WHAT IS...
a Paraproduct?

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The term *paraproduct* is nowadays used rather loosely in the literature to indicate a bilinear operator that, although noncommutative, is somehow better behaved than the usual product of functions. Paraproducts emerged in J.-M. Bony's theory of paradifferential operators [1], which stands as a milestone on the road *beyond* pseudodifferential operators pioneered by R. R. Coifman and Y. Meyer in [3]. Incidentally, the Greek word $\pi\alpha\rho\alpha$ (para) translates as *beyond* in English, and *au délà de* in French, just as in the title of [3]. The defining properties of a paraproduct should therefore go beyond the desirable properties of the product. As a first step and to illustrate these properties, let us consider the bilinear operator

$$\Pi_0(f,g)(s) = \int_{-\infty}^{s} f'(t)g(t)\,dt,\ f,g \in C_0^1(\mathbb{R}).$$

By Leibniz's rule, $fg = \Pi_0(f,g) + \Pi_0(g,f)$, that is, Π_0 *reconstructs the product* fg. In addition, Π_0 provides an exact *linearization formula*, that is, if $H \in C^\infty(\mathbb{R})$, then

$$H(f) = H(0) + \Pi_0(f, H'(f)),$$

as opposed to the one given by the product

$$H(f) = H(0) + fH'(f) + error.$$

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Π_0 also satisfies a *Leibniz-type rule*,

$$\Pi_0(f,g)' = f'g,$$

but it fails to obey one of the main properties of the product, namely *Hölder's inequality*.

So what is a paraproduct? A bilinear, noncommutative operator Π that satisfies product reconstruction and linearization formulas (up to smooth errors), a Hölder-type inequality, and a Leibniz-type rule such as

$$\partial^\alpha \Pi(f,g) = \tilde{\Pi}(\partial^\alpha f, g),$$

where $\tilde{\Pi}$ satisfies a Hölder-type inequality. For Π_0, $\tilde{\Pi}(f,g)$ equals fg when $\alpha = 1$. Π_0 comes close to being a paraproduct, but it is not well suited for L^p-spaces as it does not satisfy a Hölder-type inequality.

We now turn our attention to the evolution of the various forms of paraproducts. Each of the paraproducts Π_l below has transformed in time into a successor Π_{l+1}, and this natural flow was motivated by the need of analysts to settle specific problems.

In retrospect, the first version of a paraproduct is implicit in A. P. Calderón's work on commutators [2]. Let $U = \{s + it : s \in \mathbb{R}, t > 0\}$ and $1 < p, q < \infty$. For $F \in H^p(U)$ and $G \in H^q(U)$ (Hardy spaces), Calderón defined the bilinear operator

$$\Pi_1(F,G)(s) = -i\int_0^\infty F'(s+it)G(s+it)\,dt.$$

Again, by Leibniz's rule, Π_1 reconstructs the product FG (on the real line), its derivative obeys Leibniz's rule (just as the product does), it satisfies an exact linearization formula, and, as Calderón showed, it verifies the following Hölder-type inequality: if $1/r = 1/p + 1/q$,

$$\|\Pi_1(F,G)\|_{L^r(\mathbb{R})} \lesssim \|F\|_{H^p(U)}\,\|G\|_{H^q(U)}\,.$$

Let us now "deconstruct" Π_1. Define f_1 and f_2 to be the real and imaginary boundary values of F, that is,

$$F(s+it) = (f_1 * P_t)(s) + i(f_2 * P_t)(s),$$

where $P_t(x) = t^{-1}P(t^{-1}x)$ are dilations of the Poisson kernel $P(x) = \pi^{-1}(1+x^2)^{-1}$. Defining $Q = P'$ and taking derivatives yields

$$F'(s+it) = \frac{1}{t}(f_1 * Q_t)(s) + i\frac{1}{t}(f_2 * Q_t)(s).$$

Similarly, if we write G in terms of its boundary values, we see that $\Pi_1(F,G)(s)$ can be expressed as a sum of four operators of the form

$$\Pi_2(f,g)(s) = \int_0^\infty (Q_t * f)(s)(P_t * g)(s)\,\frac{dt}{t}.$$

In n dimensions and based on a real-variable approach, J.-M. Bony [1] considered bilinear operators of the form

