ([20], pp. 55–56).

Khinchin's proof does not make it into the class of legends, since it eventually got written down and published with the correct (I presume) attribution. The other proofs alluded to by Lyusternik no doubt lived on in the memory of those who heard them, but I do not know of any written exposition of them. They can be classified as legendary but well authenticated. If anyone has heard them and remembers what they are, writing them down would be a small service to people interested in the minutiae of mathematics.

A referee has pointed out that Khinchin's proof wasn't a new result. Dickson ([7], pp. 413–415) gives two pages of citations of standard and exotic proofs of the infinitude of primes. In particular, he notes a pamphlet by J. Braun (dates and first name unknown) [2], in which Khinchin's argument was ascribed to Jacob Hacks (1863–??). Moreover, the principle used in the proof has a long history. Euler had used the formula

$$\prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1} = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

taking $s = 1$, to argue that if the number of primes were finite, the right-hand side would be infinite and the left-hand side finite. Of course, the formula asserts only that one infinite quantity equals another when $s = 1$. But it *would be* a true equality between finite real numbers if the number of primes were finite, and so Euler was right: The fact that the right-hand side is infinite does prove that the number of primes is infinite. Kronecker pointed out that Euler's proof can be rescued in another way by simply letting s decrease to 1 on both sides ([7], p. 413).

Aphorisms

One type of legend in the history of mathematics is the species known as the Famous Quotation. Every mathematician has a stock of these: Pythagoras's "Number is all," Archimedes' "Give me a place to stand and I will move the earth," Galileo's "Nevertheless, it *does* move," Laplace's "I have no need for that hypothesis," and the like. These quotations seldom, if ever, arise from eyewitness testimony. Aphorisms are sometimes attributed to more than one person, and who can say whether they might not have been spoken by more than one person? They float around in the mathematical community like ions in a solution, opportunistically fastening on any attractive object. The following is a modern example, which again relies on my own far-from-infallible memory.

I took William Feller's 1963 course in probability, in which he mentioned that obviously every collection of sets generates a minimal σ-field, obtained as the intersection of all σ-fields containing the collection. He then said, as nearly as I can remember, "I hope you understand this type of argument. When it was first given, Kronecker said, 'That is not mathematics, that is theology!'" Given the reputation Kronecker has acquired among mathematicians as a champion of traditional mathematics and an opponent of abstraction—a reputation not entirely deserved, by the way, as he clearly saw the advantages of abstract group theory as early as 1870—this does seem typical of the kind of thing he *would* say, like his aphorism, "The good Lord made the integers; everything else is a human creation." [1]

Later, when I read Gerhard Kowalewski's mathematical autobiography *Bestand und Wandel* (*Permanence and Change*), I encountered the quotation again, this time ascribed to Paul Gordan. According to Kowalewski, it had nothing to do with σ-fields or ethereal set-theoretic arguments. Gordan supposedly said it when he first looked at Hilbert's proof of the Hilbert basis theorem in 1888. Here is what Kowalewski wrote ([17], pp. 24–25):

> Hilbert then came along and proved that the theorem holds in general, not only for forms in one variable, but for systems of forms in n variables…That was a mathematical achievement of the very first rank. Gordan said at the time, "That is *no longer* mathematics [Das ist nicht mehr Mathematik], that is theology." Whenever such a mighty discovery is made, one has the sensation that a ray of light from a higher realm has penetrated our mundane darkness. That is probably what Gordan meant by his exclamation.

The importance of statements like Gordan's lies in what they tell us about the motives and attitudes of mathematicians. Kowalewski's view was that, since theology brings heavenly knowledge down to earth, it becomes the standard by which knowledge is judged, and rarely does a human discovery such as the Hilbert basis theorem merit comparison with it. Kowalewski was a deeply religious man, and his view of theology probably led him to this interpretation.

