From Cartan to Tanaka: Getting Real in the Complex World

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It is well known from undergraduate complex analysis that holomorphic functions of one complex variable are fully determined by their values at the boundary of a complex domain via the Cauchy integral formula. This is the first instance in which students encounter the general principle of complex analysis in one and several variables that the study of holomorphic objects often reduces to the study of their boundary values. The boundaries of complex domains, having odd topological dimension, cannot be complex objects. This motivated the study of the geometry of real hypersurfaces in complex space. In particular, since all established facts about a particular hypersurface carry over to its image via a biholomorphic mapping in the ambient space, it is important to decide which hypersurfaces are equivalent with respect to such mappings—that is, to solve an equivalence problem for real hypersurfaces in a complex space.

In the case of one complex variable, the Riemann mapping theorem says that any simply connected domain is either $\mathbb{C}$ or equivalent to the unit disc. In contrast, Henri Poincaré [17] showed that in higher dimensions even the ball and the bidisc are not equivalent, which implies that their boundaries cannot be equivalent.

In the same article Poincaré posed the local equivalence problem, i.e., to decide when two hypersurfaces are equivalent in the neighbourhoods of given points. He sketched a heuristic argument that any two real hypersurfaces in $\mathbb{C}^2$ cannot be expected to be locally equivalent.

In order to solve this equivalence problem for real hypersurfaces in $\mathbb{C}^2$, Élie Cartan [6], [7] constructed in 1932 a "hyperspherical connection" by applying his method of moving frames. The technique of Cartan has been further developed by introducing modern geometric and algebraic tools, mainly in the groundbreaking work by Noboru Tanaka (see [22], [23], [24]). These powerful and elegant methods are widely used in conformal geometry and have led to the development of parabolic geometry (see [5]), while Cartan's original approach, applied to hypersurfaces in higher dimensional complex space by Shiing-Shen Chern and Jürgen Moser [8], is still dominant in complex analysis (see, e.g., [12], [13]).

An alternative approach to invariants of boundaries of complex domains has been developed by Charles Fefferman [10]. A recent result by Andreas Čap and Rod Gover [3], based on parabolic geometry, shows the relation between the Cartan-Tanaka and the Fefferman calculi.

In practice it remains difficult even to decide if a given hypersurface is locally equivalent to a sphere. Sidney Webster [25] solved this problem for an ellipsoid in $\mathbb{C}^n$ ($n \geq 3$) in 2000. In this context he states that the methods developed are at least as interesting as the results.

The purpose of this article is to illustrate the basic ideas of Tanaka's theory by deriving the

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principal curvature invariants for hypersurfaces in $\mathbb{C}^2$. This solves the equivalence problem if one of the hypersurfaces is a sphere.

While there is extensive literature on both Cartan connections (see, e.g., [21] and [5]) and CR-geometry (see, e.g., [2], [13], [12] and [9]), the authors feel that there is a need to draw these two subjects together.

Geometric Structures

Before we introduce the concept of CR-geometry, which is the suitable setting for the study of real hypersurfaces in $\mathbb{C}^n$, we would like to take a broader view of a geometric structure as a class of manifolds with additional data given on the tangent spaces. Examples of such structures include Riemannian geometry and almost complex structures.

In his Erlangen programme (1872), Felix Klein observed that geometric structures can be studied by their symmetry groups. These are the groups of the self-mappings that preserve the structure data. The study of a geometric structure hence leads to such problems as:

- What are the symmetries of a given object?
- Which objects have a given symmetry?
- What are the most symmetric objects?
- Is there a transformation between two manifolds M and N that preserves the data (equivalence problem)?
- Determine a complete system of invariants that identifies classes of equivalent manifolds.

A giant leap toward solving these problems was taken by Sophus Lie, who systematically studied, described, and classified transformation groups. It was one of his great discoveries that transformation groups can be decomposed into so-called 1-parametric subgroups. In modern terminology these are smooth homomorphisms from $\mathbb{R}$ into the transformation group. With such a 1-parametric subgroup, $\phi_t : M \to M$, one can associate a vector field χ on M by differentiating with respect to the parameter t and setting $t = 0$:

$$\chi(x) = \left.\frac{d\phi_t(x)}{dt}\right|_{t=0} \in T_x M.$$

Geometrically speaking, the vector field χ assigns to each point $p \in M$ the tangent vector to the 1-dimensional orbit $\phi_t(p)$.

The 1-parametric group can then be recovered from χ by solving the ODE

$$\frac{d\phi_t}{dt} = \chi(\phi_t), \quad \phi_0 = \mathrm{id}\,.$$

These vector fields are called infinitesimal automorphisms. Being linear objects, they are rather easy to handle, and they are extremely useful in the study of the transformation group itself.

