Cartan and Complex Analytic Geometry

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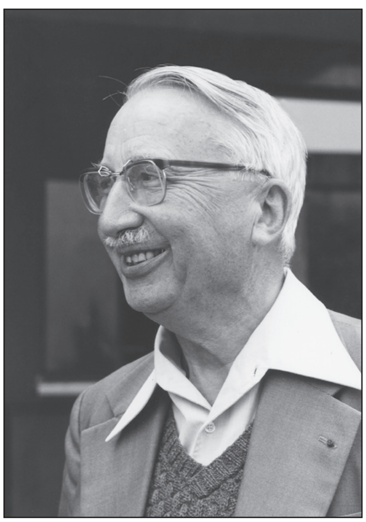
On the Mathematical Heritage of Henri Cartan

Henri Cartan left us on August 13, 2008, at the age of 104. His influence on generations of mathematicians worldwide has been considerable. In France especially, his role as a professor at École Normale Supérieure in Paris between 1940 and 1965 led him to supervise the Ph.D. theses of Jean-Pierre Serre (Fields Medal 1954), René Thom (Fields Medal 1958), and many other prominent mathematicians such as Pierre Cartier, Jean Cerf, Adrien Douady, Roger Godement, Max Karoubi, and Jean-Louis Koszul.



Henri Cartan at Oberwolfach, September 3, 1981.

However, rather than rewriting history that is well known to many people, I would like here to share lesser known facts about his career and work, especially those related to parts I have been involved with. It is actually quite surprising, in spite of the fact that I was born more than half a century later, how present Henri Cartan still was during my studies. My first mathematical encounter with Cartan was when I was about twelve, in 1969. In the earlier years, my father had been an elementary school teacher and had decided to go back to Lille University to try to become a math teacher in secondary education; there was a strong national effort in France to recruit teachers, due to the much increased access of pupils and students to higher education, along with a strong research effort in technology and science. I remember quite well that my father had a book with a mysterious title: *Théorie Élémentaire des Fonctions Analytiques d'une ou Plusieurs Variables Complexes* (Hermann, 4th edition from 1961) [Ca3], by Henri Cartan, which contained magical stuff such as contour integrals and residues. I could then, of course, not understand much of it, but my father was quite absorbed with the book; I was equally impressed by the photograph of Cartan on the cover pages and by the style of the contents, which had obvious similarity to the "New Math" we started being taught at school—namely set theory and symbols like $\cup, \cap, \in, \subset, \ldots$ My father explained to me that Henri Cartan was one of the leading French mathematicians and that he was one of the founding members of the somewhat secretive Bourbaki group, which had been the source of inspiration for the new symbolism and for the reform of education. In France, the leader of the reform commission was A. Lichnerowicz, at least as far as mathematics was concerned, and I got myself involved with the new curriculum in grade ten in 1970. Although overly zealous promoters of the "New Math" made the reform fail less than fifteen years later, for instance by pushing abstract set theory even down to kindergarten—a failure which resulted in very bad counter-reforms around 1985—I would like to testify that in spite of harsh criticism sometimes geared toward the reform, what we were taught appeared well thought out, quite rigorous, and even very exciting. In the rather modest high school I was frequenting at the time, the large majority of my fellows in the science class were certainly enjoying the menu and taking a large benefit. The disaster came only later,

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from the great excess of reforms applied at earlier stages of education.

In any case, my father left me from that period three books by Henri Cartan, namely the one already described and two other textbooks: *Differential Calculus* and *Differential Forms* [Ca4] (also by Hermann, Paris), which I never ceased using. These books are still widely used and are certainly among the primary references for the courses I have been delivering at the University of Grenoble since 1983. I find it actually quite remarkable that French secondary school teachers of the 1960–90 era could be taught mathematics in the profound textbooks by such mathematicians as Cartan, Dieudonné, or Serre, especially in comparison with the general evolution of education in the last two or three decades in France, and other Western countries as well, about which it seems that one cannot be so optimistic….

In 1975 I entered École Normale Supérieure in Paris, and although Henri Cartan had left the École ten years earlier, he was still very much in the background when I began learning holomorphic functions of one variable. His role was eminently stressed in the course proposed to first-year students by Michel Hervé, who made great efforts to introduce sheaves to us, for example, as a means to explain analytic continuation and the maximal domain of existence of a germ of a holomorphic function.