$$\Pi_3(f,g) = \int_0^\infty (\psi_t * f)(\phi_t * g)\,\frac{dt}{t}.$$

In analogy with Π_2, we have $\phi_t(x) = t^{-n}\phi(x/t)$, $\psi_t(x) = t^{-n}\psi(x/t)$, where ϕ is a Schwarz function in $\mathbb{R}^n$ such that its Fourier transform $\hat{\phi}$ is real, radially symmetric, and supported in the ball $B_1(0)$, $\hat{\phi} = 1$ in $B_{1/2}(0)$, and ψ is defined (on the Fourier side) by $\hat{\psi}(\xi) = \hat{\phi}(\xi/2) - \hat{\phi}(\xi)$. The discrete version of Π_3 (think $t = 2^{-j}$) takes the form

$$\Pi_4(f,g) = \sum_{j\in\mathbb{Z}} (\psi_j * f)(\phi_j * g),$$

with $\psi_j(x) = 2^{jn}\psi(2^jx)$, $\phi_j(x) = 2^{jn}\phi(2^jx)$. The properties of ψ give us the equality

$$\begin{aligned} fg &= \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} (\psi_j * f)(\psi_k * g) \\ &= \Pi_4(f,g) + \Pi_4(g,f) + \sum_{j\in\mathbb{Z}} (\psi_j * f)(\psi_j * g), \end{aligned}$$

which is a product reconstruction formula with an error term. A convenient modification of Π_4 is

$$\Pi_5(f,g) = \sum_{j\in\mathbb{Z}} (\psi_j * f)(\phi_{j-2} * g),$$

which now yields

$$fg = \Pi_5(f,g) + \Pi_5(g,f) + R(f,g),$$

$R(f,g)$ being the tridiagonal sum in $|j-k| \le 1$. The operator Π_5 is called *Bony's paraproduct*, and it has a number of outstanding properties. As the last identity shows, there is a reconstruction formula for the product with an error term. It is not an exact reconstruction, so why is it so useful? Consider $f \in C^\alpha$ and $g \in C^\beta$ (Hölder spaces) with $0 < \alpha < \beta$. Since a product is as smooth as its roughest factor, we will have $fg \in C^\alpha$. However, Bony showed that $\Pi_5(g,f) \in C^\alpha$, $\Pi_5(f,g) \in C^\beta$, and $R(f,g) \in C^{\alpha+\beta}$, thus identifying the bad, the good, and the best part of fg. Moreover, for $H \in C^\infty(\mathbb{R})$, Π_5 linearizes H at a function f in such a way that

$$H(f) = H(0) + \Pi_5(f, H'(f)) + e_H(f),$$

where the error $e_H(f)$ is smoother than f. Another advantage is that, as straightforward calculations on the Fourier side show, Π_5 can be rewritten as

$$\Pi_5(f,g) = \sum_{j\in\mathbb{Z}} \Psi_j * ((\psi_j * f)(\phi_{j-2} * g)),$$

where $\hat{\Psi}$ is supported in an appropriate annulus. The acute reader will notice that this is not possible with Π_4, and this is why Π_5 was introduced! Letting $\langle\cdot,\cdot\rangle$ denote the usual Schwarz function-tempered distribution dual pairing, if h is another Schwarz function, then

$$\langle \Pi_5(f,g), h\rangle = \sum_{j\in\mathbb{Z}} \langle (\psi_j * f)(\phi_{j-2} * g), \Psi_j * h\rangle,$$

and this relation provides immediate access to the Littlewood-Paley pieces of h. Mapping properties for Π_5, including the ones of Hölder-type $L^p \times L^q \to L^r$, follow by duality.