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Élie Cartan

Subsequent work by Lie, Engel, Killing, E. Cartan, and others led to the theory of Lie groups, which solves the problems stated above for manifolds that possess "many symmetries", namely so-called *homogeneous* manifolds, i.e., any manifold M such that for any two points $a, b \in M$ there is a transformation ϕ of M with $\phi(a) = b$ and respecting the geometric structure.

The technique of Cartan connections, which is a refinement of Cartan's method of moving frames, permits the study of the geometry of manifolds with few or no symmetries. It is based on the idea to first construct a homogeneous model manifold with a similar geometric structure and then to consider the original manifold as a perturbed version of the model. The resulting notion of Cartan curvature and their invariant derivatives provides the desired complete system of invariants. Moreover, the algebraic nature of Tanaka's construction has led to deep structural results for general types of parabolic geometry and to new techniques such as tractor calculus and general Fefferman spaces (see [4] and references therein). These powerful tools help to give more complete answers to the questions that arise from the Erlangen programme.

Cartan Connections

For a homogeneous manifold Q let G be the group of its symmetries and P be the subgroup of the symmetries that fix a point $a \in Q$. Then Q can be represented as the quotient manifold G/P, and we have the canonical mapping $\pi\colon G \to Q$ that makes G a fiber bundle with fiber P over Q. The total space G is equipped with a canonical 1-form ω_{MC} that takes values in the Lie algebra $\mathfrak{g} = Lie(G)$ of left-invariant vector fields on G. It is called the *Maurer-Cartan form* and assigns to a vector $X_g \in T_gG$ the left-invariant vector field generated by X_g interpreted as an element of $\mathfrak{g} = T_eG$, the tangent space at the identity of G. That is, it establishes the tautological isomorphy of $\mathfrak{g}$ in both interpretations, namely as the space of the left-invariant vector fields and as T_eG. For a matrix Lie group $G \subseteq \mathrm{GL}(n,\mathbb{R}) \subset \mathbb{R}^{n\times n}$, the left-invariant vector fields have the form

$$X(g) = g^i_j A^j_k \left.\frac{\partial}{\partial g^i_k}\right|_g = gA,$$

where g is the matrix-valued coordinate in $\mathbb{R}^{n\times n}$ (restricted to G), $\frac{\partial}{\partial g^i_k}$ are the coordinate vector fields in $\mathbb{R}^{n\times n}$ and A is a constant matrix that can be identified with $X(e)$. It follows for $X = gA$ and $Y = gB$

$$[X,Y] = g(AB - BA).$$

In this case the Maurer-Cartan form is

$$\omega_{MC} = g^{-1}dg,$$

and

$$\omega_{MC}X(g) = g^{-1}gA = A.$$

Differentiation yields the structure equation

$$\begin{aligned} d\omega_{MC} &= -g^{-1}dg \wedge g^{-1}dg = -\omega_{MC} \wedge \omega_{MC} \\ &= -\frac{1}{2}[\omega_{MC}, \omega_{MC}]. \end{aligned}$$

This is equivalent to the tautological statement that the commutator bracket of the left-invariant vector fields is the same as the Lie algebra product.

A *Cartan connection* is a "curved" version of the bundle $\pi\colon G \to G/P$ with a $\mathfrak{g}$-valued form ω.

To construct a Cartan connection that depends only on the geometric structure of the underlying manifold M one needs first to find the relevant groups G and P and hence the model G/P. This requires the notion of graded Lie algebras, i.e., Lie algebras that split into a direct sum

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\ell$$

so that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$. The general procedure is as follows:

(1) Determine a graded Lie algebra $\mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_0$ solely from the geometric data of M.
(2) Apply Tanaka's algebraic prolongation process to obtain $\mathfrak{g}_+ = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\ell$.
(3) Define G as the connected and simply connected Lie group that corresponds to $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_\ell$ and P as the subgroup that corresponds to $\mathfrak{p} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_\ell$.

We demonstrate below how the following three-step procedure is applied in the case of CR geometry of real hypersurfaces in $\mathbb{C}^2$.

