Every Waking Moment
Ky Fan (1914–2010)

Bor-Luh Lin

Photographs courtesy of the author.

Ky Fan

Ky Fan passed away on March 22, 2010, at the age of ninety-five in Santa Barbara, California. He was born in Hangzhou, China, on September 19, 1914. He enrolled in National Peking University in 1932. Despite an interest in engineering, he pursued studies in mathematics due in part to the influence of his uncle, Zuxun Feng, who was chair of the Department of Mathematics at Peking University. As a junior in college, Fan was inspired by a visit of E. Sperner and translated into Chinese the book by O. Schreier and E. Sperner, *Einführung in die Analytische Geometrie und Algebra*. This translation, published in 1935 with the title *Analytical Geometry and Algebra I, II*, became a standard textbook in China. In fact, in 1953, almost twenty years after its initial publication, I used the same text as an undergraduate at National Taiwan University for a course on advanced geometry. This book so inspired me that I sought to join Fan at the University of Notre Dame to pursue a Ph.D. in mathematics. By the time Fan graduated from Peking University in 1936, he had also translated a book of Landau on number theory and ideal theory and had coauthored with a colleague a book on number theory. He was an instructor at Peking University in 1936–1939. In 1939 he was selected by the China-France Education Foundation to receive a Boxer Scholarship. This national competition provided one student with a chance to pursue a degree in mathematics in Europe. Working with Maurice Fréchet, he received his D.Sci. from the University of Paris in just two years, with a thesis with the title *Sur quelques notions fondamentales de l'analyse générale*. He was a French National Science Fellow at Centre National de la Recherche Scientifique in 1941–1942 and a member of the Institut Henri Poincaré in 1942–1945. By 1945 he had already published twenty-five papers on abstract analysis and topology, including the monograph *Introduction à la topologie combinatorire, I. Initiation* (Vubert, Paris), written with M. Fréchet.

Fan was at the Institute for Advanced Study at Princeton in 1945–1947. As an assistant of John von Neumann, and inspired by H. Weyl, he developed an interest in operator theory, matrix theory, minimax theory, and game theory. Later he extended his interests to systems of inequalities and fixed point theory. At that time many in these fields were studying finite games and optimizations in finite-dimensional spaces. Fan, however, pioneered the fields of infinite games and inequalities in infinite-dimensional linear spaces. He made fundamental and groundbreaking contributions and continued to be a leader in these areas throughout his career. He also made major contributions to the geometry of Banach spaces, convex analysis, combinatorial topology, topological groups, and analytical function theory.

Fan taught at the University of Notre Dame from 1947 to 1960. For many summers during this period, he was at the National Bureau of Standards, Oak Ridge National Laboratory, and Argonne National Laboratory. He taught at Wayne State University (1960–1961), Northwestern University (1961–1965), and the University of

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California at Santa Barbara starting in 1965. Upon his retirement from UC Santa Barbara in 1985, an international symposium was held to celebrate his many contributions to mathematics. The symposium resulted in the publication of a monograph with the title *Nonlinear and Convex Analysis: Proceedings In Honor of Ky Fan* (Lecture Notes in Pure and Applied Mathematics, Dekker, **107**(1987)). He was chair of the Department of Mathematics at UC Santa Barbara in 1968–1969. He held visiting positions at the University of Texas-Austin, Universität Hamburg, the Université Paris IX, and Università degli Studi di Perugia. He was elected a member of Academia Sinica in 1964 and served as the director of the Institute of Mathematics at the Academia Sinica from 1978–1984. Two issues of the *Bulletin of the Institute of Mathematics, Academia Sinica*—V. 2, No. 2, 1974, and V. 3, No. 1, 1975—were dedicated to him for his sixtieth birthday. In 1989 he was a Distinguished Visiting Scholar at the Chinese University in Hong Kong and received an Honorary Professorship from Peking University. After the visit, he donated his whole collection of mathematics books and treatises to Peking University. In 1990 he was awarded the degree Docteur Honoris Causa from the Université de Paris-Dauphine and was the featured speaker at the Conference on Directions in Matrix Theory (Fourth Auburn Linear Algebra Conference). His lecture at the Auburn conference, his publication lists up to 1992, a brief bibliography, and a list of his Ph.D. students appeared in *Linear Algebra and its Applications* 162–164:1-2(1992), 1–22. In 1993 the T.I.Tec/K.E.S. Conference on Nonlinear and Convex Analysis in Tokyo was dedicated to Fan in recognition of his fundamental contributions to the field. In 2011 the seventh International Conference on Nonlinear and Convex Analysis in Hirosaki will be dedicated to his memory. In 1994 an entire issue of *Topological Methods in Nonlinear Analysis* was dedicated to him for his eightieth birthday. He served on many editorial boards. He was a distinguished editor of *Linear Algebra and its Applications*, and he was one of the founding editors of the *Journal of Mathematical Analysis and Applications* and the *Journal of Nonlinear and Convex Analysis*. The latter will publish a special issue in memory of Fan.