Two years later I started a Ph.D. thesis under the supervision of Henri Skoda in Paris, and it is only at this period that I began realizing the full extent of Cartan's contributions to mathematics, in particular those on the theory of coherent analytic sheaves and his fundamental work in homological algebra and in algebraic topology [CE, CS1]. Taking part of its inspiration from J. Leray's ideas and from the important work of K. Oka in Japan, the celebrated Cartan seminar [Ca2] ran from 1948 to 1964, and as an outcome of the work by its participants, especially H. Cartan, J.-P. Serre, and A. Grothendieck, many results concerning topology and holomorphic functions of several variables received their final modern formulation. One should mention especially the proof of the coherence of the ring of holomorphic functions $\mathcal{O}_X$ in an arbitrary number of variables, after ideas of Oka, and the coherence of the ideal sheaf of an analytic set proved by Cartan in 1950. Another important result is the coherence of the sheaf of weakly holomorphic meromorphic functions, which leads to Oka's theorem on the existence of the normalization of any complex space. In this area of complex analysis, Henri Cartan had a long record of collaboration with German mathematicians, in particular H. Behnke and P. Thullen [CT] already before World War II, and after the dramatic events of the war, during which Cartan's brother was beheaded, a new era of collaboration started with the younger German generation represented by K. Stein, H. Grauert, and R. Remmert. These events were probably among the main reasons for Cartan's strong engagement in politics, especially toward human rights and the construction of Europe; at age eighty, Henri Cartan even stood unsuccessfully for election to the European Parliament in 1984, as head of list for a party called "Pour les États-Unis d'Europe", declaring himself to be a European Federalist.

In 1960, pursuing ideas and suggestions of Cartan, Serre [CS2, Se], and Grothendieck [Gt], H. Grauert proved the coherence of direct images of coherent analytic sheaves under proper holomorphic morphisms [Gr]. Actually, a further important coherence theorem was to be discovered more than three decades later as the culmination of work on L^2 techniques by L. Hörmander, E. Bombieri, H. Skoda, Y. T. Siu, A. Nadel, and myself: if φ is a plurisubharmonic function, for instance a function of the form $\varphi(z) = c \log |\sum g_j(z)|^2$ where $c > 0$ and the g_j are holomorphic on an open set Ω in $\mathbb{C}^n$, then the sheaf $\mathcal{I}(\varphi) \subset \mathcal{O}_\Omega$ of germs of holomorphic functions f such that $|f|^2 e^{-\varphi}$ is locally integrable is a coherent ideal sheaf [Na]. The sheaf $\mathcal{I}(\varphi)$ is now called the Nadel multiplier ideal sheaf associated with φ; its algebraic counterpart plays a fundamental role in modern algebraic geometry. The main philosophical reason is probably that L^2 theory is a natural framework for duality and vanishing theorems. It turns out that I got the privilege of explaining this material to young students of École Normale Supérieure around 1992. It was therefore a considerable honor to me that Henri Cartan came to listen to this lecture along with the younger members of the audience. Although he was close to being ninety years old at that time, it was a rare experience for me to have somebody there not missing a word of what I was saying—and sometimes raising embarrassing questions about insufficiently explained points! I remember that the lecture actually had to be expanded at least half an hour beyond schedule, just to satisfy Cartan's pressing demands….

During the 1990s, my mathematical interests went to the study of entire curves drawn on projective algebraic varieties, especially in the direction of the work of Green-Griffiths [GG] on the "Bloch theorem"—for which they had provided a new proof in 1979. Henri Cartan had also taken an eminent role in this area, which is actually the subject of his Ph.D. thesis [Ca1] under the supervision of Paul Montel, although these achievements are perhaps not as widely known as his later work on sheaves. In any case, Cartan proved after A. Bloch [Bl] several important results in the then nascent Nevanlinna theory, which, in his own terms, can be stated by saying that sequences of entire curves contained in the complex projective n-space minus (n+2) hyperplanes in general position form an "almost normal family": namely, they either have a subsequence that has a limit contained in the complement or a subsequence that approaches more and more closely a certain union of the "diagonal" hyperplanes. These results were put much later in geometric form by Kobayashi and Kiernan [KK] in terms of the concepts of taut and hyperbolically embedded domains. Very recently, M. Ru and P. M. Wong [RW], E. Nochka and P. Vojta [Vo] found various generalizations and improvements with a more arithmetic flavor. It is remarkable that Cartan's early work already contains many important ingredients, such as the use of Nevanlinna estimates for Wronskians, that are still at the heart of contemporary research on the subject, for example in the form of the study of the geometry of jet bundles [De1, De2]. I had once again the privilege of explaining some of these modern developments in front of Henri Cartan in 1997, still as vigilant as ever, on the occasion of a celebration of his work by the French Mathematical Society.

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My Memory of Professor Henri Cartan

In 1953, the year I graduated from the University of Tokyo, I had the good fortune to spend a year in France as a boursier of the French government. On the hottest day on record in August, I left the port of Yokohama aboard "Viet-Nam" of the Messageries Maritimes for a four-week journey to Marseilles. I was twenty-one, not sure of myself. I was interested in differential geometry and several complex variables. During my senior year I was a member of Professor Yano's seminar, giving talks on harmonic integrals. At the same time, I was fascinated by the Cartan seminar notes, 1951/1952, on several complex variables.