The curved analog of the bundle $\pi\colon G \to Q$ is a principal P-fiber bundle $\pi\colon \mathcal{G} \to M$ endowed with a $\mathfrak{g}$-valued *Cartan connection form* ω on $\mathcal{G}$ with the following properties:

(1) $\omega_x\colon T_x\mathcal{G} \to \mathfrak{g}$ is an isomorphism of linear spaces for any $x \in \mathcal{G}$. This permits us to distinguish a space of vector fields $\hat{X}(x) = \omega_x^{-1}X$ for $X \in \mathfrak{g}$, which we will refer to as *constant* vector fields. For G/P with the Maurer-Cartan form the constant vector fields are identical to the left-invariant vector fields on G.
(2) If $\mathfrak{p} \subset \mathfrak{g}$ is the Lie algebra of the fiber group P, then for any $X \in \mathfrak{p}$ the corresponding constant vector field has the form

$$\hat{X}(x) = \left.\frac{d}{dt}\right|_{t=0} R_{\exp tX}(x),$$

where R is the right action of P on $\mathcal{G}$. This means that the constant vector fields on the fibers are the exact analogs of the left-invariant vector fields on G.
(3) For $p \in P$

$$R_p^*\omega = \mathrm{Ad}_{p^{-1}}\,\omega,$$

i.e., the *analytic* pull-back of the Cartan connection form with respect to the action of p on $\mathcal{G}$ is identical to the *algebraic* adjoint action of p^{-1} on the values of ω. Differentiating this identity yields

$$[\hat{X}, \hat{Y}] = [X, Y]_{\mathfrak{g}}$$

for $X \in \mathfrak{p}$ and $Y \in \mathfrak{g}$, i.e., the commutator relations of the constant vector fields $\hat{X} = \omega_x^{-1}(X)$ and $\hat{Y} = \omega_x^{-1}(Y)$ are the same as for X and Y in $\mathfrak{g}$ as soon as one of X, Y is from $\mathfrak{p}$.

It is easy to verify these properties for the Maurer-Cartan form. We point out that the properties (1)–(3) guarantee neither existence nor uniqueness of a Cartan connection. Further conditions are needed to make the Cartan connection form unique. There are sufficient conditions that guarantee the existence of the Cartan connection.

As an application of the Cartan connection construction we consider the equivalence problem: Given two manifolds M and M' with the same geometric structure, does there exist a local diffeomorphism from M to M' that respects the geometric structure? It turns out that a diffeomorphism ϕ would lift to a diffeomorphism Φ of the respective Cartan bundles $\mathcal{G}$ and $\mathcal{G}'$ that transforms the Cartan connection form ω into the

Cartan connection form ω', i.e., $\Phi_* \omega = \omega'$.

$$\begin{array}{ccc} (\mathcal{G}, \omega) & \overset{\Phi}{\dashrightarrow} & (\mathcal{G}', \omega') \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi'} \\ M & \xrightarrow{\ \phi\ } & M' \end{array}$$

This implies that all partial derivatives of Φ are determined and the equivalence problem reduces to the solubility of an initial value problem

$$\frac{\partial \Phi^i}{\partial x^j} = a_{ij}(\Phi, x)$$
$$\Phi(x_0) = x_0'.$$

Note that x_0 and x_0' are points in the principal bundles. As a consequence, the solubility of the initial value problem will depend on the choice of some parameters in the fibre, similar to the accessory parameters in the Christoffel-Schwarz formula. This makes it difficult to solve concrete equivalence problems.

Finally we remark that invariance property (3) of the Cartan connection form is not required for solving the equivalence problem, but it is essential for deriving the curvature invariants.

The Cartan Curvature

According to the construction of groups G and P, the structure algebra $\mathfrak{g}$ is a graded Lie algebra and $\mathfrak{p}$ is its non-negative component. Thus $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{p}$. We call a constant vector field $\hat{X}$ vertical or horizontal if $X \in \mathfrak{p}$ or $X \in \mathfrak{g}_-$, respectively. As stipulated by property (3) of the Cartan connection, the commutator of a constant vector field with a vertical vector field coincides with the commutator in $\mathfrak{g}$. This is not necessarily the case for two *horizontal* constant vector fields. In fact, it follows from Frobenius's theorem that if this were the case it would imply that M is locally equivalent to the model space G/P. The Cartan curvature measures how much such commutators deviate from the commutators in $\mathfrak{g}$ and, thus, "how far" M is from G/P. Consequently, the vanishing of the Cartan curvature can be used to decide if a given geometric structure is equivalent to the homogeneous model.