Unfortunately, what Kowalewski reported is not quite what Gordan said, as set down in Max Noether's obituary notice ([24], p. 18), and his interpretation is certainly not what Gordan meant. Gordan supposedly said "That is *not* mathematics" ("Das ist keine Mathematik"). He thought Hilbert's

[1] *This quotation first appeared in print in Heinrich Weber's obituary of Kronecker in the 1893* Jahresbericht der deutschen Mathematiker-Vereinigung, *where it was said that Kronecker had made the statement in 1886. The German is "Die ganzen Zahlen hat der liebe Gott gemacht. Alles andere ist Menschenwerk." I have somewhat inaccurately translated "Der liebe Gott" as "The good Lord" since the German exclamation "Lieber Gott!" has approximately the force of the English "Good Lord!".*

arguments were not sufficiently robust to establish what Hilbert was claiming, and he strongly advised Felix Klein not to publish Hilbert's result in the *Mathematische Annalen*. Fortunately for all concerned, Klein did not heed that advice, and even Gordan later became reconciled to Hilbert's methods.

While I have written citations asserting that Gordan made this statement, I have only my memory of what Feller said as evidence that Kronecker said it in connection with set-theoretic arguments. (He could not have said it in reference to σ-fields, which were not invented until after his death.) That by no means proves that Feller got the story wrong. If Gordan's statement was well known in the years following 1888, one might imagine that Kronecker repeated it in disparaging the kind of reasoning one finds in set theory. But I don't think he did. Kronecker had a high opinion of theology and a low opinion of set theory; that would seem inconsistent with his using the former to disparage the latter. Although I am confident that Feller related this anecdote in connection with measure theory, the person quoted may not, after all, have been Kronecker.

Some Ancient Examples

Here is a more famous example of the free migration or association of an aphorism. It forms part of that stock of quotations we all carry around with us. The first version of the quotation comes from the commentaries on Euclid's *Elements* by Proclus, written in the fifth century:

> The latter [Euclid] lived in the time of the first King Ptolemy. For Archimedes, who came after the first Ptolemy, mentions Euclid, while on the other hand it is also said that Ptolemy once asked Euclid if there was some way to geometry shorter than the *Elements*, and he replied that there was no king's highway to geometry.

The second version comes from an anthology of literary excerpts compiled by Stobaeus (Book 2, Ch. 31, § 115) at the same time that Proclus was writing, or a little later:

> Alexander demanded that the geometer Menaechmus give him a short course in geometry. But he said, "O king, in the physical world there are private roads and royal roads, but in geometry there is only one road for everybody."

Before I comment on these quotations, let me provide a bit of exegesis to clarify the highway metaphor. The word I have translated as *highway* in the first quotation (ἀτραπός) means literally *without turning*. The royal highways in the Hellenistic world were the equivalent of the U.S. interstate highway system, with straight roads that connected cities remote from one another, in contrast to the private roads (ἰδιωτικαὶ ὁδοί), which were more like county roads meandering around and connecting nearby villages with one another. If Menaechmus and Euclid had known about railroads, they might have said, "There is no express train to geometry." Both quotations warn that one cannot simply proceed directly to the major results in geometry. It is necessary to make many stops, learning the definitions and lemmas before the theorems.[2]

Which, if either, of these statements was really spoken by the person alleged to have done so? Did Menaechmus make this reply to Alexander, and Euclid copy him in his reply to Ptolemy? Or was the story told about one of them, after which it became inaccurately attached to the other? Or were both quotations simply invented by imaginative biographers? What really matters is that, after many centuries of repetition, what they assert has become folklore, expressing a sentiment shared by mathematicians.

The image of a road in geometry may have been common in the ancient intellectual world. In his commentary on Plato's *Phaedo* (Ch. 11, § 13), the sixth-century philosopher Olympiodorus inverts this metaphor in disparaging the use of diagrams in geometry, saying, "For among the geometers there is a distinction [between a true line and one drawn with instruments], since there is only one road, a narrow one, and it is the highway [ἀτραπός] and not the public road [λεωφόρος, the "people-bearer"] that masterfully holds it together." Earlier (Ch. 5, § 4), Olympiodorus had reported that it was a Pythagorean precept to use the highway and eschew the local road. He explained that to take the highway meant to seek with purity (with the intellect and not the senses) and that the "public road" was the way chosen by the multitude, meaning apparently that they rely on objects of sense such as diagrams.