In 1999 Ky Fan and his wife, Yu-Fen Fan, made a gift of approximately US$1 million to the American Mathematical Society. The fund was used to establish the Ky and Yu-Fen Fan Endowment to support and to foster collaborations between Chinese mathematicians and mathematicians in other parts of the world, especially North America, and to support mathematically talented high school students in the United States. The AMS uses the fund from the Fan Endowment to fund the China Exchange Program, which provides grants to Chinese mathematics departments to bring visitors from the rest of the world, as well as grants to North American departments to bring in visitors from China. The program also supports occasional conferences in China and improvement of library holdings in Chinese institutions. The Fan Endowment has also provided grants to assist programs in the United States that nurture mathematically talented high school students and has supported the Ky and Yu-Fen Fan Scholarships within the AMS Epsilon Scholarships Program for high school students. As noted by AMS past president Felix Browder, "The impact of Ky and Yu-Fen's generosity will be felt for years to come."

Ky Fan, China, 1939.

Fan published about 130 papers, many of which made fundamental contributions to several fields in pure and applied mathematics. From the famed Ky Fan inequalities (1951) to the Fan condition (1956), to the Ky Fan minimax inequality (1972) his papers shaped the fields of linear and nonlinear functional analysis, linear algebra, convex analysis, and optimization. Fan had a knack for identifying the central problems in a field of study and presenting them in the most general concise statements, with new approaches and elegant proofs. He brought seeming unrelated areas together to create new mathematics. As a result, his papers were often cited and pointed to new research directions. His contributions concerning fixed points and minimax inequalities have had a major impact in the development of nonlinear functional analysis. He made significant contributions to locations and identified maximum eigenvalues of matrices. His works found wide applications to mathematical economics, differential equations, potential theory, and numerical analysis. His research demonstrated the beauty of combining pure and applied mathematics. There are a large number of theorems, lemmas, inequalities, equalities, conditions, norms, etc., that bear the name of Ky Fan. The bibliography lists some major reference books that contain many of his contributions. We now briefly discuss the Ky Fan inequality, the Fan condition, and the Ky Fan minimax inequality.

Ky Fan Minimax Inequality (K. FAN, A minimax inequality and applications, *Inequalities III, Proc. of the Third Symposium on Inequalities*, Acad. Press (1972), 103–113).

> Let X be a nonempty compact convex set in a Hausdorff topological vector space E.

Suppose f is a real-valued function defined on $X \times X$ such that:

(a) For each fixed x in X, $f(x,y)$ is a lower semicontinuous function of y on X,

(b) For each fixed y in X, $f(x,y)$ is a quasi-concave function of x on X.

Then the minimax inequality

$$\min_{y\in X}\max_{x\in X} f(x,y) \le \sup_{x\in X} f(x,x)$$

holds.

Applications of this inequality to fixed points, differential equations, and potential theory were given in the paper. It was later found to be equivalent to the Brouwer fixed point theorem but much more powerful and easier to use. Many important equilibrium theorems in mathematical economics follow quickly from the Ky Fan inequality.