We introduce the *Cartan curvature* as the $\mathfrak{g}$-valued 2-form on $\mathcal{G}$

$$K_x(\hat{X}, \hat{Y}) = \omega_x[\hat{X}, \hat{Y}] - [\omega_x \hat{X}, \omega_x \hat{Y}].$$

This can be rewritten as the structure equation

$$d\omega + \frac{1}{2}[\omega, \omega] = -\frac{K}{2}.$$

Notice that $K = 0$ if one of its arguments is vertical. It is convenient to work with the *curvature function* $\kappa\colon \mathcal{G} \to \mathrm{Hom}(\mathfrak{g}_- \wedge \mathfrak{g}_-, \mathfrak{g})$ that is defined by

$$\kappa_x(X, Y) = K_x(\omega_x^{-1} X, \omega_x^{-1} Y).$$

The Algebraic Structure of the Curvature

To explain the algebraic structure of the curvature we introduce the notion of the cochain operator ∂. We consider $\mathrm{Hom}(\mathfrak{g}_- \wedge \mathfrak{g}_-, \mathfrak{g})$ as the space C^2 of 2-cochains in the sequence

$$\cdots C^1 \xrightarrow{\partial_1} C^2 \xrightarrow{\partial_2} C^3 \cdots$$

where $C^k = \mathrm{Hom}(\Lambda^k \mathfrak{g}_-, \mathfrak{g})$ and

$$\partial_k c(X_0, \dots, X_k) = \sum_i (-1)^i [X_i, c(X_0, \dots, \hat{\imath}, \dots, X_k)] + \sum_{i,j} (-1)^{i+j} c([X_i, X_j], X_0, \dots, \hat{\imath}, \hat{\jmath}, \dots, X_k),$$

where $\hat{\imath}, \hat{\jmath}$ stand for terms that have been omitted.

We denote $\mathcal{Z}^k = \ker \partial_k$ and $\mathcal{B}^{k+1} = \mathrm{im}\, \partial_k$. The cohomology H^k is defined by $H^k = \mathcal{Z}^k / \mathcal{B}^k$.

The spaces C^k split into homogeneous components, where a cochain is said to be of *homogeneity* i when it sends arguments of grade $j_1, \dots, j_k$ to a value of grade $i + j_1 + \cdots + j_k$. The splitting into homogeneous components is respected by ∂. We denote the homogeneity i components of $\mathcal{Z}^k, \mathcal{B}^k$ by $\mathcal{Z}^k_{(i)}, \mathcal{B}^k_{(i)}$, respectively, and the curvature component of homogeneity i by $\kappa^{(i)}$.

A Cartan connection is called *regular* if the curvature components of homogeneity ≤ 0 vanish. This condition ensures that the Cartan connection encodes the underlying geometry on M.

The Bianchi identity (see, e.g., [4]) relates the ∂-image of a homogeneous component of the curvature to components of lower homogeneity in the following way:

$$\partial \kappa^{(i)}(X, Y, Z) = -\sum_{cycl} \sum_{j=1}^{i-1} (\kappa^{(i-j)}(\kappa_-^{(j)}(X, Y)), Z) + \hat{Z} \kappa^{(i+|Z|)}(X, Y)),$$

where κ_- denotes the $\mathfrak{g}_-$ component of κ, $|Z|$ is the grade of Z, and $\sum_{cycl}$ is the sum over all cyclic permutations of X, Y, Z.

In particular, it follows that the lowest order nonvanishing curvature must be ∂-closed, and, more generally, any homogeneous curvature component is determined by the lower components up to a ∂-closed component. In this sense we will refer to the ∂-closed components of the curvature as the essential curvature.

Finally, according to Tanaka's results, the choice of the Cartan connection is controlled by the ∂-exact components of the curvature. In order to make the Cartan connection unique, we have to choose a complement to the space $\mathcal{B}^2$ of ∂-exact 2-cochains in the space $\mathcal{Z}^2$ of ∂-closed cochains. This complement is isomorphic to the cohomology group $H^2 = \mathcal{Z}^2 / \mathcal{B}^2$.

If $\mathfrak{g}$ is semisimple, there is an adjoint cochain operator ∂^*, and we have the Hodge decomposition

$$\mathcal{Z}^2 = \mathcal{B}^2 \oplus \ker \partial \cap \ker \partial^*,$$

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Noboru Tanaka

i.e., the space $\mathcal{H}^2 = \ker \partial \cap \ker \partial^*$ of harmonic 2-cochains is a canonical complement to $\mathcal{B}^2$ in $\mathcal{Z}^2$. The corresponding unique Cartan connection is determined by the *normalization* condition $\partial^* \kappa = 0$.

CR Structures and Graded Lie Algebras

A *CR structure*[1] is a combination of two structures:

(1) an even-dimensional distribution D in the tangent bundle of a manifold M, and

(2) a complex structure J_x on each D_x, i.e., a smooth field of endomorphisms J_x with $J_x^2 = -\mathrm{id}$.