For mathematicians who are not seriously interested in history, there is no need to address the problem that legends pose for the historian, which is to reconstruct what most likely happened in the past based on what is told in the surviving documents. It is in this area, mostly ancient history, that resorting to logic and imagination can lead to

[2] *Another, less plausible, interpretation comes from the phrase I translated as* in the physical world, *which is* κατὰ μὲν τὴν χώραν, *literally* now, according to space. *The phrase* κατὰ τὴν χώραν εἶναι *means* to be in one's proper place. *I would like to think that Menaechmus said, "O king, there are private and royal roads to keep people in their proper places, but in geometry there is only one road for everybody." That would credit geometry with creating a democracy of the mind working in opposition to the class divisions in society. However, I doubt if this interpretation is correct. On the other hand, most ordinary people were too poor to undertake long journeys and therefore were unlikely to be on the highways.*

interesting but usually unprovable reconstructions. No harm is really done unless the imaginative version gets repeated many times and hardens into a "fact".

Reconstructed Legends: The Pythagoreans

The richest vein of legend in all of mathematics is associated with the name of Pythagoras. The theorem to which we now attach his name has appeared in many times and places, and nationalistic priority disputes for the credit of having discovered it continue. Both the prehistory of the theorem and Pythagoras's connection with it have given rise to legends. Besides the main legend that Pythagoras discovered and proved this theorem, sacrificing an ox in thanksgiving for the discovery, there are also conjectures that have become "facts" about the role this theorem played in ancient Egypt and in the discovery of irrational numbers.

Much of the standard view of Pythagoras and the Pythagoreans was upset nearly fifty years ago with a lengthy study by Walter Burkert ([3],[4]), in which the efflorescence of the Pythagorean legend over the centuries was thoroughly documented. Given that the best testimony for early Greek mathematics is the summary by Aristotle's pupil Eudemus, used as a source by Proclus, Burkert ([4], pp. 449–451) inferred that the Pythagorean portion of Euclid's *Elements* is limited to some isolated results (including, however, all of Book IV, which gives the constructions for inscribing regular polygons of 3, 4, 5, 6, and 15 sides in a circle and converse constructions).

As a result of Burkert's work, Cuomo ([6], p. 30) is rightly cautious when broaching the subject of the Pythagoreans:

> Indeed, everything we know about their *mathematical* discoveries and interests comes from later, often much later, centuries and is generally thought to be unreliable—which is why the reader will not find much about Pythagoras and the Pythagoreans in this chapter.

Burkert's analysis seems to be the current consensus on the Pythagoreans (see [14], pp. 68–69). It is a commonplace, however, that any historical consensus is liable to be overturned. For example, Burkert ([4], p. 454), in discussing the supposedly Pythagorean theory of application of areas in Book VI of the *Elements*, says:

> Scholars now agree that the point of these exercises is primarily algebraic; they provide an equivalent for quadratic equations. In Babylonian mathematics they had been solved algebraically, and the individual examples of the application of areas correspond exactly to the methods developed there. Thus the "geometrical algebra" of the Greeks is revealed as the transposition of Babylonian techniques of calculation into a geometrical form.

That passage was correct when it was written. At the time *some* scholars had embraced the "geometrical algebra" interpretation of Books II and VI. For example, van der Waerden ([29], p. 63) considered a problem from the cuneiform tablet AO 8862, in which the following cut-and-paste geometric problem is solved: *The area plus the excess of length over width is* 183*; the sum of length and width is* 27*. What are the length, width, and area?* Van der Waerden decided it would be permissible to ignore the geometric meaning of these terms, since they are indeclinable, and he concluded that "We can therefore safely put the problem in the form of 2 algebraic equations" (namely $xy + x - y = 183$ and $x + y = 27$). Because there are such beautiful correspondences between the algebraic operations performed on the cuneiform tablets and the propositions in Book VI of Euclid, van der Waerden ([29], p. 119) enthusiastically endorsed the name *geometric algebra* proposed by a number of historians, saying that "this geometric algebra is the continuation of Babylonian algebra." Thus, even though there are no Greek records indicating any familiarity with these cuneiform tablets, imagination and logic provided a very attractive hypothesis of transmission from Babylon to Alexandria.

However, in 1975, this hypothesis was savaged by Unguru [28], and it has been on the defensive since that time. Where earlier scholars were eager to find evidence of "transmission" from ancient Iraq to the Greek world, scholars today (see [26], p. 281) tend to minimize the use of explanation by transmission, emphasizing the distinctness of Babylonian and Greek mathematics.

