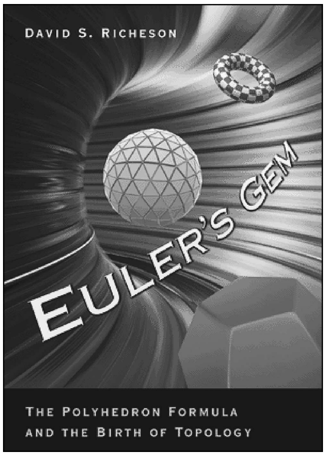
Book Review



Euler's Gem

Reviewed by Jeremy L. Martin

Euler's Gem
David S. Richeson
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When you admit to a stranger on an airplane that you are a professional mathematician, what happens next? "I never liked mathematics." "I'm no good with numbers." "I used to be good at math until I got to calculus." "Isn't math boring?" "What do you guys actually *do* all day?" Why not take such a response as an invitation to pull out a scratchpad and do a little teaching? One of my favorite topics is Euler's polyhedral formula: it is simple and elegant, it is not just about arithmetic or calculus, and it requires hardly any technical background to understand. Even strangers on airplanes are capable of looking at five or six examples, conjecturing that $V - E + F = 2$ for all polyhedra, and asking good questions: "But how can you prove that it's always true?" "Does it work if the polyhedron has a hole in it?" Admittedly, the scene doesn't always end this happily, but we mathematicians need to be able to communicate our discipline to strangers on airplanes, not to mention prospective students, deans, members of Congress, and small children.

David Richeson's *Euler's Gem* does an outstanding job of explaining serious mathematics to a general audience, and I plan to recommend it to the next stranger I meet on an airplane. The book is structured as a "tour guide" to the history of geometry and topology, revolving around Euler's formula and organized roughly chronologically: from the study of polyhedra in ancient Greek geometry to the discovery, proofs, and generalizations of Euler's formula in the seventeenth, eighteenth, and nineteenth centuries, to such diverse modern topics as knot theory, fixed-point theorems, curvature, the classification of surfaces, homology theory, and the Poincaré conjecture. The book is primarily intended for a lay audience, but there is also much of interest to professional mathematics students, teachers, and researchers. While a few of the book's generalizations about mathematical history and aesthetics are a bit simplistic or even one-sided, the wealth of clear and engaging exposition outweighs these occasional flaws.

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The historical development of geometry, from its Greek roots to its modern form, is a recurring theme. An example is the discussion of the attempts of Lhuilier and others, in the early nineteenth century, to generalize Euler's formula to nonconvex polyhedra. Lhuilier's approach, incorporating contributions for the number of "tunnels", "cavities", and "inner polygons" (Richeson's terms), as well as vertices, edges, and faces, might seem a little misguided with the benefit of hindsight; wouldn't it be easier to phrase everything in terms of cell complexes and homology? Yes, it would, but in 1810 no one knew what a cell complex (or for that matter a manifold) was. The reader can see the origins of these modern ideas by comparing Lhuilier's work with that of Listing (whose "spatial complexes" Richeson describes briefly on pp. 249–250) and finally modern topology as developed by Poincaré. Even an expert should be able to benefit from seeing the evolution of "standard" mathematical definitions from simple and intuitive to complex and precise; this evolution is something not found in most graduate courses or textbooks.

Some of the generalizations about mathematics history may be oversimplifications: for instance, "Euler's predecessors were so focused on metric properties that they missed this fundamental interdependence. Not only did it not occur to them that they should count the features on a polyhedron, they did not even know which features to count" (p. 85). Kepler and Descartes did in fact count pieces of polyhedra, and their work is described elsewhere in the book. Kepler had observed the phenomenon of polar duality for regular solids and the fact that dualizing reverses the ordered triple (V, E, F), what a modern combinatorialist would call the f-vector. Meanwhile, a century before Euler, Descartes had observed[1] a formula closely related to Euler's: $P = 2F + 2V - 4$, where P is the number of plane angles. Each angle contains two edges, and each edge belongs to four angles (two on each end), so $P = 2E$, implying Euler's formula. Whether or not you think that Descartes deserves equal naming rights with Euler (a topic that has been debated), it seems clear that Kepler and Descartes were at least counting *something.* No doubt Euler's work was a major turning point and had far more direct impact than that of Descartes, but it is an overstatement to claim that Euler was the first to realize the applicability of counting in geometry. (On the other hand, Euler's original proof was indeed, as Richeson says, "a precursor to modern combinatorial proofs" (p. 67): calculate $V - E + F$ for a given polyhedron by slicing off a tetrahedron at a time and total the contributions from the individual tetrahedra. For those, like me, who think of Euler's formula as completely combinatorial, it is interesting to learn that the first rigorous proof, due to Legendre, is fundamentally geometric: project the polyhedron onto a sphere and then apply the Harriot-Girard theorem, which says that the area of a geodesic triangle on the unit sphere equals its angle sum minus π.)