The number $n = \dim_{\mathbb{C}} D_x$ is called the CR-dimension of M ,and $k = \dim M - 2n$ is called the CR-codimension.

A CR structure naturally appears on a real hypersurface M of $\mathbb{C}^N$. We may define $D_x = T_xM \cap \mathrm{i}\, T_xM$ for any $x \in M$ where multiplication by i means the restriction of the complex structure $\mathfrak{J}_x$ on $T_x\mathbb{C}^N$ to T_xM. Thus D_x is the biggest subspace of T_xM that is closed with respect to the complex structure of the ambient $\mathbb{C}^N$. The restriction of $\mathfrak{J}_x$ to D_x serves as J_x.

The (local) *equivalence problem* for two CR structures (M, D, J) and $(\tilde{M}, \tilde{D}, \tilde{J})$ can be formulated as follows: Is there a (local) diffeomorphism $\phi\colon M \to \tilde{M}$ such that $d\phi_x D_x = \tilde{D}_{\phi(x)}$ and $\tilde{J}_{\phi(x)} \circ d\phi_x = d\phi_x \circ J_x$?

[1]*Here "CR" stands for Cauchy-Riemann, sometimes also interpreted as Complex-Real. Here we ignore the usual integrability condition of the CR structure, as it is not relevant to this article.*

The Levi Form

The Levi form at a point $x \in M$ is the skew symmetric form

$$\mathcal{L}_x(X, Y) : D_x \wedge D_x \to T_xM/D_x$$

that is defined as follows: extend X, Y to local sections $\tilde{X}, \tilde{Y}$ of D, evaluate $[\tilde{X}, \tilde{Y}]$ at x, and project this vector to the factor space T_xM/D_x.

The Levi form measures the *non*involutivity of the distribution D. It is one of the simplest and most important invariants of a CR manifold. Many nondegeneracy conditions in CR geometry reflect in one or another way the noninvolutivity of D.

With the choice of a basis in D_x and T_xM/D_x, the Levi form identifies with an $\mathbb{R}^k$-valued skew-symmetric form $L_x(\xi, \eta)$ on $\mathbb{C}^n$. A different choice of the basis leads to the form $\rho L_x(C^{-1}\xi, C^{-1}\eta)$ where $\rho \in \mathrm{GL}(k, \mathbb{R})$ and $C \in \mathrm{GL}(n, \mathbb{C})$.

Denote by $[L_x]$ the class of all $\mathbb{R}^k$-valued skew-symmetric forms on $\mathbb{C}^n$ that are equivalent to L_x with respect to the action of matrices C and ρ as above. This equivalence class is a CR-invariant. If $k = 1$ the equivalence class is completely characterised by the rank and the signature of the then scalar Hermitian form.

Furthermore, we will assume that the following compatibility condition of the Levi form and the complex structure holds:

$$\mathcal{L}_x(J_xX, J_xY) = \mathcal{L}_x(X, Y).$$

This condition is automatically satisfied for embedded CR manifolds and is weaker than the usual integrability condition.

Levi-Tanaka Algebra and Tanaka's Prolongation Procedure

We assume that the Levi form at each point is surjective, i.e., $D_x + [D_x, D_x] = T_xM$. Then the direct sum $T_x^{\mathrm{grad}}M = D_x \oplus T_xM/D_x$ has the same dimension as T_xM and is called the associated graded tangent space. Denote $\mathfrak{g}_{-1} = D_x$ and $\mathfrak{g}_{-2} = T_xM/D_x$. The Levi form defines a graded Lie algebra

$$\mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$$

by setting $[X, Y] = \mathcal{L}(X, Y)$ if $X, Y \in \mathfrak{g}_{-1}$ and $[X, Y] = 0$ if one of $X, Y \in \mathfrak{g}_{-2}$. The subalgebra $\mathfrak{g}_-$ is the *Levi-Tanaka algebra* at x in the case when the Levi form is surjective.

Tanaka's prolongation of $\mathfrak{g}_-$ is an algebraic procedure to generate a graded Lie algebra $\mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots$ that determines to a large extent the geometric properties of the CR structure. The first prolongation component, $\mathfrak{g}_0$, is a Lie algebra whose elements act linearly on $\mathfrak{g}_-$. Hence the elements of $\mathfrak{g}_0$ can be represented as pairs of endomorphisms $(C, \rho) \in \mathrm{End}(\mathfrak{g}_{-1}) \times \mathrm{End}(\mathfrak{g}_{-2})$ that define the Lie algebra product $[(C, \rho), \xi_{-1} \oplus \xi_{-2}] = C\xi_{-1} \oplus \rho\xi_{-2}$