In particular, Pythagoras's connection with the theorem that bears his name is highly doubtful. Burkert says ([4], p. 429), "There is no testimony that he gave a strict proof of the theorem, and this cannot be made to seem probable." And, in a footnote:

> The classic "windmill proof" comes from Euclid, but more primitive proofs are possible…It is pure guesswork to suggest that Pythagoreans tried anything of the sort.

Yes indeed. And pure guesswork blossoms into legend very easily. Van der Waerden, who made a point of debunking one of these legends, was cited by Unguru as one of the principal offenders in the propagation of the "geometric algebra" legend. (While I have no opinion that matters on the subject of transmission from Mesopotamia to Greece, I must confess that I find the geometric algebra hypothesis an attractive one and relinquish it very reluctantly.)

The Pre-Pythagorean History of the Theorem

A cuneiform tablet (BM 85 196) dating back more than a millennium before Pythagoras describes a problem involving a beam leaning against a wall, whose solution is found by implicitly invoking what we know as the Pythagorean theorem. The geometric formulation of this problem is crucial, since the mere occurrence of what we call *Pythagorean triples*, that is, integers x, y, and z such that $x^2 + y^2 = z^2$, proves nothing about any knowledge of right triangles. Such number sets might have arisen in solving certain problems in pure number theory having no connection at all with geometry. The cuneiform tablet VAT 8402, for example ([23], p. 76), contains a table of values of $n^2 + n^3$. What it was used for is not known, but one does not immediately suspect a geometric application. On the other hand, what is perhaps the most famous cuneiform tablet containing Pythagorean triples, Plimpton 322 ([26], pp. 110–115), provides several indications that its application is cut-and-paste geometry. The claim one sometimes hears that it contains a *proof* of the Pythagorean theorem, however, strikes me as very implausible. The cuneiform tablets simply don't contain proofs in that sense. We know the writers of cuneiform tablets knew the theorem because they applied it. We don't know what convinced them that it was a reliable principle.

It is a reasonable inference that the geometric fact expressed by the Pythagorean theorem was known, perhaps empirically or perhaps through some simple abstract geometric considerations, at least a thousand years before Pythagoras. This theorem was also discovered in other places. An analysis of a rectangle occurs in the Chinese classic *Zhou Bi Suan Jing*, which dates back at least as far as the time of Euclid, and shows that the diagonal c of a rectangle of sides a and b satisfies the relation that we would write as $c^2 = (a+b)^2 - 2ab$. It is also said ([25], p. 18) that the *Sulva Sutras*, which predate Pythagoras, contain the statement that "the diagonal of a rectangle produces both areas which the legs produce separately," exactly the form of the Pythagorean theorem we now recognize. In these cases, we can be sure we are dealing with the geometric theorem.

Egyptian "Use" of the Pythagorean Theorem

No legend that is utterly without foundation has so well stood the test of time as the story that the ancient Egyptians not only knew the Pythagorean theorem but actually used it in constructing right angles. This legend has legs! You can find it all over the Internet and even in literature published by mathematical societies. Valiant attempts by historians of mathematics (most recently [15], pp. 791–793) to give this story the plausibility it deserves (not very high) have been powerless to stop the juggernaut of long-repeated rumor.

It is not beyond the realm of possibility that the ancient Egyptians knew about the 3-4-5 right triangle. For one thing, Plutarch says they did in his essay *Isis and Osiris* (373f–374b). In addition, some Egyptian papyri contain Pythagorean triples. What is legendary here is not Egyptian *knowledge* of the theorem but its *use* in construction. Plutarch mentions this triangle only in connection with the mystical properties of numbers, Osiris being the upright side of length 3, Isis the base of length 4, and Horus the hypotenuse of length 5, and he goes on to relate all this to the properties of male and female numbers, in other words, exactly the kind of mystical numerology we have come to associate with Pythagoreanism and with Platonism, for example, in the arranging of marriages in the *Republic* (546c). But Plutarch does not mention any connection with engineering or construction.

What seems to have triggered the connection with construction is a statement in the *Stromata* (*Miscellanies*) by Clement of Alexandria in the second century (Book 1, Ch. 5, § 69.5), in which he quotes a reference by Democritus to the "rope-fasteners" of Egypt. Ropes and chains were among the standard tools of surveyors for millennia, and there are wall paintings in Egypt showing ropes being used to measure fields. But there are no such paintings of Egyptian engineers stretching ropes into the shape of a triangle.