As Professor Iyanaga had written to Professor Cartan about me, I was to visit Professor Cartan to pay my respects upon my arrival in Paris. Unfortunately, as soon as I got settled in Maison du Japon of Cité Universitaire, I became ill with typhoid fever, which I had picked up on my way to France in spite of the vaccination. After five weeks in Cité's hospital, I returned to Maison du Japon and went to see Professor Cartan. He said that he had had the same illness years ago and that, from his own experience, I would become healthier than before the illness—very encouraging words.

The Cartan seminar in 1953/1954 was fortunately again on several complex variables.

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Although I could not follow some of the lectures, such as those on automorphic functions, I faithfully attended the seminar. This is partly because every talk was written up in a complete form within a week and was distributed at the time of the following seminar. During my stay in Paris, I had an opportunity to listen to talks by Karl Stein, whom I had known only by name. His two lectures in March were the last seminar talks I attended.

In the meantime, I also attended a series of lectures by Lichnerowicz at Collège de France, and I took part in a private geometry seminar with Marcel Berger (about to finish his thesis), Paulette Libermann (already with a doctorate), Warren Ambrose (on sabbatical from MIT), and Katsumi Nomizu (on CNRS).

After Stein's lectures, I left Paris for Strasbourg to spend the remaining four months of the fellowship under Ehresmann, and I did not see Professor Cartan for more than a decade.

When *Foundations of Differential Geometry* with Nomizu appeared, I sent a copy to Professor Cartan as a token of my gratitude. When the second volume appeared in 1969, he wrote me to the effect that he was happy to see that our volume 2 really came out, since the promised second volume of some books had never come out. I found out much later which books he was referring to.

Around 1967 I shifted my focus from differential geometry to several complex variables. When I saw Professor Cartan (if my memory is correct) in the late 1960s in Berkeley, I mentioned to him the then newly discovered invariant pseudo-distance. He immediately asked whether the topology defined by the new distance gives the manifold topology, and I realized that I had taken for granted that that was the case. (This fact was later proved by T. Barth). I suspect that he must have raised this question because of his experience with the Carathéodory distance, which he had made use of in his work on transformations of bounded domains. (It is as recently as 1984 that Vigué constructed a bounded domain whose natural topology is not given by its Carathéodory distance.)

In the late 1960s I became interested in the Picard theorems in higher dimension and, as a consequence, in the hyperbolicity question for complements of hyperplanes. This led me to old papers of Emile Borel and André Bloch and then to the thesis of Cartan. In 1953 when I went to Paris, I did not dream of one day ever reading Cartan's 1928 thesis. In 1973, Peter Kiernan, one of my former students, and I wrote a paper reinterpreting Cartan's main results in terms of the invariant pseudo-distance. In his *Collected Works*, Cartan wrote a brief analysis of his own thesis and kindly mentioned our paper. We felt very honored.

To me, the 1950s still seem like yesterday. But many of the people I mentioned here, Ambrose, Ehresmann, Iyanaga, Libermann, Lichnerowicz, Nomizu, and Yano, are all gone, and now Professor Cartan. I must admit that my year in Paris was indeed long ago.

Raghavan Narasimhan

Henri Cartan

Henri Cartan came to the Tata Institute of Fundamental Research, Bombay (now Mumbai), in January 1960 to take part in an International Colloquium on Function Theory. There were many well-known participants besides Cartan, including C. L. Siegel, H. Grauert, L. Bers, L. Nirenberg, H. E. Rauch, W. Baily, M. Kuranishi, and others. I had joined the Tata Institute as a beginning member in July 1957 and had become interested in several complex variables.

The colloquium was immediately followed by a conference on mathematical education in South East Asia. Again, there were many well-known participants, including E. Artin, M. H. Stone, E. E. Moise, A. D. Alexandrov, and Y. Akizuki, as well as several of those who had come for the colloquium. In the course of the two weeks that these meetings lasted, one thing became very clear: Cartan's standards of mathematical quality and precision were perhaps equalled, but they were not exceeded.

Let me recount an incident from these two weeks. During the education conference, Cartan and I attended a lecture on topological 3-manifolds. Near the end of the lecture, the speaker said that he would conclude the proof with some hand-waving. Cartan obviously did not approve. He turned to me and said: "Now I understand why Indian Gods have so many hands; they want to give proofs in n dimensions."

