Hadwiger's Conjecture and Seagull Packing

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The four-color theorem [1, 2] is one of the most well-known results in graph theory. Originating from the question of coloring a world map, posed in the middle of the nineteenth century, it has since fascinated hundreds of researchers and motivated a lot of beautiful mathematics. It states that every planar graph (that is, a graph that can be drawn in the plane without crossings) can be properly colored with four colors. Kuratowski's theorem [15] connects the planarity of a graph with the absence of certain topological structures in it. Thus, the four-color theorem tells us that if a graph lacks a certain topological obstruction, then it is four-colorable. In 1941 Hadwiger made a conjecture that generalized this idea, from four-colorability to general t-colorability. Since then, Hadwiger's conjecture has received a great deal of attention, but only a few special cases have been solved. In this article we survey some of the known results on the conjecture and discuss recent progress.

To make this more precise, let us say that a graph G contains a *K_t-minor*, or a *clique minor of size t*, if there exist t pairwise vertex-disjoint connected subgraphs $F_1, \dots, F_t$ of G, such that for every distinct $i, j \in \{1, \dots t\}$, there is an edge with one end in $V(F_i)$ and the other in $V(F_j)$. (For a graph F, we denote by $V(F)$ the vertex set of F, and by $E(F)$ the edge set of F.) For coloring problems, it is necessary to assume that graphs are *loopless*, that is, every edge has two distinct ends, and we adopt this convention here. We can now state Hadwiger's conjecture [10].

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Conjecture 1. *For every integer $t \geq 2$, every graph that is not $(t-1)$-colorable contains a K_t-minor.*

Let us examine this conjecture for small values of t. For $t = 2$, it states that if a graph is not one-colorable, then it contains a K_2-minor. Indeed, if G is not one-colorable, then there is an edge in G with ends u, v, say; now setting $V(F_1) = \{u\}, V(F_2) = \{v\}$, and $E(F_1) = E(F_2) = \emptyset$, we observe that F_1, F_2 satisfy the conditions of the definition of a K_2-minor. For $t = 3$, it is not difficult to see that if a graph G is not two-colorable, then it contains a cycle C with at least three vertices. Let the vertices of C be $c_1, \dots, c_k$ in (cyclic) order. Set $V(F_1) = \{c_1\}, V(F_2) = \{c_2\}, E(F_1) = E(F_2) = \emptyset$ and make F_3 be the path c_3-...-c_k. Again, F_1, F_2, F_3 satisfy the definition of a K_3 minor, and the conjecture holds. The situation for $t = 4$ is a little more complicated, but still quite simple. Graphs with no K_4-minor are known as *series-parallel graphs*. Series-parallel graphs can be built starting from single vertices by applying a few simple operations. What is important for our purposes is that all series-parallel graphs have a vertex of degree at most two (after deleting parallel edges) and therefore can be colored with at most three colors. Thus Hadwiger's conjecture holds for $t = 4$.

Things become more complicated when we turn to the case of $t = 5$. In this case, Hadwiger's conjecture asserts that every graph with no K_5-minor is four-colorable. Together with Kuratowski's theorem [15], which states that a graph is planar if and only if it has no K_5-minor and no $K_{3,3}$-minor (it is not really important for this article what exactly a $K_{3,3}$-minor is), this implies the four-color theorem. Thus the case of $t = 5$ is quite deep. Wagner [18] showed that Hadwiger's conjecture for $t = 5$ is equivalent to the four-color theorem, and so Hadwiger's conjecture is true for $t = 5$.

Hadwiger's conjecture is also known to be true for $t = 6$. This is a theorem of Robertson, Seymour, and Thomas [17], and the proof also uses the four-color theorem. More precisely, the authors show that every minimal counterexample to Hadwiger's conjecture with $t = 6$ is obtained from a planar graph by adding one new vertex with arbitrary neighbors (these are called *apex graphs*). Since every planar graph is four-colorable, it follows that every apex graph is five-colorable, which contradicts the fact that we were looking at a counterexample to Hadwiger's conjecture with $t = 6$. For $t \geq 7$, Hadwiger's conjecture is open.

Since the proofs of all the known cases of Hadwiger's conjecture rely on the fact that for small t graphs with no K_t-minor are close to being planar, it is interesting to consider the opposite extreme. What about very dense graphs, with very large chromatic number, where we therefore should expect to find very large clique minors? Studying these cases may lead to new proof techniques, since such graphs are certainly not close to being planar. This may also be a good place to look for a counterexample.