To further see some of the paraproduct properties in action, we will prove the classical fractional Leibniz rule, which states that

$$\begin{aligned} \|D^\alpha(fg)\|_{L^r} &\lesssim \|D^\alpha f\|_{L^{p_1}}\, \|g\|_{L^{q_1}} \\ &\quad + \|f\|_{L^{p_2}}\, \|D^\alpha g\|_{L^{q_2}}, \end{aligned}$$

where $\widehat{D^\alpha h}(\xi) = |\xi|^\alpha \hat{h}(\xi)$ for $\alpha > 0$, and $1 < p_1, p_2, q_1, q_2, r < \infty$ with $1/r = 1/p_1 + 1/q_1 = 1/p_2 + 1/q_2$. The following short argument (see, for instance, Muscalu, Pipher, Tao, and Thiele (2004)) exploits the product reconstruction and Leibniz's rule for paraproducts:

$$\begin{aligned} &\|D^\alpha(fg)\|_{L^r} \\ &= \|D^\alpha(\Pi(f,g)) + D^\alpha(\Pi(g,f))\|_{L^r} \\ &= \|\tilde{\Pi}(D^\alpha f, g) + \tilde{\Pi}(D^\alpha g, f)\|_{L^r} \\ &\lesssim \|D^\alpha f\|_{L^{p_1}}\, \|g\|_{L^{q_1}} + \|f\|_{L^{p_2}}\, \|D^\alpha g\|_{L^{q_2}} \quad \square \end{aligned}$$

More flexible versions of Π_5 arise when considering

$$\Pi_6(f,g) = \sum_{j\in\mathbb{Z}} \phi_j^1 * ((\phi_j^2 * f)(\phi_j^3 * g)),$$

for general functions ϕ^m, $m = 1,2,3$. For example, $\tilde{\Pi}_5 = \Pi_6$ for suitable functions ϕ^m's. In turn, Π_6 has evolved as follows. Let us write $Q \in \mathcal{D}$ if Q is a dyadic cube, that is,

$$Q = \{x \in \mathbb{R}^n : k_i \le 2^j x_i \le k_i + 1; i = 1, \ldots, n\},$$

for some $k \in \mathbb{Z}^n$ and $j \in \mathbb{Z}$. In this case, we write $Q = Q_{jk}$. Let also $x_Q = 2^{-j}k$ denote the lower left corner of Q. Simple computations show that Π_6 can be written as

$$\Pi_6(f,g)(x) = \int\int K(x,y,z) f(y) g(z)\, dy dz,$$

where $K(x,y,z)$ is the bilinear kernel given by

$$\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{Z}^n}} |Q_{jk}|^{-\frac{1}{2}} \fint_{Q_{jk}} \phi^1_{j,x}(w)\phi^2_{j,y}(w)\phi^3_{j,z}(w)\,dw,$$

with $\phi^m_{j,x}(w) = 2^{jn/2}\phi^m(2^j(x-w))$, $m = 1,2,3$. If we replace the average above by the value of the integrand at x_Q, we can rewrite the kernel as

$$\sum_{Q\in\mathcal{D}} |Q|^{-\frac{1}{2}}\phi^1_Q(x)\phi^2_Q(y)\phi^3_Q(z) + E(x,y,z),$$

where $\phi^m_Q(x) = \phi^m_{j,x}(x_Q)$, $m = 1,2,3$, and the error term $E(x,y,z)$ is the bilinear kernel of a smoothing operator. The functions ϕ^m_Q are examples of the so-called *molecules associated* with a cube Q. For general families of molecules $\{\phi^m_Q\}_{Q\in\mathcal{D}}$, $m = 1,2,3$, which are not necessarily dilations and translations of a fixed profile but have suitable cancellation properties, the associated *molecular paraproduct* Π_7 has kernel

$$\sum_{Q\in\mathcal{D}} |Q|^{-\frac{1}{2}}\phi^1_Q(x)\phi^2_Q(y)\phi^3_Q(z),$$

that is,

$$\Pi_7(f,g)(x) = \sum_{Q\in\mathcal{D}} |Q|^{-\frac{1}{2}}\langle\phi^2_Q, f\rangle\langle\phi^3_Q, g\rangle\phi^1_Q(x).$$

Π_7 represents one of the modern versions of the paraproducts. Molecules based on the Haar system yield so-called *dyadic paraproducts.*

Since their 1965 debut, paraproducts have played a central role in analysis and PDEs. They are connected to the bilinear Calderón-Zygmund theory and constitute the building blocks of many other bilinear operators. Their applications include, just to name a few, the celebrated $T1$ and Tb theorems, the boundedness of Calderón commutators and the bilinear Hilbert transform, and the theories of pointwise multipliers of function spaces and of compensated compactness.

References

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