I spent two months in Paris in the fall of 1960 and was able to meet with Cartan several times. He had received (for publication in the *American Journal of Mathematics*) a paper by Errett Bishop entitled "Mappings of partially analytic spaces". The paper contained a beautiful theorem (that a Stein manifold of dimension n admits a proper holomorphic map into $\mathbb{C}^{n+1}$) but was formulated in terms of a somewhat complicated generalization of complex spaces. Cartan seemed a little skeptical about its correctness. Since I was very interested in mapping problems on Stein spaces, he asked me to look at the paper. Only after I had explained the details of the proof to him would he recommend publication; he was not satisfied with a statement that the proof was correct.

Cartan showed me great kindness both in Bombay and in Paris. Although I was just a beginner,

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he spent time with me discussing mathematics, opening vistas and suggesting improvements to the work that I was trying to do. I think that he had this nurturing quality with any young person who had a serious interest in mathematics with whom he came in contact.

Cartan's first major achievements dealt with (biholomorphic) automorphisms of bounded domains in $\mathbb{C}^n$. This work was extremely influential, and I shall say more about it below.

He introduced new methods and ideas into the study of domains of holomorphy, Stein manifolds, and global problems (such as the Cousin problems). These were complemented by work of K. Oka; this body of work transformed the entire field. There is a story, which I first heard from Karl Stein: When, during the Brussels Colloquium of 1953, Serre presented the results that he and Cartan had obtained, one of the German participants said: "We have bows and arrows; the French have tanks."

A good part of Cartan's work on these topics appeared in the Séminaire Cartan notes for the years 1951/1952 and 1953/1954. A large part of his seminal ideas concerning *real* analytic spaces appeared in two joint papers with F. Bruhat. This latter work took a more satisfactory form when H. Grauert proved that any real analytic manifold (connected, countable at infinity) admits a real analytic imbedding as a closed submanifold of $\mathbb{R}^N$ for some N. The work that he initiated on *complex* analytic spaces, especially in the Séminaire Cartan notes mentioned above, was developed further by Serre, Grauert, Remmert, Grothendieck, and others in the years that followed.

All this work forms but a small part of his contribution to complex analysis. His many fundamental contributions to other fields (such as algebraic topology and homological algebra) have not even been hinted at.

I shall describe just two of his contributions to complex analysis which, I believe, demonstrate both his great influence and his penetrating insight.

Automorphisms of Bounded Domains

Let D be a bounded domain (connected open set) in $\mathbb{C}^n$, $n \geq 1$. Denote by Aut(D) the set of biholomorphic maps $g : D \to D$ (i.e., g is holomorphic, bijective and g^{-1} is holomorphic). We provide Aut(D) with the topology of uniform convergence on compact subsets of D. Cartan proves the following: Let $\{g_p\}$, $(p \geq 1)$ be a sequence of elements of Aut(D), such that $\{g_p\}$ converges to a map $f : D \to \mathbb{C}^n$ as $p \to \infty$, uniformly on compact subsets of D. Then, either $f \in \mathrm{Aut}(D)$, or f is degenerate in the sense that $f(D) \subset \partial D$ (the boundary of D). This implies that Aut(D) is a locally compact group acting properly on D.

In connection with earlier studies of his concerning automorphisms of so-called "circled" domains, Cartan had proved the following beautiful theorem (by a remarkably simple iteration argument). Let $f : D \to D$ be a holomorphic map. Assume that there is a point $a \in D$ such that $f(a) = a$ and $f'(a) =$ identity (where $f'(a)$ is the tangent map of f at the point a). Then f *is* the identity map of D. These results show that for any $a \in D$, the map $\mathrm{Aut}(D) \to D \times GL(n, \mathbb{C})$ given by $g \mapsto (g(a), g'(a))$ is a homeomorphism of Aut(D) onto a closed subset of $D \times GL(n, \mathbb{C})$.

In 1935, Cartan published a fascicule entitled *Sur les groupes de transformations analytiques,* which contains the following theorem:

Aut(D) is a real Lie group, acting real analytically on D.

In the proof, Cartan proceeds as follows. Consider the set V of (holomorphic) vector fields $z \mapsto X(z) = \sum_{k=1}^{n} X_k(z) \frac{\partial}{\partial z_k}$, which can be obtained in the following way: there exist sequences $\{g_p\} \subset \mathrm{Aut}(D)$ and $\{m_p\} \subset \mathbb{N}$, $g_p \to$ identity, $m_p \to \infty$ as $p \to \infty$ such that $m_p(g_p(z) - z) \to (X_1(z), \cdots, X_n(z))$ as $p \to \infty$ (uniformly on compact sets).

Cartan proves that V is a finite dimensional real Lie algebra of vector fields on D whose corresponding local Lie group of transformations of D is isomorphic to the germ of Aut(D) at the identity.

