WHAT IS... a Linear Algebraic Group?

Skip Garibaldi

From a marketing perspective, algebraic groups are poorly named. They are not the groups you met as a student in abstract algebra, which I will call *concrete groups* for clarity. Rather, an algebraic group is the analogue in algebra of a topological group (from topology) or a Lie group (from analysis and geometry).

Algebraic groups provide a unifying language for apparently different results in algebra and number theory. This unification can not only simplify proofs, it can also suggest generalizations and bring new tools to bear, such as Galois cohomology, Steenrod operations in Chow theory, etc.

Definitions

A *linear algebraic group over a field F* is a smooth affine variety over F that is also a group, much like a topological group is a topological space that is also a group and a Lie group is a smooth manifold that is also a group. (For nonexperts: it is useful to think of an affine variety G as a natural assignment—i.e., a functor—that takes any field extension K of F and gives the set $G(K)$ of common solutions over K of some fixed family of polynomials with coefficients in F.)

Properly speaking, for each of the three types of groups in the previous paragraph, one needs to require that the group operations are morphisms in the appropriate category, so, e.g., for a topological group G, the multiplication $G \times G \to G$ and inversion $G \to G$ are required to be continuous. In more categorical language, a linear algebraic group over F, a Lie group, or a topological group is a "group object" in the category of affine varieties over F, smooth manifolds, or topological spaces, respectively. For algebraic groups, this implies that the set $G(K)$ is a concrete group for each field K containing F.

Examples

The basic example is the general linear group GL_n for which $\mathrm{GL}_n(K)$ is the concrete group of invertible n-by-n matrices with entries in K. It is the collection of solutions (t, X)—with $t \in K$ and X an n-by-n matrix over K—to the polynomial equation $t \cdot \det X = 1$. Similar reasoning shows that familiar matrix groups such as SL_n, orthogonal groups, and symplectic groups can be viewed as linear algebraic groups. The main difference here is that instead of viewing them as collections of matrices over F, we view them as collections of matrices over every extension K of F.

Roughly, the theory of linear algebraic groups generalizes that of linear Lie groups over the real or complex numbers to give something that makes sense over an arbitrary field. The category of linear algebraic groups over $\mathbb{R}$ contains a full subcategory equivalent to the compact Lie groups; see [3, §5]. And the parameterization of the irreducible finite-dimensional representations of a complex reductive Lie group in terms of dominant weights holds more generally for so-called "split reductive" groups over any field.

Algebraic groups allow one to deal systematically with familiar matrix groups and their generalizations in a way that works over arbitrary fields, whether they are the rationals for number theory, finite fields for finite group theory, or the real or complex numbers for geometry. This is not just a language. There is enough theory available that one can often avoid computing with actual

---

Skip Garibaldi is Winship Professor of Mathematics at Emory University. His email address is `skip@member.ams.org`.

The author thanks B. Conrad, G. McNinch, R. Parimala, G. Prasad, and J.-P. Serre for their valuable suggestions.

matrices, or at least with matrices larger than 2-by-2!

Local-Global Principles

Quadratic forms and division algebras over a global field[1] F are determined by their properties at the completions F_v of F; this is roughly the content of the Hasse-Minkowski theorem and the Albert-Brauer-Hasse-Noether theorem, respectively. These theorems can be viewed as saying that the natural map

$$(*) \qquad H^1(F,G) \to \prod_{\text{places } v \text{ of } F} H^1(F_v,G)$$

in Galois cohomology is injective, where G is an orthogonal group or PGL_n. This reformulation of the problem in terms of algebraic groups has two advantages. First, we see that the two theorems are two faces of the same phenomenon, and second, we get a natural general question: *For which linear algebraic groups G is $(*)$ injective for every global field F?* The Hasse principle (proved by Kneser, Harder, and Chernousov) says that $(*)$ is injective if G is connected and absolutely simple. (A linear algebraic group is *absolutely simple* if, when viewed as an algebraic group over an algebraic closure of F, it is not commutative and does not contain any proper, connected, and normal algebraic subgroups besides the identity.) This broadly generalizes the two results mentioned at the start of the paragraph.

If we weaken our request that $(*)$ be injective, we arrive at a substantial result due to Borel and Serre (early 1960s) that $(*)$ has finite kernel for every linear algebraic group G in the case in which F is a number field. This result has recently been extended to global fields of prime characteristic by Brian Conrad. Because these latter fields are not perfect, this extension relies on the classification of "pseudo-reductive" linear algebraic groups recently completed by Conrad, Ofer Gabber, and Gopal Prasad.

Another way to generalize the local-global question is to weaken the hypotheses on F, for example, to assume that F has cohomological dimension at most 2, which holds for totally imaginary number fields and $\mathbb{C}(x,y)$. For such fields and G simply connected, Serre's "Conjecture II" (1962) asserts that $H^1(F,G)$ is zero. This is known to hold if F is the function field of a complex surface (de Jong, He, and Starr, 2008) or G is a classical group (Bayer and Parimala, 1995). There are many other results on this conjecture and generalizations such as the Hasse principle conjecture II; Google can provide details.

[1] A global field *is a finite extension of* $\mathbb{Q}$ *or* $k(t)$ *for* k *a finite field.*

Group Theory

In the list of finite simple (concrete) groups, most are *of Lie type*. That is, take a linear algebraic group G that is absolutely simple, simply connected, and defined over a finite field F that is not very small. Then the concrete group $G(F)$ modulo its center is finite and simple. This is helpful because one can use the general framework of algebraic groups to prove theorems about these finite groups. One example of this is Deligne-Lusztig theory, which is the most effective approach to the complex representations of the finite groups of Lie type.

The construction of simple concrete groups in the previous paragraph works for many algebraic groups G and many fields F, not just for finite fields. But for precisely which G does it work? The *Kneser-Tits problem* (1964) asks: *Let G be a linear algebraic group that is simply connected, is absolutely simple, and contains* GL_1*. Is $G(F)$ modulo its center simple?* Much like the Hasse principle discussed above, Kneser-Tits is a generalization in terms of algebraic groups of earlier problems—such as the Tannaka-Artin problem—regarding more classical algebraic structures.

The answer to Kneser-Tits seems to depend on the arithmetic complexity of the field F. The answer can be "no" for fields of dimension at least 4 (Platonov, 1975).

In contrast, the answer is "yes" for global fields, which "are 2-dimensional". This was an open question for some time, until Philippe Gille settled the last remaining case in 2007 by discovering the following interesting criterion that works over every field F: for a group G as in Kneser-Tits, the concrete group $G(F)$ modulo its center is simple (a purely algebraic criterion) if and only if the variety G is, roughly speaking, "path connected" (a purely geometric criterion). See [1] for details.

We still don't know the answer to Kneser-Tits for other fields of dimension 2 and don't even have strong indications of what the answer should be for dimension 3.

In closing, I urge you: Please do not be misled by the short list of topics in this article. There are many other areas of mathematics in which algebraic groups play an essential role, such as the Langlands program, geometric invariant theory and Schubert varieties in algebraic geometry, Tits's theory of buildings... Algebraic groups deserve more attention.

References

[1] P. Gille, Le problème de Kneser-Tits, *Astérisque* **326** (2009), 39–82.

[2] V. P. Platonov and A. Rapinchuk, *Algebraic Groups and Number Theory*, Academic Press, Boston, 1994.

[3] J.-P. Serre, Gèbres, *L'Enseignement Math.* **39** (1993), 33–85. (Reprinted in *Oeuvres*, vol. IV, pp. 272–324.)