Let us therefore turn to graphs with no *stable set* of size three (meaning that there do not exist three vertices, all pairwise nonadjacent). Let G be such a graph, and let $|V(G)| = n$. Since in every proper coloring of G, all color classes have size at most two, it follows that the chromatic number of G is at least $\lceil \frac{n}{2} \rceil$, and so, if true, Hadwiger's conjecture would tell us that G has a clique minor of size at least $\lceil \frac{n}{2} \rceil$. This problem will be the focus of the rest of this article.

Conjecture 2. *Let G be a graph with n vertices and no stable set of size three. Then G contains a clique minor of size $\lceil \frac{n}{2} \rceil$.*

At first, Conjecture 2 looks weaker than the case of Hadwiger's conjecture for graphs with no stable sets of size three. However, it was shown in [16] that Conjecture 2 implies that Hadwiger's conjecture holds for graphs with no stable set of size three.

First let us remark that Conjecture 2 becomes easy if we replace $\lceil \frac{n}{2} \rceil$ with $\lceil \frac{n}{3} \rceil$:

Theorem 1. *Let G be a graph with n vertices and no stable set of size three. Then G contains a clique minor of size $\lceil \frac{n}{3} \rceil$.*

Proof. Take a maximal collection, $F_1, \ldots, F_k$, of vertex-disjoint induced two-edge paths in G pairwise with edges between them (that means that each F_i has three vertices a_i, b_i, c_i, and two edges $a_i b_i$ and $b_i c_i$; and a_i is nonadjacent to c_i in G). Let H be the graph obtained from G by deleting the union of the vertex sets of $F_1, \ldots, F_k$. It is not difficult to see, using the maximality of the collection $F_1, \ldots, F_k$, that $V(H)$ is the union of two cliques, say H_1 and H_2. We may assume that $|H_1| \geq |H_2|$. Let $F_{k+1}, \ldots, F_{k+|H_1|}$ be the one-vertex graphs consisting of the vertices of H_1. Then $F_{k+1}, \ldots, F_{k+|H_1|}$ pairwise have an edge between them. Moreover, since G has no stable set of size three, and a_i is nonadjacent to c_i for every $i \in \{1, \ldots, k\}$, it follows that for every $i \in \{1, \ldots, k\}$, every vertex of $V(G) \setminus V(F_i)$ has a neighbor in $V(F_i)$. Therefore, $F_1, \ldots, F_{k+|H_1|}$ is a clique minor in G. Now an easy calculation shows that $k + |H_1| \geq \lceil \frac{n}{3} \rceil$. Thus we produced a clique minor of size at least $\lceil \frac{n}{3} \rceil$ in G, as required. □

Notice that each of the subgraphs F_i is either a single vertex or an induced two-edge path. Let us call an induced two-edge path in G a *seagull* (in G). Seagulls seem to be a good tool for attacking Conjecture 2, and we will come back to them later in this text. Moreover, in [12, 13] a similar but more general technique was used, in a very elegant way, to obtain large clique minors in graphs with arbitrarily large stable sets.

If Conjecture 2 is true, then there exist $\lceil \frac{n}{2} \rceil$ disjoint connected subgraphs of G, pairwise with edges between them. It therefore follows that at least $\lceil \frac{n}{4} \rceil$ of these subgraphs have at most two vertices. Let us call a clique minor an *edge clique minor* if all the subgraphs $F_1, F_2, \ldots$ have at most two vertices. Seymour made the following stronger conjecture:

Conjecture 3. *Let G be a graph with n vertices and no stable set of size three. Then G has an edge clique minor of size $\lceil \frac{n}{2} \rceil$.*

If Conjecture 3 is false, Hadwiger's conjecture may still be true, but, in order for Hadwiger's conjecture to be true, Conjecture 3 needs to hold with $\lceil \frac{n}{2} \rceil$ replaced by $\lceil \frac{n}{4} \rceil$. Unfortunately, no general results are known for $\lceil \frac{n}{2} \rceil$ replaced by $\lceil cn \rceil$ for any constant $c > 0$. However, recently, Blasiak [4] was able to make progress on Seymour's conjecture by adding a simplifying assumption. He assumed that $V(G)$ is the union of three cliques, and then he could prove that G contains an edge clique minor of size $\lceil \frac{n}{2} \rceil$. For even n this gives Seymour's conjecture (in the case when $V(G)$ is the union of three cliques). The idea of Blasiak's proof is as follows. Let $V(G) = M_1 \cup M_2 \cup M_3$, where M_1, M_2, M_3 are cliques. Let us call an edge of G *big* if its ends are in two different M_i's. Thus for every two big edges e and f, there exists $i \in \{1, 2, 3\}$ such that both e and f have an end in M_i, and therefore there is an edge between e and f. This means that any matching of big edges is an edge clique minor in G (a *matching* is a collection of disjoint edges). Blasiak showed (using Tutte's theorem [3]) that, except in cases that had already been understood, G has a perfect matching of big edges (or a matching of big edges covering all but one vertex, if G has an odd number of vertices), thus producing an edge clique minor of size $\lceil \frac{n}{2} \rceil$.