for $\xi_{-1} \in \mathfrak{g}_{-1}$ and $\xi_{-2} \in \mathfrak{g}_{-2}$. The Jacobi identity requires that

$$\mathcal{L}(C\xi, C\eta) = \rho\mathcal{L}(\xi, \eta)$$

for all $\xi, \eta \in \mathfrak{g}_{-1}$, i.e., (C, ρ) is a derivation of $\mathfrak{g}_-$. This defines a linear subspace $\hat{\mathfrak{g}}_0 \subset \mathrm{End}(\mathfrak{g}_{-1}) \oplus \mathrm{End}(\mathfrak{g}_{-2})$. The Lie algebra product in $\hat{\mathfrak{g}}_0$ is the usual commutator in $\mathrm{End}(\mathfrak{g}_{-1}) \oplus \mathrm{End}(\mathfrak{g}_{-2})$.

The component $\mathfrak{g}_0$ is the subspace of pairs (C, ρ) for which C preserves the complex structure J on $\mathfrak{g}_{-1}$, i.e., C is a complex endomorphism of $\mathfrak{g}_{-1}$. Thus $\mathfrak{g}_0$ carries a part of the geometric structure. This is in contrast to the higher prolongation components $\mathfrak{g}_\ell$ for $\ell \geq 1$ that are the maximal possible spaces obtained by the following inductive procedure. Assume that the components $\mathfrak{g}_m$ for $m < \ell$ have been constructed. Then $\mathfrak{g}_\ell$ is defined as the set of linear mappings

$$A\colon \mathfrak{g}_{-1} \to \mathfrak{g}_{\ell-1}$$

and

$$a\colon \mathfrak{g}_{-2} \to \mathfrak{g}_{\ell-2}$$

such that $a[X, Y] = [AX, Y] + [X, AY]$ for all $X, Y \in \mathfrak{g}_{-1}$. This implies that a is determined by A. In particular, $A = 0$ implies $a = 0$.

The Lie bracket in the prolongation is defined by

$$[[A, B], X] = [A, [B, X]] - [B, [A, X]],$$

where $X \in \mathfrak{g}_-$. The method of Cartan connections requires that this prolonged algebra remains finite dimensional. This can be guaranteed by imposing nondegeneracy on the Levi form.

Hypersurfaces in $\mathbb{C}^2$

As an illustration we compute the Cartan curvature for real hypersurfaces in $\mathbb{C}^2$. While this is the simplest instance of CR geometry and the approach is elementary and straightforward, the computations are arduous but can be successfully carried out with the help of computer algebra.

In this case $\mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ is the 3-dimensional Lie algebra with generators $X_1 \in \mathfrak{g}_{-2}$, $X_2, X_3 = JX_2 \in \mathfrak{g}_{-1}$ and such that $[X_2, X_3] = 4X_1$ (all other commutators vanish). The Lie algebra $\mathfrak{g}_0$ is isomorphic to $\mathbb{C}$ with generators X_4, X_5 such that $\gamma X_4 + \mathrm{i}\,\delta X_5$ acts on $\mathfrak{g}_-$ by

$$(X_1, X_2, X_3) \mapsto (2\gamma X_1, \gamma X_2 - \delta X_3, \delta X_2 + \gamma X_3).$$

The Tanaka prolongation yields

$$\begin{aligned}
\mathfrak{g}_1 = \{&\alpha X_6 + \beta X_7 \in \mathrm{Hom}(\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}, \mathfrak{g}_{-1} \oplus \mathfrak{g}_0)\colon \\
&(X_1, X_2, X_3) \mapsto ((\alpha + \mathrm{i}\,\beta)X_2, -2\beta X_4 - 6\alpha X_5, \\
&2\alpha X_4 - 6\beta X_5)\} \\
\mathfrak{g}_2 = \{&rX_8 \in \mathrm{Hom}(\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}, \mathfrak{g}_0 \oplus \mathfrak{g}_1)\colon \\
&(X_1, X_2, X_3) \mapsto (-rX_4, -rX_6, -rX_7)\}.
\end{aligned}$$

The full prolongation is isomorphic to the semisimple Lie algebra $\mathfrak{su}(2,1)$. According to Tanaka's theory there exists a unique regular normal Cartan connection. We will outline its construction using a local (noncanonical) trivialization of the Cartan bundle $\mathcal{G}|_U = U \times P$ over some neighbourhood U of 0.