This theorem was the first major result concerning a question that belongs to the circle of ideas around Hilbert's fifth problem. Hilbert asked: To what extent is the assumption of differentiability essential to Lie's theory of continuous groups? This is usually interpreted as asking whether a locally Euclidean topological group is actually a Lie group. It is, however, natural to ask the following more general question (which, as far as I am aware, is still open in this form): if a locally compact group acts effectively as a topological transformation group on a manifold, is it necessarily a Lie group?

As mentioned above, Cartan's theorem was the first result in this direction. Moreover, Bochner and Montgomery added some new techniques to Cartan's method (to prove finite dimensionality) and obtained the following theorem: A locally compact group acting effectively as diffeomorphisms of a smooth manifold is, in fact, a Lie group acting smoothly.

Another major development in the study of the geometry of Lie groups was directly influenced by Cartan's work on Aut(D). Élie Cartan's important paper "Sur les domaines homogènes bornés de l'espace de n variables complexes" was also published in 1935. In the introduction to this paper, Élie Cartan calls Henri Cartan's work a remarkable contribution to the pseudo-conformal (=biholomorphic) representations of domains in the space of $n \geq 2$ variables. He says that Henri

Cartan's theorem suggested to him that it might be possible to classify bounded homogeneous domains in $\mathbb{C}^n$. He succeeded in doing this for $n = 2$ and $n = 3$, and he classified all bounded *symmetric* domains in $\mathbb{C}^n$ for $n \geq 4$. He found that all bounded homogeneous domains in $\mathbb{C}^2$ and $\mathbb{C}^3$ are symmetric and raised the question of whether this was true in general (without really expressing an opinion). We now know, thanks to the work of I. Piatetski-Shapiro, that, for $n \geq 4$, there exist bounded homogeneous domains in $\mathbb{C}^n$ which are not symmetric.

A Theorem on Holomorphic Matrices

As mentioned earlier, the work of Cartan and Oka transformed the study of global problems on Stein manifolds into an extensive theory with powerful tools. There are two major results that are crucial in this theory. One, due to Oka, is the coherence of the structure sheaf of $\mathbb{C}^n$. The other, chronologically the first, is a theorem on holomorphic matrices published by Cartan in 1940.

Let R be a closed rectangle, $a_k \leq Rez_k \leq b_k$, $c_k \leq Im\ z_k \leq d_k (k = 1, 2, \cdots, n, z = (z_1, \cdots, z_n) \in \mathbb{C}^n)$. Let $R_1 = \{z \in R | Rez_1 \geq 0\}$, $R_2 = \{z \in R | Rez_1 \leq 0\}$ and set $R_0 = R_1 \cap R_2$. We assume that $R_0 \neq \emptyset$, and, as usual, denote by $GL(q, \mathbb{C})$ the group of invertible $q \times q$ matrices with entries in $\mathbb{C}$ ($q \geq 1$ being a given integer).

Cartan's theorem is as follows.

Let f_0 be a holomorphic map of a neighborhood of R_0 into $GL(q, \mathbb{C})$. Then, there exist holomorphic maps f_ν of neighborhoods of R_ν into $GL(q, \mathbb{C})$ $[\nu = 1, 2]$ such that $f_0 = f_1 \cdot f_2^{-1}$ on some neighborhood of R_0.

It is this result that makes it possible to pass from the local to the global in the theory of coherent analytic sheaves on Stein spaces.

It is natural to try to prove this result as an implicit function theorem by solving the linearized problem $h_1 - h_2 = h_0$ (in a neighborhood of R_0). Today, one does this by working with bounded holomorphic functions on open rectangles and an implicit function theorem in Banach spaces. Cartan deals directly with Fréchet spaces. The solution of the linearized problem (with bounds) involves shrinking the domain of definition of the functions h_ν. In general, implicit function theorems in Fréchet spaces involve the loss of some kind of smoothness at each stage of the iteration, and a smoothing operator is required to restore fast convergence (so-called Nash-Moser technique). Cartan's iteration scheme produces fast convergence without the need for a smoothing operator and compensates for the shrinking of the domain of definition.

Thus, as early as 1940, Cartan had recognized the use of fast convergence in studying iteration in Fréchet spaces.

I believe that Cartan's work and the standards of quality and precision in mathematics that he set have influenced most mathematicians in the second half of the twentieth century.

Yum-Tong Siu

Tribute to Henri Cartan from a Complex Analyst

Henri Cartan was an intellectual giant in the world of mathematics in the twentieth century. His fundamental contributions spanned a wide range of fields: complex variables, algebraic topology, potential theory, homological algebra, and many others. This tribute is from the point of view of a complex analyst and touches only the field of complex variables. Even within complex analysis the work of Henri Cartan is very broad. We choose here only two areas.