The two ingredients of the legend seem to have lain dormant and separate from each other for centuries until they were combined in a history of mathematics published by Moritz Cantor in 1880. Here is what Cantor wrote ([5], Vol. 1, p. 56). I urge careful attention to the first sentence, in which I have added the emphasis:

> Let us suppose, *without any actual evidence, however*, that the Egyptians knew that when three sides of lengths 3, 4, and 5 are joined in a triangle, they form a right angle between the two shorter sides. Further let us suppose that the pegs [marking the corners of one side of a building] are 4 units apart along a meridian. Finally, let us suppose that the rope is 12 units long and divided into lengths 3, 4, and 5 by knots. It is then clear…that when the rope is pulled taut at one knot while the other two are laid down at the pegs, it will necessarily form a right angle with the meridian.

Cantor had preceded that quotation with an essay on the supreme importance of constructing accurate right angles for such important buildings as temples and concluded that some precise geometric construction must have been necessary. But why this particular construction? One could also stretch a rope taut to form a straight line and

(with one end fixed) a circle. By doing that, one could draw right angles as we were all taught to do in school.

If we wish to dream up a way in which Egyptian surveyors and contractors might have laid out a rectangular building, ancient tradition provides plenty of material for conjecture. Thales, for example, is said to have learned geometry in Egypt and is also said to have known that an angle inscribed in a semicircle is a right angle. Putting these two attributes together, we could assume that he learned the geometric fact from the Egyptians. Perhaps he saw them laying out the foundation of a large rectangular building by drawing a circle whose diameter was a diagonal of the building, then simply drawing a chord along one of the sides of the projected building. The connection with "rope-fasteners" would also hold for such a construction.

This legend was summarized by B. L. van der Waerden (in the preface to [29]):

> Is this not incredible? Not that Cantor at one time formulated this hypothesis, but that repeated copying made it a "universally known fact".

What most likely happened is that the interesting conjecture began to be repeated out of context, and Cantor's warning that there was no direct evidence for the conjecture simply got left out. I first saw it in its full developed form in my high-school algebra text over fifty years ago. Van der Waerden gives some speculation of his own on the line of reasoning that led Cantor to this hypothesis. I shall quote it here, calling it a *pseudo-syllogism of type* 1:

(1) These right angles must have been constructed by the rope-stretchers.
(2) I (Cantor) cannot think of any other way of constructing a right angle by means of stretched ropes than by using three ropes of lengths 3, 4, and 5, forming a right triangle.
(3) *Therefore* the Egyptians must have known this triangle.

The source of the legend. Now that this legend has grown to its present extent, it begins to take on an interest of its own, independent of its accuracy as history. Until recently, it never occurred to me that it might have originated anywhere except in Cantor's history. However, in a book [19] giving 367 proofs of the Pythagorean theorem, Elisha Scott Loomis (1852–1940) quoted in English translation a passage from a German book [30] by one Jury Wipper that gave 46 proofs of the same ([19], p. 11):

> Fifteen hundred years before the time of Pythagoras…the Egyptians constructed right angles by so placing three pegs that a rope measured off into 3, 4, and 5 units would just reach around them, and for this purpose professional "rope fasteners" were employed.

I have not been able to find a copy of the German book by Wipper cited by Loomis. However, I find it very intriguing that it was published in the same city and year as Cantor's book and reports Cantor's conjecture as fact. It may be that Cantor had published this conjecture earlier. Jury Wipper was the Russian scholar Yurii Vipper (1824–1891), a generalist rather than a specialist, who wrote books on a variety of scientific and religious topics. The 1880 German book is a translation of the Russian original, published in Moscow in 1876 under the title *Forty-five Proofs of the Pythagorean Theorem* (evidently, one proof was added in translation). At my request, S. S. Demidov checked the Russian original out of the Russian State Library in Moscow and reported to me what Vipper had to say about the history of this theorem (p. 2):

> Pythagoras, who lived in Egypt for 21 years and in Babylon for 12, could have learned the properties of the [3-4-5] right triangle in either country. As for the geometric proof, it most likely is due to Pythagoras himself.