Ky Fan Inequality (K. FAN, Maximum properties and inequalities for the eigenvalues of completely continuous operators, *Proc. Natl. Acad. Sci. USA*, 37 (1951), 760–766).

Let $A_1, A_2, \ldots, A_m$ be m ($m \ge 1$) completely continuous operators in Hilbert space H. For each $j = 1, 2, \ldots, m$, let $\lambda_{j1}, \lambda_{j2}, \ldots, \lambda_{ji}, \ldots$ be the eigenvalues of $A_j^* A_j$. Then for every positive integer n:

$$\max \left| \sum_{i=1}^{n} (U_1 A_1 \cdots U_m A_m x_i, x_i) \right| = \sum_{i=1}^{n} (\lambda_{1i}\lambda_{2i}\cdots\lambda_{mi}),$$

$$\max \left| \det_{1\le i,k\le n} (U_1 A_1 \cdots U_m A_m x_i, x_k) \right| = \prod_{i=1}^{n} (\lambda_{1i}\lambda_{2i}\cdots\lambda_{mi}),$$

where, for both maxima, $U_1, U_2, \ldots, U_m$ independently run over all unitary operators and $\{x_1, x_2, \ldots, x_n\}$ runs over all n orthonormal elements in H.

Let A, B be completely continuous operators in Hilbert space H. If $\{\lambda_i\}$, $\{\kappa_i\}$, and $\{\sigma_i\}$ are the eigenvalues of A^*A, B^*B, and $(A+B)^*(A+B)$ respectively, then

$$\Phi(\sqrt{\sigma_1}, \sqrt{\sigma_2}, \ldots, \sqrt{\sigma_n}) \le \Phi\left(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_n}\right) + \Phi(\sqrt{\kappa_1}, \sqrt{\kappa_2}, \ldots, \sqrt{\kappa_n})$$

holds for every symmetric gauge function Φ of any number n variables.

These inequalities generalized inequalities of von Neumann and Weyl and were listed by J. Dieudonné in *A Panorama of Pure Mathematics, as seen by N. Bourbaki*, Acad. Press (1982) as a major contribution to operator theory.

Fan Condition (K. FAN, *On systems of linear inequalities, linear inequalities and related systems*, Annals of Math. Studies, Princeton Univ. Press, vol. 38 (1956), 99–156).

Let X be a real linear space. For any linear functionals $f_i, i = 1, 2, \ldots, n$, and real numbers $c_i, i = 1, 2, \ldots, n$, there exists $x \in X$ such that $f_i(x) \ge c_i, i = 1, 2, \ldots, n$, if and only if for any nonnegative numbers $a_i, i = 1, 2, \ldots, n$, the relation $\sum_{i=1}^n a_i f_i = 0$ implies $\sum_{i=1}^n a_i c_i \le 0$.

The Fan condition has many applications and is a fundamental result in linear programming. The following is a nonlinear version of the Fan condition and is one of the major tools in nonlinear programming and convex analysis. It appeared in the paper "Systems of inequalities involving convex functions", by K. Fan, I. Glicksberg, and A. J. Hoffman, *Proc. Amer. Math. Soc.* **8** (1957), 617–622.

Let K be a nonempty convex set in a real vector space X. Let $f_i: K \to \mathbb{R}, i = 1, 2, \ldots, n$, be convex functions. Then $f_i(x) < 0, i = 1, 2, \ldots, n$, has a solution $x \in K$ if and only if there exist real numbers $\lambda_i \ge 0$, $i = 1, 2, \ldots, n$, not all zero, such that $\sum_{i=1}^n \lambda_i f_i(x) \ge 0$ for all $x \in K$.