The first area is value distribution theory in which he wrote his thesis [2]. His thesis, though written so long ago, is still one of the most fundamental and most elegant results in value distribution theory in higher dimension. To the general mathematical community this result of his, being overshadowed by his many other achievements, is not as well known. In recent years, because of the parallelism with diophantine approximation pointed out by Vojta [18], value distribution theory has taken on a new dimension. Cartan's thesis is being highlighted here to make the general mathematical community aware of this very beautiful piece of work.

The second area is what is now known as the theory of Cartan and Oka concerning Stein manifolds. In his interview with Allyn Jackson in March 1999 [11], to the question posed by Jackson, "You have worked in many areas of mathematics. Do you feel equally at home in analysis, in algebra, in geometry…?" Cartan replied, "Geometry—not exactly geometry. Topology, I would say. But I could also see the relations between them. One day I discovered that topological notions, and in particular sheaf theory, could be applied to analytic functions of several variables. This was very important. One can use results from topology in order to get some important results for analytic functions. I think that is interesting." When Cartan recalled his wide-encompassing work in many fields of mathematics, this second area seems to occupy a special position.

As a way of paying tribute to one of the first-ranked mathematicians of the twentieth century, without going too much into the technical details we explain here his contributions to the two areas

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so that the general mathematical community can appreciate and enjoy more the wonderful mathematical legacy he left us.

Value distribution theory was started by R. Nevanlinna [12] to relate the zero-set of a holomorphic function f on $\mathbb{C}$ to its growth, as a way of generalizing the fundamental theorem of algebra which states that the number of zeroes of a polynomial P is equal to its degree.

The degree of the polynomial P is replaced by the growth behavior of the *characteristic function* $T(r,f)$ which is defined as the average of $\log^+|f| := \max(\log|f|, 0)$ over the circle C_r of radius r centered at 0. For any complex number a the number of zeroes of the polynomial $P - a$ is replaced by the growth behavior of the *counting function* $N(r,f,a) := \int_{\rho=0}^{r} \frac{n(\rho,a)d\rho}{\rho}$, where $n(\rho,f,a)$ is the number of zeroes of $f - a$ inside the disk Δ_ρ of radius ρ centered at 0.

Unlike the case of a polynomial, the closeness of the value of f to a has to be counted together with the value of f actually equal to a. The growth behavior of $N(r,f,a)$ is not yet the same as $T(r,f)$ and has to be compensated by a function $m(r,f,a)$ known as the *proximity function,* which is defined as the average of $\log^+\frac{1}{|f-a|}$ over C_r. The sum $m(r,f,a) + N(r,f,a)$ is equal to $T(r,f)+O(1)$ as $r \to \infty$ for some bounded term $O(1)$, which is known as the *First Main Theorem*. The theory works also for the case of a meromorphic function f on $\mathbb{C}$.

The infimum $\delta(f,a)$ of $\frac{m(r,f,a)}{T(r,f)}$ as $r \to \infty$ is the *defect* of f for a, which measures the shortfall in the number of hits of a by f relative to the growth behavior of f. Nevanlinna's defect relation states that the sum of all defects is at most 2, which nowadays is interpreted as the Chern class of the projective line $\mathbb{P}_1$ when the meromorphic function f is interpreted as a holomorphic map from $\mathbb{C}$ to $\mathbb{P}_1$. The defect relation is a consequence of the *Second Main Theorem,* which states that

$$\sum_{j=1}^{q} m(r,f,a_j) \le (2+\varepsilon)\,T(r,f) - \sum_{j=1}^{q}\left(N(r,f,a_j) - N_1(r,f,a_j)\right) + O(1)$$

for any $\varepsilon > 0$ and any set of distinct points $a_1, \cdots, a_q$ when $r \to \infty$ avoids a subset of $\mathbb{C}$ which has finite measure with respect to $\frac{dr}{r}$, where $N_k(r,f,a)$ is the k-truncated counting function $\int_{\rho=0}^{r} \frac{n_k(\rho,a)d\rho}{\rho}$ with $n_k(\rho,f,a)$ counting any zero of order $\ge k$ in Δ_ρ only as of order k.

Cartan's thesis obtains the *Second Main Theorem*

$$\sum_{j=1}^{q} m(r,f,H_j) \le (n+1+\varepsilon)\,T(r,f) - \sum_{j=1}^{q}\left(N(r,f,H_j) - N_n(r,f,H_j)\right) + O(1)$$

for a holomorphic map $f : \mathbb{C} \to \mathbb{P}_n$ and any collection of hyperplanes $H_1, \cdots, H_q$ in general position. Later Cartan's theorem was obtained with much longer proofs involving so-called *associated curves* by Hermann and Joachim Weyl [19] and Lars Ahlfors [1].

