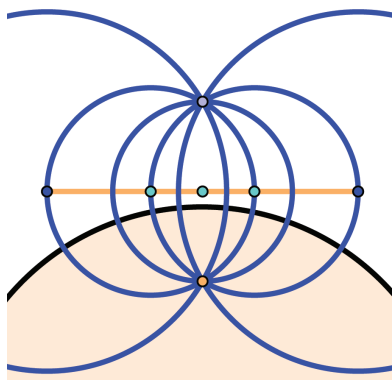


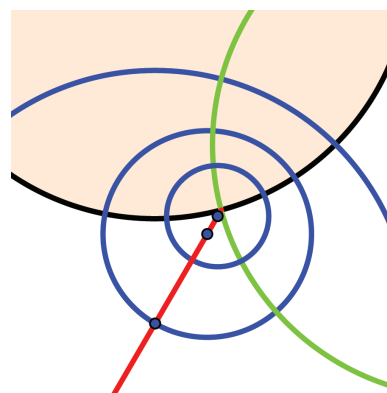
Coxeter and Escher

As Doris Schattschneider mentions briefly in her article on the mathematics of M. C. Escher in this issue, the response of H. S. M. Coxeter to some queries from Escher about hyperbolic tessellations was not entirely satisfactory. Coxeter had sent Escher a reprint of an article on symmetry that included as its Figure 7 the tessellation of the Poincaré disc associated to a $2 - 4 - 6$ reflection group (reproduced as Figure 6 in Schattschneider's article). Escher was much intrigued by this diagram, and wanted to know how to reproduce the figure in his own terms, presumably by compass and straightedge. Relatively elementary considerations based on Pythagoras' Theorem and symmetry enabled him to construct the non-Euclidean arcs bounding several larger triangles of the tessellation, but he did not see how to continue. Coxeter side-stepped the matter rather abruptly. Figure 7 of Schattschneider's article tells much of the story—Escher sent Coxeter a hand-drawn diagram of what he had been able to reproduce, along with the query, "...how are to be constructed further circles with radii shorter again and again ...?" Coxeter sent the diagram back to Escher with some added markings in red, on its reverse side, that indicated only how to draw a line that contained somewhere along it the center of one of the circles that Escher had not been able to find. He gave absolutely no indication of exactly where on that line the center was actually located. His comment that "This can be continued in the same manner" wasn't all that helpful. We are led to ask, why didn't he say more?

The cover image reproduces Coxeter's Figure 7 overlaid with a diagram that makes some attempt to indicate what is going on. Most of what's there is just another version of Schattschneider's figure, which is in turn a version of what Escher included in the drawing sent to Coxeter together with the red point and line that Coxeter added and sent back. There are two basic principles that seem to be required in drawing this tessellation. One is the basic duality between points P inside the Poincaré disc and lines ℓ outside it: *The non-Euclidean arcs through P have their centers on ℓ .*



It is not clear whether Escher was aware of this principle or not. He does draw several lines compatible with this assertion, but we don't know exactly what he deduced from this. To tell the truth, we don't really know whether Coxeter's red additions told him anything he didn't already know. This principle did not alone give him sufficient information to add new circles, and he may have thought there was no point in drawing lines unless they really contributed. The second principle is what he needed: *Suppose given a non-Euclidean arc with endpoints P and Q on the unit circle. The centers of all circles orthogonal to the arc lie on the straight line through P and Q .* This allows one to find centers by intersecting lines drawn according to either or both of these principles, as is shown for the red circle on the cover.



We do not know where this principle was first formulated in the literature, but it is likely that it was used to construct many diagrams to be found in Coxeter's own writings. For example, secant lines—i.e., lines intersecting the unit circle obliquely—apparently used for this purpose can be found in the first figure of his article on Escher's *Circle Limit III* in volume 12 of the journal *Leonardo*. It is unlikely that Coxeter drew those figures himself, however. Somewhere he makes a remark about secant lines in this and similar figures, but it is not clear from what he says that he knows what their purpose is. There is therefore a mystery—what exactly was the reason that Coxeter didn't tell Escher about this technique?

We wish to thank both Coxeter's daughter, Susan Thomas, and the Royal Society of Canada for permission to reproduce his Figure 7, which was originally published in volume 51 of the *Transactions of the Royal Society of Canada*. We also wish to thank Doris Schattschneider for taking part in correspondence involved in figuring out how Coxeter's diagrams were constructed.

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