Topological Methods for Nonlinear Oscillations

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Introduction

Periodic phenomena play a pervasive role in natural and in man-made systems. They are exhibited, for example, in simple mathematical models of the solar system and in the observed circadian rhythms by which basic biological functions are regulated. Electronic devices producing stable periodic signals underlie both the electrification of the world and wireless communications. My interest in periodic orbits was heightened by research into the existence of oscillations in nonlinear feedback systems. While these kinds of applications are illustrated in Examples 2.2 and 4.2, a more detailed expedition into this important application area is omitted here for the sake of space and focus.

Periodic orbits have played a prominent role in the mathematics of dynamical systems and its applications to science and engineering for centuries, due to both the importance of periodic phenomena and the formidable intellectual challenges involved in detecting or predicting periodicity. As a first step toward addressing this challenge, Poincaré developed his method of sections, beginning with the observation that, if a periodic orbit γ for a smooth vector field X exists, if $x_0 \in \gamma$ and if $\mathcal{H}$ is a hyperplane complementary to the tangent line $T_{x_0}(\gamma)$ to x_0 at γ, then on a sufficiently small neighborhood $S \subset \mathcal{H}$ of x_0 one can define, by the implicit function theorem, a (least) positive time $t_x > 0$ so that for each $x \in M$ the solution to the differential equation defined by X with initial condition x returns to $\mathcal{H}$. In particular, one can define a smooth Poincaré, or "first-return", map $\mathcal{P}$ on S, which sends the initial condition x to the solution of the differential equation at time t_x. Moreover, the dynamics of the iterates of $\mathcal{P}$ on S are then intimately related to the dynamics of X, near γ, in positive time. Conversely, if a "local section" S transverse to X exists for which there exists a Poincaré map $\mathcal{P}$, the existence of periodic points for $\mathcal{P}$ implies the existence of periodic orbits for X, allowing for the use of powerful topological fixed point and periodic point theorems in the study of nonlinear oscillations. The importance of Poincaré's method of sections led G. D. Birkhoff to develop two sets of necessary and sufficient conditions [1] for the existence of a section for a differential equation evolving in $\mathbb{R}^n$. One of these was formulated in terms of what Birkhoff called an "angular variable", and the other involved what, in modern terminology, would be called an "angular one-form". Both concepts are reviewed in this article.

The existence of a section is, of course, both one of the standard paradigms for the existence of nonlinear oscillations and one of the grand tautologies of nonlinear dynamics, since to know whether S is section for X is to know a lot about the long-time behavior of the trajectories of the corresponding differential equation—in which case one might already know whether there are periodic orbits. Nonetheless, this paradigm has actually been used with great success in applications, most notably beginning with Birkhoff's proof of Poincaré's Last Theorem, which arose in the restricted three-body problem in celestial mechanics. An easier paradigm is provided by the

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principle of the torus, which has been widely used in applications to biology, chemistry, dynamics, engineering, and physics [13]. In this literature, if $\mathbb{D}^N$ denotes the closed unit disc in $\mathbb{R}^N$, a submanifold $M \subset \mathbb{R}^n$ that is diffeomorphic to $\mathbb{D}^{n-1} \times S^1$ is called a *toroidal region*, and the principle of the torus asserts that, if a smooth vector field X leaves a toroidal region positively invariant and has a section S that is diffeomorphic to $\mathbb{D}^{n-1}$, then X has a periodic orbit in M by Brouwer's fixed point theorem. Of course, among the limiting features of the principle of the torus is the need not only to find a section but also to have the ability to characterize familiar topological spaces such as toroidal regions, disks, and spheres. Fortunately, remarkable advances in dynamics and topology since Poincaré's time now allow us to effectively address both of these technical issues.

Among the tools from dynamics that play a role in the results described in this paper are the properties of nonlinear dynamical systems that dissipate energy. This has been developed in two separate schools, one pioneered by Liapunov and the other beginning with Levinson and significantly developed by Hale, Ladyzhenskaya, Sell, and others. One of the main results of the latter school concerns the existence of Liapunov stable global attractors for dissipative systems. The topological methods described here are also global, allowing one to bring techniques such as the classical combination of cobordism and homotopy theory, as described in [14], to bear on the study of nonlinear oscillations. The early use of topological methods in the study of nonlinear dynamics dates back to the work of Poincaré, Birkhoff, Lefschetz, Morse, Krasnosel'skiĭ, Smale, and many others. The results described here also rely on global topological methods developed by F. W. Wilson Jr. in his study of the topology of Liapunov functions for global attractors. For the case of periodic orbits, Wilson's results form the starting point for the derivation of necessary conditions, derived in [5], for the existence of an asymptotically stable periodic orbit for a smooth vector field defined on an orientable manifold. The proof uses a very general cobordism theorem of Barden, Mazur, and Stallings [10] in dimensions bigger than 5. In lower dimensions, crucial use is also made of the solution of the Poincaré Conjecture in dimensions 3 and 4 by Perelman and Freedman and a result of Kirby and Siebenmann on smoothings of 5-manifolds. The remarkable fact that the necessary conditions are sufficient for the existence of a periodic orbit follows from an explicit cobordism argument [5] involving the period maps of one-forms, introduced by Abel.

In this paper, I give a brief overview of the proofs of these results and then describe how to combine them to derive a new sufficient condition, which replaces a topological assumption with an assumption that the dynamical system dissipates energy. These sufficient conditions are easier to use in practice and are illustrated in several ways, including examples taken from population dynamics and feedback control systems. This exposition concludes with an existence theorem that is valid for a much more general class of smooth manifolds but that requires a more restrictive hypothesis. Among several applications, this result is illustrated in the case of the existence of periodic orbits for smooth vector fields on compact 3-manifolds, with or without boundary.

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Stability of Equilibria, Periodic Orbits, and Compact Attractors

In this section, some basic results about asymptotic stability of compact sets that are invariant with respect to a smooth vector field X are reviewed and illustrated for a feedback design problem presented in Example 2.2. Except for the last two sections of this survey, I will only need these results for vector fields defined on $\mathbb{R}^n$, on the "toroidal cylinder", $\mathbb{R}^n \times S^1$, or on the solid torus, $\mathbb{D}^n \times S^1$. In this section, I will confine the discussion to the case of vector fields on the toroidal cylinder, which in this section is denoted by M. In Sections 3 and 4, vector fields on solid tori are studied in more detail. It should be noted, however, that these results, suitably formulated, do hold for smooth paracompact manifolds, with or without boundary, and with careful modification they also hold in infinite dimensions [9].

Any point in M has coordinates (x, θ), where $x \in \mathbb{R}^n$ and $\theta \in S^1$, and therefore any smooth vector field X on M has the form

$$X = \begin{pmatrix} f_1(x,\theta) \\ f_2(x,\theta) \end{pmatrix}$$

where f_1 takes values in $\mathbb{R}^n$ and f_2 takes values in $\mathbb{R}$. The vector space of smooth vector fields on M is denoted by $\mathrm{Vect}(M)$. In particular, a smooth vector field defines, and is defined by, an ordinary differential equation (ODE)

$$\dot{x} = f_1(x,\theta) \tag{2.1}$$
$$\dot{\theta} = f_2(x,\theta) \tag{2.2}$$

to which the local existence, uniqueness, and smoothness theorem for solutions to ODEs applies, since small variations in an initial condition $z_0 = (x_0, \theta_0)$ take place in an open subset of $\mathbb{R}^n \times \mathbb{R}$. $\Phi(t, z_0)$, defined for sufficiently small t, will denote the solution initialized at z_0 at time $t_0 = 0$. In this paper, only vector fields for which $\Phi(t, z_0)$

is defined, for each $z_0 \in M$ and for all $t \geq 0$, are considered. Any such X defines a *semiflow*

$$\Phi : [0, \infty) \times M \to M. \tag{2.3}$$

When t is fixed, it is often convenient to use the notation $\Phi_t(z) := \Phi(t, z)$. In particular, Φ defines a semigroup of smooth embeddings Φ_t of M.