The constant vector fields $\hat{X}_k = \omega^{-1}X_k$ on $\mathcal{G}|_U$ have the form

$$\begin{aligned}
\hat{X}_1 &= a_{11}\zeta + a_{12}\xi + a_{13}\eta + a_{14}\hat{X}_4 + \cdots + a_{18}\hat{X}_8 \\
\hat{X}_2 &= a_{22}\xi + a_{23}\eta + a_{24}\hat{X}_4 + \cdots + a_{28}\hat{X}_8 \\
\hat{X}_3 &= a_{32}\xi + a_{33}\eta + a_{34}\hat{X}_4 + \cdots + a_{38}\hat{X}_8 \\
\hat{X}_4 &= -\gamma\frac{\partial}{\partial\gamma} - \delta\frac{\partial}{\partial\delta} - \alpha\frac{\partial}{\partial\alpha} - \beta\frac{\partial}{\partial\beta} - 2r\frac{\partial}{\partial r} \\
\hat{X}_5 &= \delta\frac{\partial}{\partial\gamma} - \gamma\frac{\partial}{\partial\delta} - \beta\frac{\partial}{\partial\alpha} + \alpha\frac{\partial}{\partial\beta} \\
\hat{X}_6 &= \frac{\partial}{\partial\alpha} - \beta\frac{\partial}{\partial r} \\
\hat{X}_7 &= \frac{\partial}{\partial\beta} + \alpha\frac{\partial}{\partial r} \\
\hat{X}_8 &= \frac{1}{2}\frac{\partial}{\partial r},
\end{aligned}$$

where ξ, η, ζ are vector fields on M with $\xi \in \Gamma(D)$, $\eta = J\xi$, $\zeta = 4[\xi, \eta]$, $\gamma + \mathrm{i}\,\delta \neq 0$, $\alpha + \mathrm{i}\,\beta$, r are coordinates of P and, according to property (2) of the Cartan connection, $\hat{X}_4, \dots, \hat{X}_8$ are the left-invariant vector fields on the fiber P. The explicit expressions of the curvature depend on the coefficients of the commutator relations

$$\begin{aligned}
[\xi, \zeta] &= c_{11}\zeta + c_{12}\xi + c_{13}\eta \\
[\eta, \zeta] &= c_{21}\zeta + c_{22}\xi + c_{23}\eta.
\end{aligned}$$

Notice that for the homogeneous model one can choose ξ, η in such a way that $[\xi, \zeta] = [\eta, \zeta] \equiv 0$. In general, after a suitable coordinate change, we can achieve $c_{11} = c_{21} \equiv 0$.

It is our task to determine the coefficients $a_{11}, \dots, a_{38}$. Property (3) of the Cartan connection allows us to determine how they depend on the fiber variables. In particular, it follows that

$$\begin{aligned}
&a_{11} = \gamma^2 + \delta^2, \\
&a_{12} = -\gamma\alpha + \delta\beta, \qquad a_{22} = \gamma, \qquad a_{32} = -\delta \\
&a_{13} = -\delta\alpha - \gamma\beta, \qquad a_{23} = \delta, \qquad a_{33} = \gamma.
\end{aligned}$$

To find the dependence of the coefficients $a_{11}, \dots, a_{38}$ on the horizontal variables x, y, u we try to make the three commutators $[\hat{X}_i, \hat{X}_j]$ with $i, j = 1, 2, 3$ as close as possible to the corresponding commutators in $\mathfrak{su}(2,1)$. This is restricted by the ∂-cohomology H^2. Direct computation shows that the cocycles form a 17-dimensional subspace $\mathcal{Z}^2$ of the 24-dimensional space C^2. The subspace $\mathcal{B}^2 \subset \mathcal{Z}^2$ is 15-dimensional, which implies $\dim H^2 = 2$. The space C^2 splits in our case into homogeneous components of order from 0 to 5. The distribution of the dimensions by homogeneity is displayed in the table below:

Hom	$\dim \mathcal{B}$	$\dim \mathcal{Z}$	$\dim C$	$\dim H^2$
0	1	1	1	0
1	4	4	4	0
2	5	5	6	0
3	4	4	6	0
4	1	3	5	2
5	0	0	2	0

The cohomology occurs in homogeneity 4, which means that all curvature of homogeneity $i < 4$ vanishes. Hence the components of $[\hat{X}_2, \hat{X}_3]$ of grade ≤ 1 and the components of $[\hat{X}_1, \hat{X}_2]$ and $[\hat{X}_1, \hat{X}_3]$ of grade ≤ 0 are exactly as in the Lie algebra $\mathfrak{g}$.

The closedness of the curvature in homogeneity 2 and 3 is a consequence of the Jacobi identity $[\xi,[\zeta,\eta]] + [\eta,[\xi,\zeta]] + [\zeta,[\eta,\xi]] = 0$.