It appears, then, that the passage quoted by Loomis was added to the book in the process of translating it into German. Interestingly, Vipper does mention Cantor (*Mathematische Beiträge zur Kulturgeschichte der Völker*), but only in connection with the origins of the Pythagorean theorem in China.

The Pythagorean Theorem

What was the connection of Pythagoras with this theorem, and how did it come to bear his name? We rely on the commentators for what we know of Pythagoras, since there is no mention of him in the writings that have come down to us from people who knew him personally, with the possible exception of Heraclitus, who sneered at him. If there is a documentary chain from us back to him, some of the early links disappeared, presumably after being partly incorporated in the writings of the commentators. Among these commentators were Aristotle, Plutarch, the third-century writers Porphyry Malchus and Diogenes Laertius, and the early fourth-century writer Iamblichus. Most of these writers do not mention the Pythagorean theorem. However, some of them do, and here is what two of them say.

Plutarch, in *Nine Books of Symposium Topics* (usually called by the Latin name *Questiones Conviviales*), Book 8, Problem 2, "Why does Plato say that God eternally geometrizes?":[3]

[3] *Here again, we encounter a legendary quotation. Plato does not make any such statement explicitly in the extant dialogues.*

> Among the most geometric theorems, or rather problems, is the following: *Given two figures, to construct a third equal [in area] to the one and similar to the other.* It is said that Pythagoras offered a sacrifice on the discovery of this theorem. It is insufficiently appreciated that this theorem is more refined and elegant than the theorem proving that the square on the hypotenuse is equal to those on the two sides enclosing the right angle.

Proclus, in his commentary on the first book of Euclid:

> Proposition XLVII, Theorem XXXIII. *In a right triangle, the square on the side opposite the right angle equals the squares on the two sides enclosing the right angle.* To hear those who like to research [*historein*] the origins of things tell of it, this theorem is to be attributed to Pythagoras, and they say that he sacrificed an ox upon the discovery of it. But while I admire those who first understood the truth of this theorem, I am much more impressed by the author of the *Elements*…

From Proclus's words, it seems there was by his day a tradition of connecting this theorem with Pythagoras. What the contribution of Pythagoras to it was, however, is not known. Proclus goes on to say that the theorem is much less impressive than the generalizations of it that Euclid produces in Book VI. That also is the place where the problem of transformation of areas, mentioned by Plutarch, can be found.

The Discovery of Incommensurables

In his book *On the Pythagorean Life* (Ch. 34, § 247), Iamblichus says that "The first who disclosed the nature of commensurability and incommensurability, making it possible for the uninitiated to take part in the discussion, was so hated that not only was he expelled from the society and its way of life, but his funeral was conducted, as if he had already departed from the life of the men whose comrade he had once been." Iamblichus does not say who this wretch was. If the sources Iamblichus used are reliable and he was not mistakenly ascribing a later result to the Pythagoreans (always a possibility), then the Pythagoreans were somehow able to discover that certain pairs of lines in geometry have no common measure. The likeliest candidates for such pairs are the simplest: the side and diagonal of a square or the side and diagonal of a pentagon.

The way in which incommensurables were discovered is not known with certainty, and as usual, certain hypotheses have been repeated so many times that they are frequently told as established fact. The reasoning seems to incorporate the principles of a pseudo-syllogism of type 1. It goes as follows:

(1) One commentator says the Pythagoreans knew about incommensurables.
(2) I personally am aware of one prominent achievement attributed to the Pythagoreans, namely the Pythagorean theorem.
(3) *Therefore*, the Pythagorean theorem must have been the route by which incommensurables were discovered.

Such reasoning would not be cogent unless there were some mathematical connection between the Pythagorean theorem and incommensurable pairs of lines, and of course there is: The theorem shows that the diagonal of a square is $\sqrt{2}$ times the side and that the diagonal of a pentagon is $\frac{1+\sqrt{5}}{2}$ times the side, and both of these constants are irrational numbers.

From these ingredients, one can prepare a rather common version of the history, found in many places, among them Chapter 3 of Bertrand Russell's *History of Western Philosophy* [27]. Apparently, the reasoning is that there was no need to deal with the number $\sqrt{2}$ until geometry, via the Pythagorean theorem, forced the confrontation.