An equilibrium for X is a point $z_0 \in M$ satisfying $X(z_0) = 0$ or, equivalently, $\Phi(t, z_0) = z_0$ for all $t \geq 0$. A solution curve of (2.3) initialized at a nonequilibrium point $z \in M$ is periodic provided $\Phi_t(z) = z$ for some $t > 0$. The minimum time $T > 0$ such that $\Phi_T(z) = z$ is its *period* and the set of points in M transcribed by a periodic solution is called a *periodic orbit*. A subset I of M is *positively invariant* for a vector field X if $\Phi(t, z) \in I$ for each $z \in I$ and every $t \geq 0$. I is *invariant* if $\Phi(t, z) \in I$ for each $z \in I$ and every $t \in \mathbb{R}$. Equilibria and periodic orbits are invariant sets. A compact invariant set K is a *maximal compact invariant set* for the semigroup (2.3) provided every compact invariant set of (2.3) is contained in K.

For any $B \subset M$, the ω-limit set of B is defined [9] as

$$\begin{aligned} \omega(B) = \{z \in B |\ \text{for}\ \ z_j \in B \text{ and } \ t_j \to +\infty, \\ \text{with } j \to +\infty, \Phi(t_j, z_j) \to z\}. \end{aligned} \tag{2.4}$$

For $B = \{z\}$, this coincides with the ω-limit set $\omega(z)$ introduced by Birkhoff in [1]. The α-limit sets, $\alpha(B)$ and $\alpha(z)$, are defined as in (2.4) with the sequence of times t_j tending to $-\infty$. Following [9], a closed set $A \subset M$ is said to *attract* a closed set $B \subset M$ provided the distance

$$\delta(\Phi_t(B), A) := \sup_{z \in B} \inf_{y \in A} d(\Phi_t(z), y) \tag{2.5}$$

between the sets $\Phi_t(B)$ and A tends to 0 as $t \to +\infty$, where d is any complete metric on M.

Definition 2.1 ([9])**.** A compact invariant subset K is said to

1. be *stable* provided that for every neighborhood V of K, there exists a neighborhood V' of K, satisfying $\Phi_t(V') \subset V$, for all $t \geq 0$;
2. *attract points locally* if there exists a neighborhood W of K such that K attracts each point in W;
3. be *asymptotically stable* if K is stable and attracts points locally.

Remark 2.1. If K is a compact invariant set, the notion of attracting a point or attracting a compact set is independent of the choice of metric, as it should be. Moreover, since K is compact, condition (3) is equivalent to the existence of a positively invariant neighborhood $K \subset L$ for which K attracts L [9, Lemma 3.3.1]. The largest open set, $\mathcal{D}$, attracted by K is called the *domain of attraction* of the attractor K.

Definition 2.2. If a compact subset $K \subset M$ satisfies conditions (1) and attracts every point of M, then K is called a *global attractor*.

Compact attractors exist in many situations in which the dynamical system dissipates energy, a notion that can be mathematized in several ways. There are two formulations of dissipativity that are very useful. In reverse chronological order, one has its roots in the work by Levinson on the forced van der Pol oscillator and is developed in [9] for the case of Banach spaces. In this exposition, it takes the following form.

Definition 2.3. $X \in \mathrm{Vect}(M)$ is *point-dissipative* provided there exists a compact set $K \subset M$ that attracts all points in M.

Remark 2.2. For any $\epsilon > 0$, the ϵ-neighborhood, $B = B_\epsilon(K)$, of a global attractor K is a relatively compact *absorbing set*; i.e., every trajectory eventually enters and remains in B. A system is point-dissipative if, and only if, there exists a relatively compact absorbing set.

Point-dissipative systems on $\mathbb{R}^n$ are also sometimes referred to as being "ultimately bounded" systems, and their origin lies in classical nonlinear analysis.

Example 2.1. Consider a C^∞ periodically time-varying ordinary differential equation

$$\dot{x} = f(x, t), \quad f(x, t + T) = f(x, t) \tag{2.6}$$

evolving on $\mathbb{R}^n$. Historically, a central question concerning periodic systems is whether there exists an initial condition $(x_0, 0)$ generating a periodic solution having period T. Such solutions are called *harmonic* solutions. Following the pioneering work of Levinson on dissipative forced systems in the plane, V. A. Pliss formulated the following general definition for periodically time-varying systems:

Definition 2.4 ([16])**.** The periodic differential equation (2.6) is *dissipative* provided there exists an $R > 0$ such that

$$\varlimsup_{t \to \infty} \|x(t; x_0, t_0)\| < R. \tag{2.7}$$

In particular, the ball $B(0, R)$ of radius R about $0 \in \mathbb{R}^n$ is an absorbing set for the time-varying system (2.7). As noted in [16], the system (2.6) defines a time-invariant vector field X on the toroidal cylinder M via

$$\dot{x} = f(x, \tau) \tag{2.8}$$

$$\dot{\tau} = 1. \tag{2.9}$$

To say that (2.6) is dissipative on $\mathbb{R}^n$ is to say that (2.8) is point-dissipative on M. For a dissipative periodic system, one can define a smooth Poincaré map $\mathcal{P} : \mathbb{R}^n \to \mathbb{R}^n$ defined via

$$\mathcal{P}(x_0) = \Phi(T; x_0, 0). \tag{2.10}$$

An important consequence [16] of (2.7) is that there exists a closed ball $0 \in B \subset \mathbb{R}^n$ and an $r \in \mathbb{N}$

such that if $x_0 \in B$, then $x(t; x_0, 0) \in B$ for $t \geq rT$; i.e.,

$$\mathcal{P}^r : B \to B. \tag{2.11}$$

Applying Brouwer's fixed point theorem to $\mathcal{P}^r$, Pliss [16] showed the existence of a forced oscillation. In fact, Browder's fixed point theorem asserts that any map satisfying (2.11) must have a fixed point in B so that there always exists a harmonic solution.

The theory of dissipative systems has been studied by many mathematicians. Indeed, dissipative systems play a central role in the work of Krasnosel'skiĭ, Hale, Ladyzhenskaya, Sell, and others for both finite- and infinite-dimensional systems. In the present context, the principal result is the following.

Theorem 2.1. *If $X \in \mathrm{Vect}(M)$ is point-dissipative, then there exists a compact attractor $\mathcal{A}$ for X on M. $\mathcal{A}$ is the maximal compact attractor and satisfies*

$$\mathcal{A} = \{z \in M : \{\Phi(t,z) : -\infty < t < \infty\} \text{ is relatively compact}\}. \tag{2.12}$$

In particular, if $B \subset M$ is relatively compact, then $\omega(B) \subset \mathcal{A}$. Moreover, $\mathcal{A}$ is connected.