In the parallelism between value distribution theory and diophantine approximation formulated by Vojta [18], without the contribution from the truncation of counting zeroes, the Second Main Theorem of Nevanlinna corresponds to Roth's theorem [15] on the impossibility of approximating an algebraic number by a rational number $\frac{p}{q}$ with error $\le \frac{1}{q^{2+\varepsilon}}$ for an infinite number of q. Cartan's theorem corresponds to Schmidt's subspace theorem [16].

The theory of Cartan-Oka originally was motivated by the two problems of Cousin [9] for a class of domains known as domains of holomorphy or Stein domains. Cousin's first problem asks whether, from locally given meromorphic functions with the property that the difference of any two is holomorphic, we can find a global meromorphic function whose difference with each of the locally given meromorphic functions is holomorphic. Cousin's second problem asks the corresponding multiplicative question whether, from locally given holomorphic functions with the property that the quotient of any two is holomorphic, we can find a global holomorphic function whose quotient by each of the locally given holomorphic functions is holomorphic.

Domains of holomorphy were introduced because Hartogs [10] observed that any holomorphic function of two complex variables (z,w) on $\{|z| < 1, |w| < 2\} \cup \{|z| < 2, 1 < |w| < 2\}$ extends always to $\{|z| < 2, |w| < 2\}$ by considering the Laurent series extension in w and the vanishing of the coefficients of negative powers of w as functions of z, first for $|z| < 1$ and automatically also for $|z| < 2$. A domain for which extension of every holomorphic function on it to a larger domain is not possible is called a domain of holomorphy. Cartan [3] introduced the notion of holomorphic convexity as a characterization of domains of holomorphy. A domain Ω is holomorphically convex if for every compact subset K its holomorphic convex hull is also compact, which is defined as consisting of all points P with $|f(P)| \le \sup_K |f|$ for any holomorphic function f on Ω. Cartan [3] proved the necessity of holomorphic convexity for any domain of holomorphy and also the sufficiency when the domain is also circular in the sense that $(z_1, \cdots, z_n)$ is in it if and only if $(|z_1|, \cdots, |z_n|)$ is in it. The full equivalence was given in the joint paper of Cartan and Thullen [8].

For an abstract complex manifold the analog of a domain of holomorphy is a Stein manifold which, besides being holomorphically convex, satisfies the condition that global holomorphic functions on it separate any pair of distinct points.

Cartan's seminal contribution is the incorporation of sheaf theory from topology into his work on complex variables to introduce the very important notion of a coherent sheaf [5, 6]. He finally crowned the success of his work in this direction by proving Theorems A and B for coherent sheaves on Stein manifolds [7]. Theorem B states that the cohomology group $H^p(X, \mathcal{F})$ of degree p over a Stein manifold X with coefficients in a coherent sheaf $\mathcal{F}$ over X vanishes if $p > 0$. Theorem A states that at every point P of X global sections of $\mathcal{F}$ generate $\mathcal{F}$ at P over the ring of holomorphic function germs on X at P.

On an open subset Ω of $\mathbb{C}^n$ a coherent sheaf is locally described as consisting of the set of all p-tuples of holomorphic function germs on Ω modulo those in the range of the homomorphism given by a $p \times q$ matrix of holomorphic functions on Ω. A global coherent sheaf on a complex manifold is obtained by piecing together locally defined coherent sheaves. In Theorem B the vanishing of $H^p(X, \mathcal{F})$, for example, when $p = 1$, means that, for an open cover $\{U_\alpha\}$ of X by Stein open subsets, local sections $f_{\alpha\beta}$ of $\mathcal{F}$ over $U_\alpha \cap U_\beta$ with $f_{\alpha\beta} = -f_{\beta\alpha}$ and $f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha} = 0$ on $U_\alpha \cap U_\beta \cap U_\gamma$ can be expressed as $f_{\alpha\beta} = f_\beta - f_\alpha$ with f_α being a section of $\mathcal{F}$ over U_α. The case of $\mathcal{F}$ being the sheaf of holomorphic function germs of X and $f_{\alpha\beta}$ being the difference of the locally given meromorphic function F_α on U_α and the one on U_β would solve immediately the additive first Cousin problem with the global meromorphic function given by $F_\alpha - f_\alpha$ on U_α.

One crucial ingredient in the proofs of Theorems A and B is the following important gluing lemma of Cartan [4]. Denote by $R_{a,b;c,d}$ the rectangle in $\mathbb{C}$ with coordinate $z = x + \sqrt{-1}\, y$ defined by $a < x < b$ and $c < y < d$. When $a_1 < a_2 < b_1 < b_2$, let $D_j = R_{a_j,b_j;c,d} \times G$ for some polydisk G and $D = D_1 \cap D_2$. Cartan's gluing lemma enables him to write a nonsingular matrix A of holomorphic functions given on the topological closure $\overline{D}$ of D as the product A_1A_2 on D, where A_j is a nonsingular matrix of holomorphic functions on D_j.