To that, one can only say "Perhaps." The number-theoretic Books VII–IX of the *Elements* have some results on square and cube integers. If these are Pythagorean in origin, it does not seem far-fetched that the Pythagoreans would have sought fractional but rational square roots of integers for purely arithmetic reasons.

The version of the discovery as described by Russell contains what I call a *pseudo-syllogism of type* 2, which has the following form:

(1) Mathematician N proved Proposition A.
(2) Proposition A implies Proposition B.
(3) *Therefore*, Mathematician N proved Proposition B.

The fallacy in this pseudo-syllogism was neatly formulated long ago by Abbé Charles Batteux (1713–1780): One should not assume that ancient authors perceived either the consequences of their principles or the principles from which their results could be derived (cited by Burkert, [4], p. 405).

One might argue that as a mathematical demonstration, this syllogism is not "pseudo", since proving A when A implies B does establish that B is true. However, that statement presumes we *know* that A implies B. Historically, that is precisely the issue. If the mathematician who proved A didn't know that A implies B, then he/she had not proved B. In the present case, one would know the Pythagorean theorem implies that $\sqrt{2}$ is irrational only if one knew that the side and diagonal of a square were incommensurable. One could prove the latter geometrically by showing that the Euclidean algorithm applied to these lengths cycles with period 2 and hence does not terminate. Otherwise, one might as well start from scratch to

prove that 2 has no rational square root; the proof doesn't at all require the Pythagorean theorem, only the properties of even and odd numbers. I'm inclined to think that that is what the Greeks did.

It was mentioned above that an unnamed Pythagorean was said to have been "made dead" to the Pythagoreans for not keeping the existence of incommensurables a secret within the Pythagorean brotherhood. Legends tend to grow and become more elaborate. This unnamed unfortunate frequently acquires the name Hippasus and is said to be the discoverer of incommensurables and also the leader of a schismatic group of Pythagoreans. It is also sometimes said that he was thrown overboard by angry Pythagoreans. There apparently was a man of this name among the Pythagoreans. According to Iamblichus (*On the Pythagorean Life*, Ch. 18, § 88), Hippasus perished at sea, but his crime was not revealing the fact that there are incommensurable pairs of lines. It was that he revealed "the sphere of the twelve pentagons" to noninitiates of the Pythagorean mysteries. Moreover, it was not merely the revelation that doomed the hapless Hippasus. His offense was taking credit for this discovery, when the Pythagoreans were supposed to attribute all their results to The Man, that is, Pythagoras himself. Iamblichus says nothing about Hippasus having been thrown overboard by outraged Pythagoreans. ("The Man" is exactly the name by which the Pythagoreans called their legendary leader. Its resemblance to modern American argot is coincidental.)

What then becomes of the alleged connection between the Pythagorean theorem and irrational numbers? It is a possibility, but it is not established beyond a reasonable doubt. There are other possible scenarios by which incommensurables could have been discovered.

The arithmetic fact that there are no integers m and n such that $m^2 = 2n^2$ is always demonstrated by very simple even-and-odd reasoning having no connection with geometry at all. There is a similar argument by which one can show that there are no integers m and n such that $m^2 = kn^2$ for any integer k not equal to a square and not congruent to 1 mod 8, since the square of any odd integer *is* congruent to 1 mod 8. Knorr ([16], Chs. III and VI) made a very good case that the original reasoning by which *some* square roots of integers were proved irrational was exactly of this type. Knorr analyzed Plato's dialogue *Theaetetus*, in which it is reported that Theodorus proved the irrationality of the square roots of all nonsquare integers less than 17, but "there he somehow got stuck." Knorr conjectured—plausibly, I believe—that Theodorus got stuck at that point because his argument broke down, 17 being the first nonsquare integer that is congruent to 1 mod 8.

It thus appears that in Plato's time a general proof that the square root of a nonsquare integer is irrational had not been found. However, the Greeks did eventually find a proof. Proposition 22 of Book VII of the *Elements* asserts that if three integers are in proportion and the first is a square, then the third is, also. In our language, if a, b, and c are integers, $a/b = b/c$, and a is the square of an integer, then c is the square of an integer. (By writing this paragraph, I have deliberately produced a pseudo-syllogism of type 2. Euclid would not have recognized the proposition in the first sentence, since the phrase *irrational number* would have been an oxymoron to him.)