Remark 2.3. For an ODE evolving on $\mathbb{R}^n$, any trajectory with initial condition z such that $\{\Phi(t,z) : -\infty < t < \infty\}$ is bounded is called *Lagrange stable*.

Example 2.2. The problem considered in this example is referred to as a "set-point control" problem in the systems and control literature and is widely used in engineering applications in which certain physical variables need to be maintained asymptotically close to a desired constant. Important examples include controlling the temperature and air quality in buildings, for which heating, ventilation, and air conditioning consume the largest portion of energy costs, and controlling the altitude or airspeed of aircraft, a problem of great importance for air traffic control. In more detail, consider the system

$$\begin{aligned} \dot{x}_1 &= -gx_1 + u_1, \quad \dot{x}_2 = -gx_2 + u_2, \\ \dot{x}_3 &= x_1u_2 - x_2u_1 - \alpha x_3, \end{aligned} \tag{2.13}$$

which models [2] the control of a rotor by an AC motor, with (x_1, x_2) being the components of the magnetic field, u_1, u_2 being the current through the armature coils, g the resistance in the coils, x_3 modeling the angular velocity of the rotor, and α the coefficient of friction. The control objective studied in [2] was to design a C^∞"control law"

$$u_1 = u_1(x_1, x_2, x_3, d), \quad u_2 = u_2(x_1, x_2, x_3, d) \tag{2.14}$$

so that the system (2.13)–(2.14) has the property that, in "steady-state," $\lim_{t\to\infty} x_3(t) = d$, for a desired constant rate of rotation $d > 0$. In the engineering literature, a system of this form, in which explicit control laws are "fed back" into a control system, such as (2.13), is called a *closed-loop system*, and the control laws are referred to as *feedback laws*.

A natural starting point is to determine necessary conditions on the control laws in order to have the closed-loop system (2.13)–(2.14) be point-dissipative on some positively invariant open set $\mathcal{D} \subset \mathbb{R}^3$ *and* solve the set-point control problem for all initial conditions $x \in \mathcal{D}$. Point-dissipativity would imply the existence of a global compact attractor $\mathcal{A} \subset \mathcal{D}$, while solving the set-point control problem would consist of ensuring that $x_3|_{\mathcal{A}} = d$. Since $\mathcal{A}$ is invariant, $\dot{x}_3|_{\mathcal{A}} = 0$, which implies $x_1\dot{x}_2 - x_2\dot{x}_1 = \alpha d$ or

$$\dot{\theta} = \frac{\alpha d}{x_1^2 + x_2^2} > 0, \quad \text{for } x \in \mathcal{A}, \tag{2.15}$$

where (r, θ) denotes polar coordinates in the (x_1, x_2)-plane. In particular, the magnetic field must rotate in steady-state, as it should in order to generate torque. In fact, for conventional AC motors, the rotational rate of the magnetic field of the AC motor should be constant, and imposing the condition $\dot{\theta} = f > 0$ yields several additional conclusions. For example, since any trajectory on $\mathcal{A}$ will be a closed curve in the affine plane, $x_3 = d$, having constant amplitude $A = \sqrt{\alpha d/f}$, the global compact attractor $\mathcal{A}$ must consist of this single periodic orbit. If one assumes that the control laws (2.14) are defined on all of $\mathbb{R}^3$ and that the rotational rate for the magnetic field of the AC motor is constant for all initial conditions in $\mathcal{D}$, one is led to the further constraint

$$x_1u_2 - x_2u_1 = f(x_1^2 + x_2^2) \tag{2.16}$$

on (2.14), which yields

$$u_1 = \kappa x_1 - fx_2 + x_1h(x_1, x_2) + H_1(x_1, x_2, x_3)(x_3 - d),$$

$$u_2 = \kappa x_2 + fx_2 + x_2h(x_1, x_2) + H_2(x_1, x_2, x_3)(x_3 - d),$$

for some $\kappa \in \mathbb{R}$. Setting $h = 0$, each of these control laws produces a closed-loop system having a periodic orbit γ with period $T = 2\pi/\kappa$ on $x_3 = d$, evolving as a classical harmonic motion, $\dot{x}_1 = -fx_2, \dot{x}_2 = fx_1$, on the circle $(x_1^2 + x_2^2) = \alpha d/f$. In [2], Brockett shows that the feedback law

$$u_1 = gx_1 - fx_2 + \beta(d - x_3)x_1 \tag{2.17}$$

$$u_2 = gx_2 + fx_1 + \beta(d - x_3)x_2 \tag{2.18}$$

where $f, \beta > 0$ solves the set-point control problem, inducing an asymptotically stable periodic orbit γ on $\mathcal{D} = \mathbb{R}^3 - X_3$, where X_3 is the x_3-axis. In fact, this system is point-dissipative on $\mathcal{D} \simeq \mathbb{R}^2 \times S^1$, with γ as its compact global attractor. This is easiest to see using Liapunov methods, which are described below.

A very powerful way to formulate dissipation of energy near an equilibrium was developed by Liapunov in his 1892 thesis and has since been extended to uniformly attractive closed invariant sets. The main results for compact invariant sets suffice for point-dissipative systems.

Definition 2.5. Suppose X leaves an open subset $D \subset M$ positively invariant and that $K \subset D$ is a compact invariant subset. A Liapunov function V for X on the pair (D, K) is a C^∞ function $V : D \to \mathbb{R}$ that satisfies

(1) $V|_K = 0$ and $V(z) > 0$ for $z \notin K$,
(2) $\dot{V} < 0$ on $D - K$, and
(3) V tends to a constant value (possibly ∞) on the boundary, ∂D, of D.

Theorem 2.2. *Suppose X leaves an open subset $D \subset M$ positively invariant and that $K \subset D$ is a compact invariant subset. If a function V exists satisfying conditions* (1) *and* (3) *of Definition* 2.5 *and if $\dot{V} \le 0$ on D, then K is stable. If V also satisfies condition* (2)*, then K is a global compact attractor on D.*

Example 2.3 (Example 2.2 (bis))**.** Consider the "closed-loop" vector field $X \in \mathrm{Vect}(\mathbb{R}^3)$ obtained by implementing the feedback law (2.17) in the system (2.13)

$$\begin{aligned} \dot{x}_1 &= -f x_2 + \beta(d - x_3)x_1, \\ \dot{x}_2 &= f x_1 + \beta(d - x_3)x_2, \\ \dot{x}_3 &= f(x_1^2 + x_2^2) - \alpha x_3. \end{aligned} \tag{2.19}$$

As noted above, (2.19) has a periodic solution γ with initial condition $x(0) = (\sqrt{\alpha d/f}, 0, d)^T$. Following [2], consider the function $V : \mathcal{D} \to [0, \infty)$ defined by

$$\begin{aligned} V(x_1, x_2, x_3) &= \beta(d - x_3)^2 + f(x_1^2 + x_2^2) \\ &\quad - \alpha d \ln(x_1^2 + x_2^2) + \alpha d(\ln(\alpha d/f) - 1). \end{aligned} \tag{2.20}$$

The function V satisfies conditions (1) and (3) of Definition 2.5 for $K = \gamma$. Moreover,

$$\dot{V}(x_1, x_2, x_3) = -2\alpha\beta(d - x_3)^2 \le 0, \tag{2.21}$$

along trajectories of X, so that γ is stable. Since V is nonincreasing along trajectories, it follows that for any $x \in \mathcal{D}$, $\omega(x_0)$ is an invariant subset of $\dot{V}^{-1}(0)$—a very useful result known as *LaSalle's invariance principle.* To say $\dot{V} = 0$ is to say that $x_3 = d$ and, as shown in Example 2.2, the only invariant set on which $x_3 = d$ is γ. This proves that γ is a global attractor on $\mathcal{D}$. A typical sublevel set of V together with a trajectory converging to γ is depicted in Figure 1.