This is a very nice result, but it is really disappointing that the clique minor it produces is one too small, compared with Hadwiger's conjecture. It is thus very tempting to try to fix this shortcoming. Now, after a few years of frustration, here is a fix and a strengthening [5]:

Theorem 2. *Let G be a graph with n vertices and with no stable set of size three. Assume that some clique of G has cardinality at least $\frac{n}{4}$, and at least $\frac{n+3}{4}$ if n is odd. Then G has a clique minor of size $\lceil \frac{n}{2} \rceil$.*

Theorem 1 asserts that Hadwiger's conjecture is true for a larger class of graphs than that considered by Blasiak, but unlike Blasiak's theorem, it does not guarantee an edge clique minor. In fact, in all the novel cases that Theorem 1 deals with, the minor it produces uses seagulls. Here is an outline of the proof of Theorem 1. Let $p = \lceil \frac{n}{2} \rceil$. By a previous result [16], we may assume that G is p-connected (in fact, it is not difficult to prove that if G is not p-connected, then it contains an edge clique minor of size p). Now let Z be a clique in G of maximum size, and let $F_1, \ldots, F_t$ be a collection of pairwise vertex-disjoint seagulls in $G \setminus Z$ with t as large as possible. Let $F_{t+1}, \ldots, F_{t+|Z|}$ be one-vertex graphs consisting of the vertices of Z. As before, $F_1, \ldots, F_{t+|Z|}$ is a clique minor in G. The main result of [5] is that $t + |Z| \geq p$, and therefore G contains a clique minor of size at least p, as required. Let us remark that here, again (as in the proof that every graph with no stable set of size three has a clique minor of size $\lceil \frac{n}{3} \rceil$), the minor consists of one-vertex graphs and seagulls.

Thus, trying to prove Theorem 1 gives rise to the following question: for an integer k, what are necessary and sufficient conditions for a graph G to have k pairwise vertex-disjoint seagulls? And, of course, there is the corresponding algorithmic question: given a graph G and an integer k, is there a polynomial time algorithm to test if G contains k pairwise vertex-disjoint seagulls? For general graphs G, Dor and Tarsi showed that this problem is NP-complete [7]. But what about graphs G with no stable set of size three? In joint work with Seymour [5], we were able to prove the following (the *five-wheel* is a graph consisting of an induced cycle C on five vertices, and a vertex complete to $V(C)$):

Theorem 3. *Let G be a graph with no stable set of size three, and let k be an integer. Assume that if $k = 2$, then G is not the five-wheel. Then G contains k pairwise vertex-disjoint seagulls if and only if*

- *$|V(G)| \geq 3k$, and*
- *G is k-connected, and*
- *for every clique C of G, if D denotes the set of vertices in $V(G) \setminus C$ that have both a neighbor and a non-neighbor in C, then $|D| + |V(G) \setminus C| \geq 2k$, and*
- *the complement graph of G has a matching with k edges.*

Now the proof of Theorem 1 boils down to checking that the conditions of Theorem 2 hold for the appropriate graph G and integer k. The proof of Theorem 2 is far too complex to be included here, but let us just mention that it uses many beautiful results from graph and matroid theory, such as Hall's theorem [11], the Tutte-Berge formula [3], and Edmonds' matroid union theorem [8].

And what about the algorithmic question? Recently, in joint work with Oum and Seymour [6], we were able to design an algorithm that checks (in polynomial time) whether each of the necessary and sufficient conditions of Theorem 2 is satisfied. The only step that was not previously known is checking the third condition (that is, an algorithm that, given an integer k and a graph G with no stable set of size three, tests whether for every clique C of G, if D denotes the set of vertices in $V(G) \setminus C$ that have both a neighbor and a non-neighbor in C, then $|D| + |V(G) \setminus C| \geq 2k$). We were able to reduce this step to the well-known question of finding a maximum matching in a bipartite graph.

However, even before the result of [6], Theorem 2 could be used in an indirect way to test if a given graph contains k pairwise disjoint seagulls. The algorithm was not as simple conceptually, but it used some interesting tools, and so it is worthwhile mentioning. Let us consider the fractional version of the question. Let G be a graph with no stable set of size three, and let S be the set of all seagulls in G. A *fractional packing* of seagulls in G is a function $f : S \to [0, 1]$ such that for every vertex v of G

$$\sum_{S \in S \text{ s.t. } v \in S} f(S) \leq 1.$$

The *value* of the packing f is $\sum_{S \in S} f(S)$. Now the question is: when does G have a fractional packing of seagulls of value at least k? This question can be easily presented as a linear programming problem, of size polynomial in $|V(G)|$, and thus it can be answered in polynomial time using the ellipsoid method [14].