The explicit computations give the nonzero curvature components in homogeneity 4:

(1)

$$\begin{aligned}\kappa(X_1, X_2) = &\left(\frac{\tau^4-\delta^4}{96}k_1 - \frac{\tau\delta^3+\tau^3\delta}{48}k_2\right)X_6\\ &-\left(\frac{\tau^4-\delta^4}{96}k_2 - \frac{\tau\delta^3+\tau^3\delta}{48}k_1\right)X_7\\ \kappa(X_1, X_3) = &-\left(\frac{\tau^4-\delta^4}{96}k_2 + \frac{\tau\delta^3+\tau^3\delta}{48}k_1\right)X_6\\ &-\left(\frac{\tau^4-\delta^4}{96}k_1 - \frac{\tau\delta^3+\tau^3\delta}{48}k_2\right)X_7\end{aligned}$$

where

(2)

$$\begin{aligned}k_1 = &-5\eta^2 c_{12} + 15\xi\eta c_{22} + 4\xi^2 c_{23} - 15\eta\xi c_{13}\\ &- 9\eta\xi c_{22} + 72 c_{22}c_{12} + 12 c_{13}c_{12} + 84 c_{13}c_{23}\\ k_2 = &-7\xi\eta c_{23} - 3\xi^2 c_{13} - 18\eta\xi c_{12} + 9\eta\xi c_{23}\\ &- 3\eta^2 c_{22} + 20\xi\xi c_{22} + 24 c_{12}c_{23} - 24 c_{23}^2\\ &+ 48 c_{12}^2 - 36 c_{22}^2 + 36 c_{13}^2.\end{aligned}$$

The X_8 component of $\kappa(X_2, X_3)$ can be made 0 by using the one-dimensional freedom from $\mathcal{B}$.

Finally, the Bianchi identity completely determines the curvature of homogeneity 5.

Ellipsoids

In [25], Webster proved that an ellipsoid

$$|z|^2 + \sum_{j=1}^{n} \frac{A_j}{2}(z_j^2 + \bar{z}_j^2) = 1 \text{ with } 0 < A_j < 1$$

in $\mathbb{C}^n$ for $n \ge 3$ has no umbilic points, i.e., the harmonic Cartan curvature does not vanish at any point. Though his methods seem to be suitable for proving a similar result for $n = 2$, this case would be different because the curvature invariants depend on higher derivatives than for $n \ge 3$.

Our computations from the previous section applied to the ellipsoid

$$|z_1|^2 + |z_2|^2 + \frac{a}{2}(z_1^2 + \bar{z}_1^2) + \frac{b}{2}(z_2^2 + \bar{z}_2^2) = 1$$

in $\mathbb{C}^2$ at the vertex $(0, \sqrt{\frac{2}{2+b}})$ give us the harmonic curvature

$$\begin{aligned}k_1 = &\frac{96(b+1)^4\left(-2(4b+7)a^3\right.}{(b+2)^2}\\ &+ \frac{\left.(1335b^2+3123b+1450)\,a - 165(b-1)b\right)}{(b+2)^2}\\ &\ne 0\\ k_2 = &\ 0.\end{aligned}$$

Here we have used the vector field

$$\begin{aligned}\xi = &\left((b+1)^2x_2^2 + (b-1)^2y_2^2\right)\frac{\partial}{\partial x_1}\\ &+ \left(-(a+1)(b+1)x_1x_2 - (a-1)(b-1)y_1y_2\right)\frac{\partial}{\partial x_2}\\ &+ \left((1-a)(b+1)x_2y_1 + (a+1)(b-1)x_1y_2\right)\frac{\partial}{\partial y_2}.\end{aligned}$$

Conclusion

Using the Tanaka prolongation procedure, we have explicitly calculated the Cartan connection and Cartan curvature of a general Levi nondegenerate real hypersurface in $\mathbb{C}^2$ and, in particular, the ellipsoid. An analysis of the algebraic structure of the curvature permits us to isolate two out of twenty-four components (1) from which all invariants can be derived. In particular, two hypersurfaces are equivalent if their two harmonic curvature components match. Though this is difficult to check in general, it provides a simple criterion for equivalence with the sphere, namely the vanishing of (2). Our computations for the ellipsoid allow us to show that the real-analytic subset of umbilic points is proper and therefore the ellipsoid cannot be locally equivalent to the sphere at any point.

The precise expressions of the coefficients of the Cartan connection form, the curvature, and the computations in Mathematica can be found at `http://turing.une.edu.au/~bmclaugh/cartan/`.

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