In fact, F. W. Wilson Jr. has shown that Liapunov functions for compact attractors always exist. In this setting, his result takes the following form.

Theorem 2.3. *Suppose that $X \in \mathrm{Vect}(M)$. A necessary condition for a compact subset $K \subset M$ to be a global compact attractor on an open positively invariant domain $D \subset M$ is that there exist a Lyapunov function V for X on the pair (D, K).*

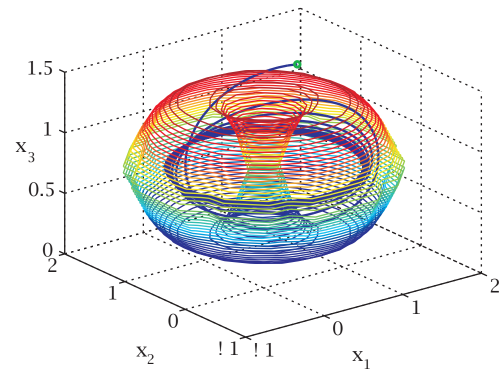

Figure 1. The sublevel set $V^{-1}[0, 1]$ and the trajectory of X for initial condition $(1, .75, 1.5)$.

Remark 2.4. Wilson [19] also studied the topology of Liapunov functions in the case that K is a smooth submanifold. For example, if $K \simeq S^1$, as in the case of an asymptotically stable periodic orbit, then the domain of attraction $\mathcal{D}$ of K always satisfies $\mathcal{D} \simeq \mathbb{R}^{n-1} \times S^1$, in harmony with Examples 2.2–2.3 and Figure 1, for which $\mathcal{D} = \mathbb{R}^3 - X_3 \simeq \mathbb{R}^2 \times S^1$.

Angular Variables and Angular One-Forms

The purpose of this section is to recast Birkhoff's seminal ideas on the existence of sections for smooth dynamical systems in modern terms and to delineate the extent to which these ideas are applicable. For the closed-loop system (2.19), an important role in the analysis of the nonlinear oscillator was played by the variable θ, measuring the rotation of the magnetic field. More formally, for the Liapunov function V defined in (2.20) and for a fixed choice of $c > 0$, denote the sublevel set $V^{-1}[0, c]$ by M and consider the function

$$J : M \to S^1, \quad J(r, \theta, x_3) = \theta, \tag{3.1}$$

which satisfies $\dot{J}(x) = \langle dJ(x), X(x)\rangle = f > 0$. The sign of $\dot{J}$ is irrelevant; only the fact that $\dot{J}$ is sign definite is important. The existence of such an "angular variable" also arises in the two-body problem with a central force field, since conservation of angular momentum implies that $\dot{\theta} = c$, for c a constant and for (r, θ) in an invariant plane of motion.

The importance of angular variables in the theory of nonlinear oscillations was elucidated by G. D. Birkhoff in his 1927 book on nonlinear dynamics. In [1, pp.143–145] Birkhoff derived two sets of necessary and sufficient conditions for the existence of what he called a "surface of section" for an arbitrary differential equation evolving in $\mathbb{R}^n$. For Birkhoff, a smooth section for $X \in \mathrm{Vect}(\mathbb{R}^n)$ is a hypersurface $S \subset M$ in some region $M \subset \mathbb{R}^n$ so that, for each $x \in M$, the "flow line" (trajectory) through x intersects S transversely at a (least) forward time $t_x > 0$ and, replacing t by $-t$, at a least time $s_x > 0$ in reverse time. In this case, he defines the map $\phi(x) = 2\pi s_x/(s_x + t_x)$ and notes that ϕ "increases along every streamline (trajectory of X) by 2π between successive intersections with S". Abstracting from this construction, he called any map ϕ satisfying $\frac{d}{dt}(\phi(x(t))) > 0$ for all trajectories $x(t) \in M$ an *angular variable* for X. Moreover, if ϕ is an angular variable, he observes that $S = \phi^{-1}(0)$ is a surface of section for X on M.

Of course, an angular variable is actually a multivalued function, but it can be made single-valued as a map with values in S^1. For example, Birkhoff's construction leads to the map

$$J(x) = \exp(2\pi i s_x/(s_x + t_x)). \tag{3.2}$$

Birkhoff's second set of conditions is the existence of smooth functions a_i such that

$$\sum_{i=1}^{n} a_i X_i > 0 \text{ and } \frac{\partial a_i}{\partial x_j} = \frac{\partial a_j}{\partial x_i}, \text{ for } i, j = 1, \ldots, n,$$

where the X_i are the coordinates of X, abstracting the observation that $\dot{J} = \sum_{i=1}^{n} \frac{\partial J}{\partial x_i} X_i$ is actually a well-defined, real-valued function on M.

More precisely, suppose $M \subset \mathbb{R}^n$ consists of an open subset together with a smooth boundary. In the language of one-forms, one may define $\omega = \sum_{i=1}^{n} a_i dx_i$ and note that Birkhoff's conditions assert that ω is closed, i.e., $d\omega = 0$, and that the natural pairing of the one-form ω and the vector field X satisfies

$$\langle \omega, X \rangle = \sum_{i=1}^{n} a_i X_i > 0. \tag{3.3}$$

It is important to note that (3.3) can be checked pointwise (in particular, without explicit knowledge of trajectories) just as in the applications of Liapunov functions. In this spirit, one may also formulate the existence of an angular variable without reference to the trajectories $x(t)$. If $\mathrm{Vect}_+(M)$ denotes the set of vector fields that point inward on the boundary of M, then M is positively invariant under any $X \in \mathrm{Vect}_+(M)$.

Definition 3.1 ([5]). Suppose $X \in \mathrm{Vect}_+(M)$. We say that a map $J : M \to S^1$ is an *angular variable* for X if it satisfies

$$\dot{J} = \langle dJ, X \rangle > 0 \tag{3.4}$$

everywhere on M. If ω is a closed one-form on M, then ω is an *angular one-form* for X provided that (3.3) holds everywhere on M.

For example, if M is the solid torus, $\mathbb{D}^{n-1} \times S^1 \subset \mathbb{R}^n$, the smooth boundary of M is $S^{n-2} \times S^1$, where S^r denotes the r-sphere. In this case, every closed one-form can be written as $\omega = \sum_{i=1}^{n-1} a_i dx_i + a_n d\theta$. In fact, there exists $c \in \mathbb{R}$ such that

$$\omega = c\,d\theta + df \tag{3.5}$$

for a smooth function $f : M \to \mathbb{R}$.