Here is another theorem from [5]:

Theorem 4. *Let G be a graph with no stable set of size three, and let $k \geq 0$ be a real number. The following are equivalent:*

- *There is a fractional packing of seagulls of value at least k.*
- *$|V(G)| \geq 3k$, and G has connectivity at least k, and for every clique C, if D denotes the set of vertices in $V(G) \setminus C$ that have both a neighbor and a non-neighbor in C, then $|D| + |V(G) \setminus C| \geq 2k$.*

Now we proceed as follows. Let k be an integer and let G be a graph with no stable set of size three. First we test if G has a fractional packing of seagulls of value at least k. If not, then G does not have k pairwise vertex-disjoint seagulls, and we can stop. If yes, Theorem 2 and Theorem 3 imply that in order to check whether G contains k pairwise vertex-disjoint seagulls, it remains to check the following two conditions:

- If $k = 2$, then G is not the five-wheel.
- The complement graph of G has a matching with k edges.

Checking the first condition is straightforward, and there is a well-known polynomial-time algorithm for the second, due to Edmonds [9]. Thus we obtain the required polynomial-time algorithm.

To conclude, in this article we described some recent progress on Hadwiger's conjecture for graphs with no stable set of size three (Conjecture 2). In general, this conjecture is still open and may even be false. Proving it would be a great step toward solving Hadwiger's conjecture in general, and it is also likely to be a good place to look for a counterexample, if one exists.

References

[1] K. Appel and W. Haken, Every planar map is four-colorable. Part I: Discharging, *Illinois J. Math.* **21** (1977), 429–490.

[2] K. Appel, W. Haken, and J. Koch, Every planar map is four-colorable. Part II. Reducibility, *Illinois J. Math.* **21** (1977), 491–567.

[3] C. Berge, Sur le couplage maximum d'un graphe, *C. R. Acad. Sci. Paris* **247** (1958), 258–259.

[4] J. Blasiak, A special case of Hadwiger's conjecture, *J. Combinatorial Theory, Ser. B*, **97** (2007), 1056–1073.

[5] M. Chudnovsky and P. Seymour, Packing seagulls, submitted.

[6] M. Chudnovsky, S. Oum, and P. Seymour, Finding minimum clique capacity, submitted.

[7] D. Dor and M. Tarsi, Graph decomposition is NP-complete: A complete proof of Holyer's conjecture, *SIAM J. Comput.* **26** (1997), 1166–1187.

[8] J. Edmonds, Matroid partition, in: *Mathematics of the Decision Sciences Part 1* (Proceedings Fifth Summer Seminar on the Mathematics of the Decision Sciences, Stanford, California, 1967; G. B. Dantzig, A. F. Vienott Jr., eds.) [Lectures in Applied Mathematics Vol. 11], American Mathematical Society, Providence, Rhode Island, 1968, 333–345.

[9] ______, Paths, trees, and flowers, *Canad. J. Math* **17** (1965) 449–467.

[10] H. Hadwiger, Über eine Klassifikation der Streckenkomplexe, *Vierteljschr. Naturforsch. Ges. Zürich*, **88** (1943), 133–142.

[11] P. Hall, On representation of subsets, *J. Lond. Math. Sci.* **10** (1935), 26–30.

[12] K. Kawarabayashi and Z-X. Song, Independence number and clique minors, *J. Graph Theory* **56** (2007), 219–226.

[13] ______, Some remarks on odd Hadwiger's conjecture, *Combinatorica* **27** (2007), 429–438.

[14] L. G. Khachiyan, Polynomial algorithms in linear programming, *Zh. Vychisl. Mat. i Mat. Fiz.* **20** (1980), 51–68, 260.

[15] K. Kuratowski, Sur le problème des courbes gauches en topologie, *Fund. Math.* **15** (1930), 271–283.

[16] M. D. Plummer, M. Stiebitz, and B. Toft, On a special case of Hadwiger's conjecture, *Discuss. Math. Graph Theory* **23** (2005), 333–363.

[17] N. Robertson, P. Seymour, and R. Thomas, Hadwiger's conjecture for K_6-free graphs, *Combinatorica* **13** (1993), 279–361.

[18] K. Wagner, Über eine Eigenschaft der ebenen Komplexe, *Math. Ann.* **114** (1937), 570–590.