Although Fan's contributions to mathematics are undisputed, I can think of no better way to honor his memory than to describe his role as a mentor and adviser. Fan was widely known to be a very rigorous teacher, one who expected the best of his students. Over his career, Fan had twenty-two Ph.D. students. When he left Notre Dame in 1960, John C. Cantwell, Ronald J. Knill, Robert E. Mullins, John O. Riedl Jr., and I followed him to Northwestern to complete our Ph.D. theses. The five of us were given thesis topics in five completely different areas. These areas were so diverse that we found no common ground among us to discuss the individual problems, and they were chosen so that none of the topics were closely related to any of Fan's published papers. He had decided the best way to train us was by pushing us to develop our own research directions rather than simply follow in his footsteps.

His style of teaching was unique, if occasionally intimidating. In his courses, we were given long lists of references to study. These references were in English, French, German, or Russian. The book of exercises he compiled to accompany these references was equally daunting. The content of his courses was usually about double that of a typical course. But, as with everything Fan did, the courses were beautifully organized. I still remember his lectures. He would start at the far corner of the blackboard and gradually cover the entire blackboard as the class went on. His lecture would always finish in the far right corner of the blackboard as class ended.

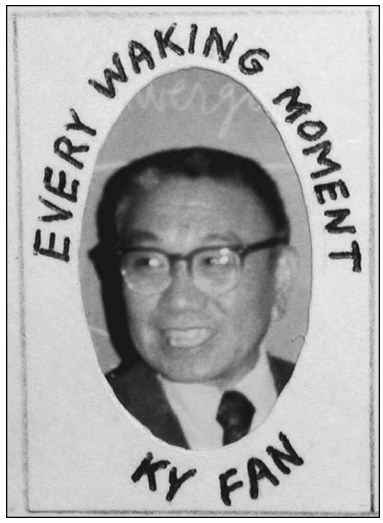
He demanded nothing less than complete dedication from his students. He considered his

lectures to be performances and demanded the students' total attention. Those who neglected to take notes would be the first to be called upon in class, and he was not shy about letting the entire class know if any answer was stupid or irrelevant. It was not long before all of the students were assiduously taking notes and preparing answers to the many possible questions that could come at any moment.

He demanded that his graduate students think about mathematics all the time. In 1975 graduate students at UC Santa Barbara surprised him on his sixtieth birthday with a T-shirt imprinted with his photo, surrounding which, in large black letters, were the words, "EVERY WAKING MOMENT". Fan, of course, was smiling in the middle.

Dennis Wildfogel, a Ph.D. student of Fan at UC Santa Barbara in 1974, provides a story about Fan: "As many people know, Dr. Fan's doctoral adviser was the noted mathematician Maurice Fréchet, and Fréchet's adviser was the even more famous Jacques Hadamard. Fréchet died while I was working on my thesis. Shortly thereafter, someone pointed out to Fan that Hadamard had lived until age ninety-seven, Fréchet until ninety-four, and concluded that Fan would live to be ninety-one. Now, for most people in their late fifties (as Dr. Fan was at that time), being told you will live until age ninety-one would be good news. But Dr. Fan resented any limitation on his opportunity to do mathematics, so he replied testily, 'How do you know it's a straight line?!? Maybe it is a parabola."' For additional stories on Fan's teaching and mentoring, see the blog `drfantales.blogspot.com`, created by Wildfogel.

As a mathematician and teacher, he expected perfection and total devotion. As a mentor and as a person, he was a traditional Chinese scholar; unfailingly kind, courteous, generous, and humble. He went out of his way to help his students or anyone who was interested in mathematics. Even when he was in a wheelchair in the later part of his life, he would still respond to my letters with advice and encouragement. Any news I or other students had on progress in our studies in mathematics would make him happy. In 1994 he sent me a preprint of a paper to be published in *Proceedings of AMS*, with a note that this was his way of celebrating his eighties birthday. He lived the way he taught his students: EVERY WAKING MOMENT was spent thinking about and working on mathematics. His love and dedication to mathematics continue to be a model for all of us, and his impact will be felt on the field of mathematics for generations to come.



T-shirt from graduate students, UC Santa Barbara, 1975.

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