This is most easily seen using the basic theory of fundamental groups, e.g., the fundamental group $\pi_1(M)$ of M satisfies $\pi_1(M) \simeq \mathbb{Z}$ with a generator given by a simple closed path γ transversing $\{0\} \times S^1$ in either direction. Choose a fixed "base point" $x_0 \in M$ and an arbitrary upper limit $x \in M$ for the integral

$$\int_{x_0}^{x} \omega. \tag{3.6}$$

Of course, this integral may depend on a choice of path joining x_0 to x. Suppose that γ_1, γ_2 are two such paths and consider the closed path $\tilde{\gamma}$ constructed by traversing γ_1 and then γ_2 in the opposite direction. To say that $\int_{x_0}^{x} \omega$ is path independent is to say that $\int_{\tilde{\gamma}} \omega = 0$ for every closed path $\tilde{\gamma}$. If $c = \int_{\gamma} \omega$, then $\int_{\gamma}(\omega - c\,d\theta) = 0$ and, since every closed path $\tilde{\gamma}$ is homotopic to a multiple of γ, $\int_{\tilde{\gamma}}(\omega - c\,d\theta) = 0$ for all closed paths $\tilde{\gamma}$. Therefore $\int_{x_0}^{x}(\omega - c\,d\theta) = f(x)$ is independent of path and $\omega = c\,d\theta + df$.

If c in (3.5) is an integer, then ω is called an integral closed one-form, and every such one-form defines a smooth *period map* $J : M \to S^1$ in the following manner. As noted above, (3.6) is only defined up to "periods" of ω, i.e., up to the elements of the subgroup

$$\Pi(\omega) = \left\{ \int_{\gamma} \omega : [\gamma] \in \pi_1(M) \right\} = (c) \subset \mathbb{Z} \subset \mathbb{R}.$$

If $c \neq 0$, then $\Pi(\omega)$ is an infinite cyclic subgroup, and therefore the *period map* J of ω defines a smooth surjection from M to a circle

$$J : M \to S^1, \tag{3.7}$$

defined via

$$J(x) = \left[\int_{x_0}^{x} \omega \right] \in \mathbb{R} \bmod (c). \tag{3.8}$$

Moreover, J satisfies

$$\dot{J} = \langle dJ, X \rangle > 0 \tag{3.9}$$

if and only if ω is an angular one-form for X.

Remark 3.1. If ω is an angular one-form, then it can be shown that $c \neq 0$. Moreover, the normalization $\omega/|c|$ is an integral angular one-form, where the constant in (3.5) is ± 1. Therefore, an angular variable exists. Conversely, since $S^1 \simeq \mathbb{R}/\mathbb{Z}$, a smooth map $J : M \to S^1$ can be regarded as a multivalued map $J : M \to \mathbb{R}$ where the value $J(x)$ is determined only up to an integer constant. Nonetheless, $\omega = dJ$ is well-defined as a one-form on M, since the derivative of a constant is zero. Moreover, if J is an angular variable for X, then $\langle \omega, X \rangle = \langle dJ, X \rangle > 0$, so that ω is an angular one-form. It is convenient to employ a synthesis of these two approaches.

Remark 3.2. The calculations made above, such as (3.5), for the solid torus also hold, without change of notation, for the toroidal cylinder $\mathbb{R}^n \times S^1$.

Periodic Orbits on Solid Tori and Toroidal Cylinders

While Birkhoff was not specific about the ambient submanifold $M \subset \mathbb{R}^n$ in which a surface of section might exist for X or what that might imply for the topology of M, his statements clearly exclude the choice $M = \mathbb{R}^n$ and specifically include surfaces of section having a boundary. Moreover, it is necessary for his construction that M be an invariant set. The development of angular variables in [5] includes the case of a smooth manifold with boundary that is only required to be positively invariant and allows for a characterization of M topologically when there exists a locally asymptotically stable orbit.

Theorem 4.1 ([5]). *Suppose that $n > 1$. If γ is an asymptotically stable periodic orbit of $X \in \mathrm{Vect}(\mathbb{R}^n)$, then there exists a smooth, positively invariant n-dimensional submanifold $\gamma \subset M \subset \mathbb{R}^n$, homeomorphic to a solid n-torus, on which X has an angular variable $J : M \to S^1$. In fact, M is diffeomorphic to $\mathbb{D}^{n-1} \times S^1$, except perhaps when $n = 4$.*

Remark 4.1. The idea underlying the proof is to build a positively invariant solid torus using a Liapunov function V on the domain of stability $\mathcal{D}$ of γ, generalizing what was observed about the system (2.19). Indeed, for c sufficiently small,

$$M_c = V^{-1}[0, c] \tag{4.1}$$

is a positively invariant compact subset, consisting of an open subset and with a smooth boundary. More succinctly, M_c is a compact orientable smooth manifold with boundary. Moreover, for c sufficiently small, M_c clearly admits an angular variable, since γ does. The rest of the proof uses several key ingredients, starting with the fact that, according to Remark 2.4, the domain of attraction, $\mathcal{D}$, for γ is diffeomorphic to $\mathbb{R}^{n-1} \times S^1$. It can also be shown that the inclusion $M_c \subset \mathcal{D}$ induces an isomorphism of homotopy groups. In other words, the inclusion is a homotopy equivalence. On the other hand the projection $p_2 : \mathbb{R}^{n-1} \times S^1 \to S^1$ onto the second factor, $p_2(x, \theta) = \theta$, is also a homotopy equivalence, since $\mathbb{R}^n$ is contractible. Consequently, M_c is homotopy equivalent to S^1.

For $n = 2$, by the classification of surfaces it then follows that $M_c \simeq \mathbb{A}$, the standard two-dimensional annulus. For $n = 3$, the solution by Perelman of the classical Poincaré conjecture [15] implies that, up to diffeomorphism, $\mathbb{D}^2 \times S^1$ is the only 3-dimensional compact, orientable manifold that is homotopy equivalent to S^1. Similarly, for $n = 4$, the solution by Freedman of the 4-dimensional Poincaré conjecture [7] implies that, up to homeomorphism, $\mathbb{D}^3 \times S^1$ is the only 4-dimensional compact, orientable manifold that is homotopy equivalent to S^1. In higher dimensions, there are infinitely many smooth compact orientable manifolds homotopy equivalent to S^1, but M_c is special. Adapting some constructions of Wilson [19], one can show that ∂M_c is homotopy equivalent to $S^{n-1} \times S^1$. This fact, along with some homotopy theory, allows one to use Freedman's proof of the 4-dimensional Poincaré conjecture [7] to show that, for $n = 5$, M_c is homeomorphic to $\mathbb{D}^4 \times S^1$, while a fundamental result due to Kirby and Siebenmann [11] implies that M_c is diffeomorphic to $\mathbb{D}^4 \times S^1$. An application of the theorem of Barden, Mazur, and Stallings [10] completes the proof [5] of Theorem 4.1 for $n \geq 6$.

Remark 4.2. It is unknown how many differentiable structures on $\mathbb{D}^3 \times S^1$ may exist.

In fact, the necessary conditions for the existence of an asymptotically stable periodic orbit are also sufficient for the existence of a periodic orbit.

Theorem 4.2 ([5]). *If $M \subset \mathbb{R}^n$ is a smooth submanifold which is diffeomorphic to $\mathbb{D}^{n-1} \times S^1$, then any $X \in \mathrm{Vect}(\mathbb{R}^n)$ leaving M positively invariant and having an angular variable $J : M \to S^1$ has a periodic orbit in M. Moreover, the homotopy class of this periodic solution generates $\pi_1(M)$.*

The idea behind the proof is to first use a level set $S_\theta = J^{-1}(\theta)$, for any regular value θ of an angular variable J, as a section for X and to next prove that, after modifying J if necessary, $S_\theta \simeq \mathbb{D}^{n-1}$. In particular, one can apply Brouwer's fixed point theorem to the Poincaré map $\mathcal{P} : S_\theta \to S_\theta$.