Oka [13, 14] contributed to Cartan's program by proving the existence of local "pseudobases" for the kernel defined by a $p \times q$ matrix A of holomorphic functions on a domain Ω in $\mathbb{C}^n$. It means that any point P of Ω admits some open neighborhood U and a finite number of q-tuples $f_1, \cdots, f_k$ of holomorphic functions on U with $Af_j \equiv 0$ for $1 \le j \le k$ such that any q-tuple g of holomorphic function germs at any point Q of U with $Ag \equiv 0$ can be written as $\sum_{j=1}^k h_j f_j$ for some holomorphic function germs $h_1, \cdots, h_k$ at Q. Oka also showed that, for the common zero-set V of a finite number of local holomorphic functions, similar local "pseudobases" exist for the ideal of function germs defined by their restrictions to V being identically zero.

Serre [17] later transported the theory of coherent sheaves to algebraic geometry. It has since become a very powerful indispensable tool in algebraic geometry.

In the early 1970s I had the good fortune of meeting Cartan in person on two occasions when I was at a relatively early stage of my career. One occasion was when I gave a talk in a seminar in the École Normale Supérieure and had dinner with him and a couple of other mathematicians afterward. Another occasion was at a big party he hosted in his house on Boulevard Jourdan. He was very kind, caring, warm, and inspiring. I still vividly remember how in mathematical discussions he chose very thoughtful and insightful questions posed with an encouraging tone to point to thought-provoking new ideas and directions.

As time goes by with further involvement in complex analysis on my part, my admiration for Cartan's work is ever elevated to higher planes. Even after eighty years of value distribution theory in higher dimension, his thesis is still being used as a starting point in lectures given in conferences on the subject. Both the result and the presentation of his thesis are so very elegant and natural.

As for the theory of Cartan and Oka, it will always be a shining gem in the crown of mathematics.

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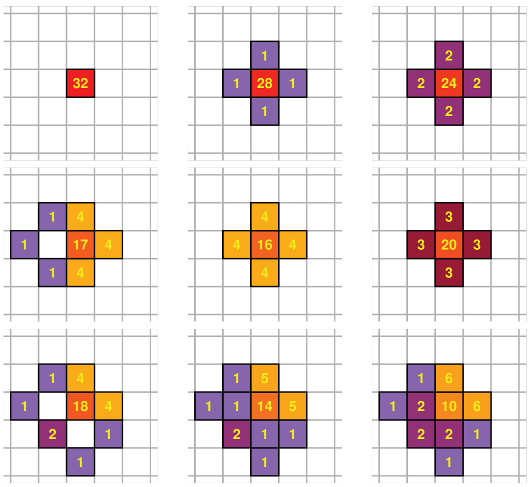
About the Cover

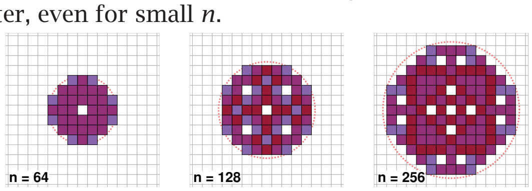
Sandpile

This month's cover theme is taken from this issue's article "What is a sandpile?" by Lionel Levine and James Propp. A sandpile associated to the 2D lattice $\mathbb{Z}^2$ is a simple cellular automaton. At any moment is given a function $f(i,j)$ on the lattice, non-zero at only a finite number of points. A point (i,j) at which $f(i,j) \geq 4$ is selected. The function at (i,j) is decremented by 4, and the value at each of its 4 neighbors is incremented by 1. This is called *toppling* the sandpile at (i,j). In general, one toppling may lead to an avalanche, but the process will eventually reach a stable state with no more topplings possible. Here are the first 9 steps of the process on the cover, which starts out with an initial value of $n = 32$ at the origin.



Here, as on the cover, this is to be read in boustrophedon mode, alternating left-to-right and right-to-left from one row to the next. The selection criterion in this example was lexicographic, but in fact this sandpile is *abelian*, which means that the state after n steps depends only on the set of nodes toppled, not the order in which they are toppled.

As the article by Levine and Propp mentions, and as it is with many cellular automata (my favorite is Langton's ant) it is the conjectural large-scale and/or long-term behavior of sandpiles that fascinates the physicist, who see them as simple examples of behavior near critical points. The conjectured scaling behavior shows up weakly in the growth of the diameter, even for small n.



I have found useful "The abelian sandpile; a mathematical introduction" by Ronald Meester, Frank Redig, and Dmitri Znamenski, which can be found at `www.cs.vu.nl/_rmeester/preprints/sandpile.ps`

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