A Modest Proposal

I would like to invite some ambitious mathematician with time on his/her hands to write a monograph on the role played by legends in the mathematical community. Or perhaps someone would be willing to set up the mathematical equivalent of the `snopes.com` website, where mathematical "urban legends" can be checked out.

The kind of investigation I have in mind is typified by the following two examples. First, Jeremy Gray's investigation [12] of a rumor I heard as a graduate student that Poincaré called set theory "a cancer on mathematics". By 1908 Poincaré probably thought set theory *had* a disease from which it would recover (namely its paradoxes), but he didn't think it *was* a disease. The incorrect quotation was spread by E. T. Bell and Morris Kline, and Gray shows how they both came to get the story slightly wrong.

Second, there is a famous story that as a boy, Gauss impressed his stodgy teacher Büttner by performing an amazing feat of mental calculation. This story occurs in the memorial volume *Gauss zum Gedächtnis* published by Wolfgang Sartorius, Baron von Waltershausen, in 1856, the year after Gauss's death. Like other rumors, it has grown with the telling, as we now know, thanks to the diligent research of Brian Hayes [13]. It seems to have been Ludwig Bieberbach who added the detail that Gauss summed the first 100 integers, although, as with so much that mathematicians believe, this version was amplified and popularized by E. T. Bell. According to Bell's version in *Men of Mathematics*, the problem had been assigned to keep the boys busy, and Gauss's solution of it so impressed Büttner that from then on he made an exception for Gauss when applying his usual harsh teaching methods.

I imagine that, as happens at the already-existing `snopes.com`, most of the legends will be refuted. However, even when a legend is refuted, there is some interest in seeing how it arose, as I think there is in the case of the rope-stretching legend. Here are some examples of such exploded myths, which still tantalize. One would like to know how they came into general circulation.

(1) Some twenty years ago, a colleague of mine (actually in the philosophy department) asked me who it was that Euler had confounded with his mathematical "proof" of the existence of God. Being well-versed in legends (but not in the truth behind them), I was able immediately to supply the name of Diderot. The story itself is surely false, however. Just about every aspect of the story arouses suspicion in the careful reader. A referee has provided me with a reference that thoroughly debunks it [11]. The legend arose in Volume 3 of Dieudonné Thiébault's 1804 book *Mes souvenirs de vingt ans de séjour à Berlin*, then was reported in a distorted form by Augustus De Morgan in his 1872 book *A Budget of Paradoxes* and further spread through the efforts of Florian Cajori, Eric Temple Bell, and Lancelot Hogben. Given the obvious fact that the story is false, only historical interest attaches to it, and the primary issue is, "How did something so blatantly implausible ever come to be believed?"

(2) We all "know" that Plato had the motto "Let no one ignorant of geometry enter" at the entrance to his Academy. This legend has been investigated ([10], Ch. 6), and it turns out that the earliest verifiable citation with anything resembling that language occurs some 1500 years after Plato and more than 500 years after the Emperor Justinian closed the Academy. How then did this motto come to be so confidently reported, one of those "universally known truths", referred to by van der Waerden?

(3) Why is the formula $e^{i\pi} + 1 = 0$ called Euler's formula? (A referee suggested this legend as an example of a pseudo-syllogism of type 2.)

(4) Under what circumstances did Newton really solve the brachistochrone problem?

The book/website I am imagining would assemble as many rumors and legends as possible, classify them, investigate their validity, and discuss their significance for the history of mathematics. Even if these "universally known truths" are not true in the straightforward sense, they often contain ideas that are worth contemplating. Rumors and legends are continually arising. In an ideal world, the proper approach is not to try to eradicate them, only to recognize and appreciate them for what they are.

Conclusion

I hope the preceding examples have at the very least amused the reader. Mathematics should have some amusing aspects. On a more serious level, we all feel that mathematics is a dignified and worthy profession; and some of the legends we tell hold up an ideal of nobility for us. Along with this sense of the dignity of the profession, we can, with caution, use legends as hypothesis generators when we try to imagine the parts of the history of our subject that have not left enough mathematical texts to tell us what happened. These three purposes motivated the examples I have presented, and I hope the examples will motivate some ambitious reader to undertake the systematic investigation I have proposed.

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