Briefly, since J is an angular variable, then $dJ = \omega = c d\theta + df$ according to (3.5) and, according to Remark 3.1, $c \neq 0$. Without loss of generality, one can assume $c = 1$ and can embed ω in the family of one-forms $\omega_\lambda = d\theta + \lambda df$, $0 \leq \lambda \leq 1$, which defines a homotopy

$$\tilde{J} : M \times [0, 1] \to S^1, \quad \tilde{J}(x, \lambda) = \int_{x_0}^{x} \omega_\lambda \tag{4.2}$$

between the period mappings $J_0 = J(\cdot, 0)$ and $J_1 = J(\cdot, 0) = J$ and therefore a deformation of $J_0^{-1}(\theta) \simeq \mathbb{D}^{n-1}$ into $J_1^{-1}(\theta) \simeq S$. The remainder of the proof in [5] uses the fruitful relationship between homotopy and cobordism, as described in [14], to show that this deformation is a diffeomorphism.

Remark 4.3. In [5], Theorems 4.1 and 4.2 are proven for the more general case in which $\mathbb{R}^n$ is replaced by an arbitrary orientable paracompact manifold N of dimension $n > 1$.

One corollary of Theorem 4.1 and the proof of Theorem 4.2 is that, except perhaps when $n = 4$, the seemingly stringent hypotheses of the principle of the torus are actually necessary for the existence of a locally asymptotically stable periodic orbit. More importantly, a combination of the proofs of Theorems 4.1 and 4.2 yields an amplification of Theorem 4.2 that combines topology and dynamics in a form that is easier to apply in practice.

Theorem 4.3. *Suppose that* $N \simeq \mathbb{R}^n \times S^1$. *If* $X \in \mathrm{Vect}(N)$ *is point-dissipative and has an angular variable* J, *then* X *has a periodic orbit* γ. *Moreover, the homotopy class determined by* γ *generates* $\pi_1(N)$.

This result was originally proved [3, Theorem 2.1] for the class of dissipative periodic systems discussed in Example 2.1, but the proof extends to the general case. The key idea is to use a Liapunov function V for the global attractor $\mathcal{A}$ for X to build a positively invariant torus for X as in Remark 4.1 and then apply Theorem 4.2. Since $\mathcal{A}$ is not in general smooth, this argument does require a bit more work than the proof of Theorem 4.1.

Theorem 4.3 generalizes Example 2.1 and gives a new proof of Browder's theorem on the existence of harmonic oscillations for dissipative periodic systems (2.6) as well. Indeed, any dissipative system (2.6) was noted to be equivalent to the point-dissipative system (2.8) evolving on the toroidal cylinder $N = \mathbb{R}^n \times S^1$. Since (2.8) has τ as an angular variable, a periodic solution $\{\Phi(t, x_0, 0)\} \subset N$ exists. That $\{\Phi(t, x_0, 0)\}$ is *harmonic* follows from the fact that the homotopy class of $\Phi(\cdot, x_0, 0)$ generates $\pi_1(N) \simeq \mathbb{Z}$.

As the next example shows, Theorem 4.3 also applies directly to the May-Leonard equations, modeling the population dynamics of three competing species with immigration [6].

Example 4.1. The May-Leonard model for three competing species with immigration ($\epsilon > 0$),

$$\begin{aligned} (4.3)\quad \dot{N}_1 &= N_1(1 - N_1 - \alpha N_2 - \beta N_3) + \epsilon \\ (4.4)\quad \dot{N}_2 &= N_2(1 - \beta N_1 - N_2 - \alpha N_3) + \epsilon \\ (4.5)\quad \dot{N}_3 &= N_3(1 - \alpha N_1 - \beta N_2 - N_3) + \epsilon, \end{aligned}$$

where $0 < \beta < 1 < \alpha$, $\alpha + \beta > 2$, leaves the positive orthant

$$\mathcal{P}^+ = \{(N_1, N_2, N_3) : N_i > 0, \ \ i = 1, 2, 3\}$$

positively invariant. Let $X \in \mathrm{Vect}(\mathcal{P}^+)$ denote the vector field defined by this differential equation.

The function $V(N_1, N_2, N_3) = N_1 + N_2 + N_3$ is positive on $\mathcal{P}^+$ and has derivative $\dot{V} = L_X V(N_1, N_2, N_3) = N_1 + N_2 + N_3 - (N_1^2 + N_2^2 + N_3^2) - (\alpha + \beta)(N_1 N_2 + N_2 + N_3 + N_1 N_3)$, which is negative for (N_1, N_2, N_3) sufficiently large in norm, so that $B(0, R) \cap \mathcal{P}^+$ is an absorbing set, for the ball of some radius R. Since $\epsilon > 0$, the vector field $X|_{\partial \mathcal{P}^+}$ points inward, and therefore there is a smaller, relatively compact absorbing set in $\mathcal{P}^+$. Hence, by Remark 2.2, X is point-dissipative.

Following [6], there is a unique equilibrium $(\nu(\epsilon), \nu(\epsilon), \nu(\epsilon)) = (1 + (1 + 4\epsilon\rho)^{1/2})/2\rho \in \mathcal{P}^+$, where $\rho = 1 + \alpha + \beta$. Immigration stabilizes the population around this equilibrium if $\epsilon > 2(\rho - 3)/(\rho + 3)^2$, but for

$$(4.6) \qquad \epsilon < 2(\rho - 3)/(\rho + 3)^2$$

the equilibrium is unstable with a one-dimensional stable manifold $W^s(0) = \{(N, N, N)\}$, where $N_i = N > 0$ for $i = 1, 2, 3$.

In this case, $M = \mathcal{P}^+ - W^s \simeq \mathbb{R}^2 \times S^1$ is positively invariant. Moreover, the one-form

(4.7)
$$\omega = \frac{(N_1 dN_2 - N_2 dN_1) + (N_2 dN_3 - N_3 dN_2) + (N_3 dN_1 - N_1 dN_3)}{N_1^2 + N_2^2 + N_3^2 - (N_1 N_2 + N_2 N_3 + N_1 N_3)}$$

is an angular one-form for X on M. Therefore, by Theorem 4.3, there exists a periodic orbit $\gamma \in \mathcal{P}^+$ whenever (4.6) is satisfied, as is illustrated in Figure 2.

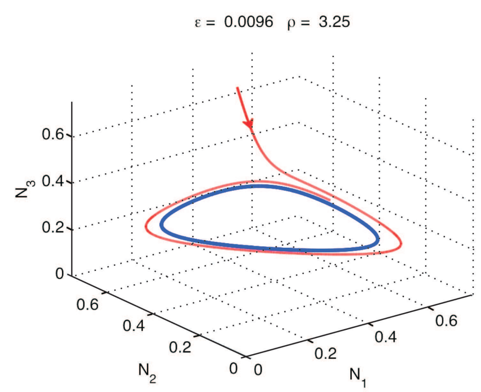

Figure 2. A periodic trajectory for $\alpha = 1.5$, $\beta = .75$.

Remark 4.4. The existence of periodic orbits for the May-Leonard equations when (4.6) is satisfied is well known, and some of our calculations were inspired by the analysis of these equations in [6], although our use of angular one-forms and dissipativity is new and more streamlined. Indeed, the treatment in [6] proves the existence of a periodic solution by checking some comparatively very restrictive hypotheses in an existence theorem due to Grasman [6]. In fact, Grasman's theorem is a corollary of Theorem 4.3.