The big historical picture may be slightly fuzzy, but the exposition of substantial mathematics is uniformly clear and concrete, with lots of pictures and examples, and sensibly organized. Several of the chapters stand on their own and would work well as self-contained reading assignments in a geometry course for mathematics majors or future secondary-school teachers. For example, the description of the classification of surfaces by their Euler characteristics and orientability (Chapters 16–17) is an absorbing, self-contained mathematical story, told at an appropriate level of technicality, in which terms such as "isomorphism invariant" and "orientable" receive clear, simple definitions that avoid unnecessary technicalities, without sacrificing accuracy. The theme of intrinsic versus extrinsic geometry (what properties of a curve or surface depend on how it is embedded in space?) is given the attention it deserves, leading into a chapter on the lovely subject of knot theory, with lots of pictures and explanations that make substantial mathematics (like the Seifert surface, an orientable surface in R^3 having a given knot as boundary) appealing and fun. If your seatmate complains that geometry is boring, here is an antidote.

Further geometry topics that receive excellent treatment include Descartes' theorem on solid angles of polyhedra and its continuous analogue, the Gauss-Bonnet theorem, which measures the total curvature of a surface in terms of its Euler characteristic. The presentation is clear and self-contained, requiring little more background than the fact that the sum of angles in an n-sided polygon is $(n - 2)\pi$—hardly too much to ask. The explanation of curvature is concrete; wisely, the calculus details are banished to footnotes. (My one small complaint: the biographical sketch of Gauss somewhat breaks up the flow of the mathematical story.) Richeson's explanation of homology (Chapter 23) was one of my favorite parts of the book, and I wish I had read it before taking algebraic topology as a graduate student—all those long exact sequences would have made a lot more sense if I had known what they were trying to measure.

Unfortunately, this outstanding section is followed by a mistaken explanation: "Kepler's observation [polar duality] is Poincaré duality in disguise. We are free to exchange the roles of i-dimensional and $(n - i)$-dimensional simplices." The intent is good, but the details are inaccurate: the f-vectors of two polar dual polytopes are the reverses of each other, whereas Poincaré duality says (among other things) that the Betti numbers of a *single* manifold form a palindrome. Although both statements are superficially concerned with symmetry, they are hardly the same thing in disguise. I cannot resist inserting a plug for combinatorics here: I would have liked to see a section about another relevant (and quite beautiful) duality for polytopes, namely the Dehn-Sommerville equations. Briefly, the f-vector of a simplicial polytope can be transformed into another invariant called the h-vector (for example, an octahedron has f-vector $(6, 12, 8)$ and h-vector $(1, 3, 3, 1)$), which carries the same information; the Dehn-Sommerville equations say that the h-vector is a palindrome. After describing homology and Poincaré duality, it would have been a natural next step to define the h-vector, to state the Dehn-Sommerville equations with an example or two, and perhaps to sketch the beautiful geometric proof by Bruggesser and Mani [1]. Perhaps this is something to look forward to in the second edition.

[1] *In a set of papers lost in a shipwreck after Descartes' death and not brought to public light until 1860—Richeson provides the juicy details in Chapter 9.*

I was disappointed by the book's discussion of the use of computers in mathematics, particularly Appel and Haken's 1976 proof of the four-color theorem (the first solution of a major open problem that relied on a computer to check a large finite number of cases):

> Although most people came to believe that [Appel and Haken's] proof was correct, most pure mathematicians found the proof inelegant, unsatisfying, and unsporting. It was as if Evel Knievel boasted that he could cross the Grand Canyon on his motorcycle, only to build a bridge and use it to make the crossing. Perhaps it is how mountain climbing purists feel about the use of bottled oxygen in high-altitude climbing. (p. 143)

This is unnecessarily dismissive and it neglects to present the other point of view: that Appel and Haken's work did us all a big favor by introducing a powerful new tool in doing mathematics and that whether it is "sporting" is a moot point, because mathematics is the richer for having any proof at all of the four-color theorem. The passage also pays scant attention to the fact that computer-aided proofs are much more widely accepted in the mathematical community today than they were in 1976. Later on the same page, we read,

> Perhaps some day someone will create a black box that proves theorems…. Some would say that this would take the fun out of mathematics and make it less beautiful.

Yet some would say the reverse: computers can help us discover and create beauty. Consider the development of automated summation techniques; they may take some of the fun out of proving hypergeometric identities, but being able to delegate such tasks to a computer frees up lots of mathematician-hours to do other things that a machine can't do. In addition, the mathematics underlying hypergeometric summation is itself quite beautiful, and it's hard to imagine any hypothetical black box being built without much more complex and beautiful mathematics (as a starting point, see the articles on formal proof in the October 2008 issue of the *Notices*). Describing the artistic and aesthetic sides of mathematics is a noble goal, but I am concerned that the quoted passages are counterproductive. We should portray ourselves not as purists who disdain the use of nontraditional tools but as scientists who are willing to be open to new methods.

It is easier to criticize a problematic sentence than to praise an entire well-written chapter. Overall, I found much more to like than to criticize in *Euler's Gem*. At its best, the book succeeds at showing the reader a lot of attractive mathematics with a well-chosen level of technical detail. I recommend it both to professional mathematicians and to their seatmates.

References

[1] H. Bruggesser and P. Mani, Shellable decompositions of cells and spheres, *Math. Scand.* **29** (1971), 197–205.