Example 4.2. (Voltage Controlled Oscillators with Nonlinear Loop Filters.) A phase-locked loop (PLL) is a basic electronic component used in wireless communication networks for the transmission of stable periodic signals. A PLL consists of three components: a *phase detector* (PD), a *voltage-controlled oscillator* (VCO), and a ("low-pass") *loop* *filter* (LF), each of which can be described in terms of a mathematical model. For example, in a very simple, commercially available form, the LF has the form $\dot{y} = -y + u$, where $u, y \in \mathbb{R}$ are the input and output of a one-dimensional system, the VCO is an integrator, $\dot{\theta} = y$ and the closed-loop system produced by a PD that compares the phase results in the feedback control $u = \alpha - a\sin(\theta)$, where $\alpha > a > 0$. In this case the region $y > \alpha - a$ is positively invariant, and using Poincaré-Bendixson theory for the pseudo-polar coordinates (y, θ) in this region shows that the interconnected feedback system results in a sustained, or self-excited, oscillation in the steady-state response of $y(t)$. In this example, I consider the 3-dimensional, nonlinear LF defined by

$$\dot{x}_1 = -2x_1 - x_1 e^{x_2} + x_2 \tag{4.8}$$
$$\dot{x}_2 = -x_1 - 3x_2 - x_2^3 + \beta y \tag{4.9}$$
$$\dot{y} = u \tag{4.10}$$

with the same VCO and feedback law as above, resulting in the interconnected feedback system on $M = \mathbb{R}^3 \times S^1$ defined by

$$\dot{x}_1 = -2x_1 - x_1 e^{x_2} + x_2 \tag{4.11}$$
$$\dot{x}_2 = -x_1 - 3x_2 - x_2^3 + \beta y \tag{4.12}$$
$$\dot{y} = -y + \alpha + a\sin(\theta) \tag{4.13}$$
$$\dot{\theta} = y \tag{4.14}$$

where $\alpha > a > 0$. When $\beta = 0$, this system is uncoupled, consisting of a two-dimensional globally asymptotically stable system on $\mathbb{R}^2$ and the classic voltage-controlled oscillator. In fact, using Theorem 4.3, one can show that there exists a sustained oscillation for any $\beta \geq 0$. First, note that since $\alpha > a$ $N = \{(x_1, x_2, y, \theta) : y > 0\}$ is a positively invariant submanifold which is diffeomorphic to $\mathbb{R}^3 \times S^1$. Moreover, θ is an angular variable for X on N since $y > 0$. Finally, using the energy function

$$V(x_1, x_2, y) = x_1^2 + x_2^2 + (y - \alpha)^2, \tag{4.15}$$

it is straightforward to see that the system is point-dissipative on N, provided $\beta \geq 0$. Therefore, by Theorem 4.3 there exists a periodic orbit, as is illustrated in Figure 3.

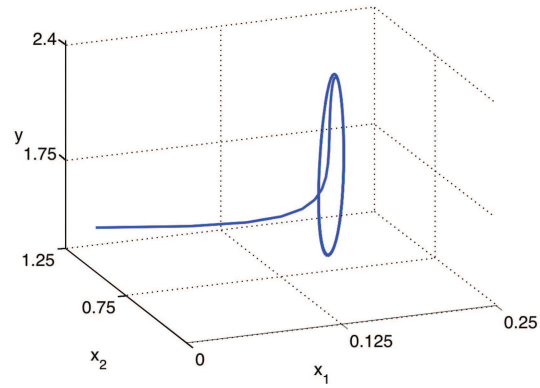

Figure 3. A periodic orbit in the case for $a = 1, \alpha = 2$, and $\beta = 1$.

The Existence of Periodic Orbits for Vector Fields on Closed Three Manifolds

In this and the next section, I will presume familiarity with the concept of a smooth manifold. An excellent introduction, and invitation, to the subject is the book [14]. As a prelude to investigating how generally the sufficient conditions in Theorem 4.2 might hold, consider the case of $X \in \mathrm{Vect}(M)$, where M is a compact orientable 3-manifold. For example, an irrational constant vector field on the 3-torus, $\mathbb{T}^3$, is nowhere vanishing and *aperiodic* by Kronecker's theorem on Diophantine approximations. Moreover, it is easy to construct a constant coefficient angular one-form for such a vector field. Another class of counterexamples can be constructed from the Heisenberg group $N = H_3(\mathbb{R})$ of nonsingular upper-triangular 3×3 real matrices and its discrete subgroups $\Gamma_k = H_3(k\mathbb{Z})$, where $H_3(k\mathbb{Z})$ consists of the upper triangular Heisenberg matrices with integer entries all divisible by $k \in \mathbb{Z}$ with $k \geq 1$. Explicitly, L. Auslander, Hahn, and L. Markus have shown that there exist (left invariant) vector fields on N that descend to vector fields on the compact 3-manifolds $N_k = N/\Gamma_k$ having N_k as their smallest closed invariant set. In particular, any such vector field is aperiodic, and I have constructed examples which also possess angular one-forms [4]. Actually, these are the only counterexamples in dimension 3.

Theorem 5.1 ([4]). *Suppose that M is a compact orientable 3-manifold without boundary. Every $X \in \mathrm{Vect}(M)$ having an angular variable has a periodic orbit except when M is a nilmanifold, i.e., except when either*

(1) $M \simeq \mathbb{T}^3$, *or*
(2) $M \simeq N/\Gamma_k$.

If J is an angular variable for a complete vector field on a compact 3-manifold without boundary, then $S = J^{-1}(0)$ is a compact surface that is a global section for X on M. Since M is orientable and X is transverse to S, S is orientable and can be shown to be connected [4]. Therefore S is a compact orientable connected surface S_g with g holes, and there is a Poincaré map $\mathcal{P} : S_g \to S_g$. In particular, periodic orbits will exist provided $\mathcal{P}$ has a periodic point, i.e., a fixed point of $\mathcal{P}^k$ for $k \in \mathbb{Z}$ with $k \geq 1$.

For what follows, I will also need to assume some familiarity with algebraic topology, particularly homology or cohomology and the notion of the Euler characteristic of a space. For example, the surfaces S_g have Euler characteristic $\chi(S_g) = 2 - 2g$. At about the same time that Birkhoff published [1], S. Lefschetz published a remarkable fixed point theorem that vastly generalized Brouwer's fixed point theorem. As a special case of the general theorem, if $f : M \to M$ is a continuous map on a smooth compact manifold, with or without boundary, Lefschetz introduced an integer $\Lambda(f)$, which can be computed in terms of f and the homology or cohomology vector spaces of M, and for which $\Lambda(f) \neq 0$ implies that f has a fixed point. In 1953, one of Lefschetz's students, F. B. Fuller, extended this result to a neat theorem that implies that if $P : N \to N$ is a homeomorphism on a compact manifold N, with or without boundary, then

$$\chi(N) \neq 0 \implies \Lambda(P^k) \neq 0, \text{ for some } k = 1, 2, \dots \tag{5.1}$$

and hence P has a periodic orbit. In particular, if $S \simeq S_g$ for $g \neq 1$, then the Poincaré map always has a periodic point ,and therefore X has a periodic orbit. In case $g = 1$, the section S is a 2-torus and the remainder of the proof of Theorem 5.1 consists of checking when $\Lambda(f) \neq 0$ by hand and what $\Lambda(f) = 0$ means geometrically, following [18]. The remarkable fact is the role played by nilmanifolds, which can also be expressed algebraically in terms of fundamental groups.

The Existence of Nonlinear Oscillations on *n*-Manifolds With or Without Boundary

In the decade following Fuller's publication of his theorem on periodic points, there were substantial applications of algebraic topology to the study of the existence of periodic orbits for dynamical systems having an angular variable. If a vector field X generates a solution backward and forward for all time, then the solutions of the corresponding differential equation define a mapping

$$\Phi : \mathbb{R} \times M \to M, \tag{6.1}$$

which is said to be a *flow*. The case of flows is more tractable than semiflows and was developed quite generally by S. Schwartzman [17]. An important special case of his results is the following.

Theorem 6.1 ([17]). *Let M be a compact manifold, with or without boundary, and suppose $X \in \mathrm{Vect}(M)$ defines a flow on M. The following conditions on a closed submanifold $S \subset M$ are equivalent:*

(1) *S is a cross section.*
(2) *The smooth map $\Phi : \mathbb{R} \times S \to M$ defines a covering space with an infinite cyclic group of covering transformations.*
(3) *The map $\Phi : \mathbb{R} \times S \to M$ is a surjective local diffeomorphism.*
(4) *There exists a smooth angular variable $J : M \to S^1$.*

In particular, if the covering space has a nonvanishing Euler characteristic, then X has a periodic orbit. On the other hand, for systems that dissipate energy, the objects of interest are often asymptotically stable invariant sets, positively invariant submanifolds, and semiflows. In this direction, Fuller [8] also used the method of angular variables in the more difficult case of a semiflow in a general setting that includes the case of a compact manifold. In this case, following [8], a smooth connected hypersurface S is said to be a *positive cross section* for $X \in \mathrm{Vect}_+(M)$ provided S is a local section for X everywhere in S and, for each $x \in M$, there is a time $t_x > 0$ such that $\Phi_{t_x}(x) \in S$. Among the additional topological challenges in the fundamental work of Fuller on the existence of nonlinear oscillations for such dissipative systems is that, while the Poincaré map is an embedding, it is typically not a (surjective) diffeomorphism, and his theorem on periodic points does not apply. Nonetheless, Fuller was able to prove the existence of periodic orbits in several interesting situations.

Fortunately, a refinement of the notion of angular one-forms provides a general approach to surmounting this technical difficulty.

Definition 6.1. When $\partial M \neq \emptyset$, ω is said to be a *nonsingular angular one-form* provided it is an angular one-form for X and $\omega|_{\partial M}$ is nonsingular.

By Sard's theorem for manifolds with boundary [14] and the compactness of M, it follows that the period map (3.7) of a nonsingular angular form ω is a fiber bundle, since both J and $J|_{\partial M}$ are submersions. Moreover, using the existence of a nonsingular angular one-form, one can show that $\mathcal{P}_*$ is homotopic to a diffeomorphism of S, and therefore $\mathcal{P}_*$ is an automorphism of the integral homology ring $H_*(S)$ of S. This key fact enables us to use a corollary of a result of Halpern, generalizing Fuller's periodic point theorem.

Theorem 6.2. *Suppose that M is a compact manifold with boundary for which $\chi(M) \neq 0$. Any continuous map $f : M \to M$ inducing an automorphism f_* on $H_*(M)$ satisfies $\Lambda(f^k) \neq 0$, for some $k \geq 1$. In particular, f has a periodic point.*

We summarize these results concerning the existence of periodic orbits as follows.

Theorem 6.3 ([4]). *Suppose that M is a smooth, compact connected orientable manifold, with or without boundary, and suppose $X \in \mathrm{Vect}_+(M)$ has a nonsingular angular one-form. There exists a smooth compact, connected and oriented submanifold $S \subset M$ having codimension one and boundary $\partial S = S \cap \partial M$ such that*

(1) *S is a global positive section for X, and*
(2) *$\mathcal{P}_* : H_*(S) \to H_*(S)$ is an automorphism.*

Consequently, if $\chi(S) \neq 0$, then X has a periodic orbit.

Remark 6.1. All compact submanifolds S satisfying conditions (1)–(2) have canonically isomorphic integral homology rings, so that $\chi(S)$ is intrinsically defined. There are counterexamples due to Fuller for $n \geq 4$ that show the inequality in (3.3) must be strict.

Denoting as before the annulus in two dimensions by $\mathbb{A}$, the *hollow torus*, $M = \mathbb{A} \times S^1$, is also the product of the torus $\mathbb{T}^2$ and an interval and therefore admits nowhere vanishing aperiodic vector fields, some of which have a nonsingular angular one-form. This is the only source of counterexamples to the existence of periodic orbits for vector fields on a 3-manifold with boundary having a nonsingular angular one-form.

Theorem 6.4 ([4]). *Suppose M is a three-dimensional manifold with boundary. Every $X \in \mathrm{Vect}_+(M)$ that has an angular one-form ω has a periodic solution whose homotopy class has infinite order in $\pi_1(M)$, except when M is diffeomorphic to a hollow torus $\mathbb{A} \times S^1$.*

There are several other corollaries of Theorem 6.3. For example, [4] contains a general result for closed 5-manifolds that implies that periodic orbits exist for vector fields with an angular one-form on any closed 5-manifold with $\pi_1(M) \simeq \mathbb{Z}$. A similar result is proven in [4] for vector fields defined on a compact manifold M with boundary that is homotopy equivalent to S^1.

Conclusions

Nonlinear oscillations are fascinating and important but hard to rigorously detect or predict. Since Poincaré's time, the best known and most accessible methods in applied mathematics and related fields rely on small parameter analysis and provide local existence criteria for periodic motions having sufficiently small amplitudes. On the other hand, since the period of nonlinear oscillations is generally not known a fortiori, the existence of nonlinear oscillations is a global phenomenon, and therefore any comprehensive theory would necessarily be global in nature. This article continues in the tradition of Poincaré, Birkhoff, and others in studying cross sections for vector fields, creating a global approach to developing criteria for the existence of periodic orbits using methods drawn from the global theory of nonlinear dynamical systems that dissipate some mathematical form of energy and methods drawn from algebraic and differential topology, particularly the fruitful combination of cobordism and homotopy theory.

One of the major points of departure for this approach is the ability to include motions of a dynamical system that leave a manifold with boundary positively invariant, rather than invariant both forward and backward in time. This enables one to discuss and characterize what must occur topologically when a locally asymptotically stable periodic orbit exists, a necessary condition that itself proves to be sufficient for the existence of periodic orbits. Using the language and methods of dissipative systems formulated by Hale, Ladyzhenskaya, and others, this sufficient condition is reformulated into a global sufficient condition that is fairly easy to apply in several specific examples. The article concludes with a formulation of stricter sufficient conditions for the existence of periodic orbits for vector fields defined, however, on general compact manifolds, with or without boundary, that fiber over a circle.

My own interest in this subject is the existence of asymptotically stable periodic motions in nonlinear feedback systems, both manmade and natural, and possible future directions of research should include the incorporation of stability criteria, including classical tools such as Hopf bifurcations and describing function methods and the intriguing possibility of extending this work to the case of invariant tori.

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