Perfect Rigor: A Genius and the Mathematical Breakthrough of the Century

Reviewed by Donal O'Shea

Perfect Rigor: A Genius and the Mathematical Breakthrough of the Century
Masha Gessen
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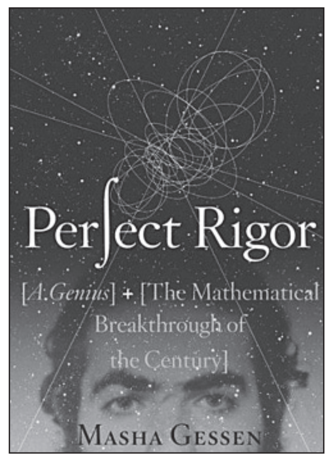
Gregory Perelman's proof of the Poincaré conjecture in 2002 and 2003 ranks as the greatest scientific achievement of the last decade. It is a great mathematical and a great human story and has been the subject of several books and will undoubtedly inspire others. The latest such book, Masha Gessen's *Perfect Rigor*, focuses largely on Perelman and the media storm that surrounded him.

The story of a lone individual dropping out of sight for nearly a decade and posting to the Internet a solution to one of the best known mathematical problems of all time would predictably interest the media. Stir in Perelman's eccentricity, a widely circulated *New Yorker* article alleging an unscrupulous attempt to wrest priority from him, the Clay Mathematical Institute's million-dollar prize for the solution, the rejection of a Fields Medal, and Perelman's refusal to talk to the press or to publish in a standard refereed journal, and you have the makings of a category five media hurricane. A journalist, Gessen brings exceptional credentials to the job. Russian-born, she emigrated with her family as a teenager to the United States and returned to Moscow to live in the early 1990s. She writes in both English and Russian and has made a name for herself for her courage and outspokenness. Unhappily, Gessen was unable to interview Perelman, as he refused all contact with her. She did, however, manage to talk with many in and out of the Russian mathematical establishment. The result is an exceedingly lively book about the schooling and society that gave rise to Perelman.



Before turning to Gessen's book, let me remind readers briefly of the mathematical story. In 1895 Poincaré wrote an extraordinary foundational paper on topology that introduced the fundamental group, generalized the notions of homology and Betti numbers to all dimensions, and focused on differentiable manifolds as objects of particular study. Although this paper and the five "complements" to it that appeared between 1899 and 1904 compose a small percentage of Poincaré's total work (measured by number or page length), they established algebraic, differential, and combinatorial topology as fields in their own right. So deeply embedded are the results of these papers into mathematical consciousness that we no longer have any awareness of the extent to which they have shaped our thinking. In fact, these papers firmly established the utility of topological concepts in analysis and geometry and thereby set the course of much of twentieth-century mathematics.

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They also gave us the first famous mathematics problem that belonged wholly to the twentieth century. In the second complement, which appeared in 1900, Poincaré announced that a compact n-dimensional manifold without boundary with the same Betti numbers as the n-dimensional sphere was necessarily homeomorphic to it. Four years later, he discovered, and in the fifth complement described, a marvelous counterexample: a three-dimensional manifold with the same homology groups as a three-sphere, but with finite fundamental group. In the very last paragraph of the fifth and final complement, however, Poincaré restates his announcement of 1900, asking what has become known as "the Poincaré conjecture". Namely, he asked whether a simply-connected, compact three-dimensional manifold (without boundary) is necessarily homeomorphic to a three-dimensional sphere.

Higher-dimensional analogues of the Poincaré conjecture were proved in dimensions greater than four by Smale in 1960 and in dimension four by Michael Freedman in 1982. Both men received Fields Medals, and their work opened new areas and touched off further advances. But the original three-dimensional conjecture, arguably the most natural question one could ask of compact three-manifolds, resisted all attempts at resolution. Its subtlety and hidden difficulty became famous as false proofs and would-be counterexamples accumulated over the century. In fact, until Bill Thurston formulated and amassed significant evidence in the late 1970s for his celebrated geometrization conjecture, no one really had any idea whether or not the Poincaré conjecture might be true. Thurston's conjecture implied the Poincaré conjecture and provided a conjectural framework indicating why it might be true. But a proof seemed completely out of reach. After some initial promise, various differential geometric approaches pioneered by Yau, Anderson, Hamilton, and others seemed hopelessly stalled. Perelman changed all that. With a mix of sheer power and dazzling geometric insight, he understood and exploited the geometry of Hamilton's Ricci flow, allowing him to prove the full geometrization conjecture. The unselfish labor of mathematicians all over the world in verifying and explaining Perelman's work shows the mathematics at its best.

Gessen set herself the tasks of trying to find out what it was about Perelman's mind that enabled him to settle the Poincaré and geometrization conjectures and, having done so, why he reportedly decided to abandon mathematics. She begins by sketching how mathematics assumed its unique role in Soviet society, beginning with the strength of the Moscow mathematical community in the early twentieth century, the arrest of Dimitri Egorov, the denunciation of Nikolai Luzin, and the conjectural, somewhat casual, decision of Joseph Stalin to head off a show trial and to partially rehabilitate Luzin. The exceedingly effective lesson underscored the dependence of the mathematical community on the goodwill of the state and severed relations between the mathematical community in Russia and the rest of the world. It did, however, leave the community's scientific strength intact, and the usefulness of mathematicians in developing the former Soviet Union's nuclear and space programs resulted in substantial state investment in mathematics and the privileging of mathematicians. Gessen recounts the charmed life of Kolmogorov, born of wealth and fabulously talented, whose influence stemmed not just from his own extraordinary mathematical prowess but also his wide interests, his ideals of a classical liberal education, and his interest in education. Kolmogorov set up a system of schools for gifted children that had a wide influence.

Gessen has a sharp eye for irony and points out that the existence of a strong mathematical school in Stalinist and Soviet Russia is nearly miraculous, since the values of the mathematical community are antithetical to either. She details the unspoken but ubiquitous anti-Semitism of the Soviet mathematical establishment and the rigid quotas on Jews admitted to elite postsecondary educational institutions (two per year to Mathmech, none-to-one in Steklov). This resulted in a strong mathematical culture outside the establishment, peopled by very talented individuals working for the sheer love of mathematics, but occupying positions unequal to their ability and accomplishment. She describes how a system of after-hour mathematics clubs in which schoolchildren would be coached in solving mathematics problems fed the two mathematics cultures, one establishment and privileged, the other not. As in other places in the book, she is able to draw on her own experience, contrasting the teaching she received in elementary school, where her teacher made her pretend to read as poorly as the other kids, with the math club in which the coach elicited from her how she might proceed to solve a particular problem and then exhorted her to do it. "Apparently, this was a place where I was expected to think for myself," she writes. "A wave of embarrassment covered me; I hunched over my piece of paper, sketched out a solution in a couple of minutes. And felt a wave of relief so total that I think I became a math junkie on the spot."

She found, and interviewed extensively, Sergei Rukshin, the idiosyncratic self-made coach who ran the mathematics club that Perelman joined and who was a decisive influence on the younger Perelman. She describes the closeness between coach and student and Rukshin's coup in getting the ablest members of his club admitted as a bloc to the storied Leningrad school 239. She interviewed high school mathematics teacher Valery Ryzhik, who inherited the bloc and who recognized Perelman's genius. She describes the careful machinations by which Rukshin arranged to have Perelman, a Jew, become a member of the team representing the Soviet Union at the International Mathematics Olympiad in Budapest in 1982.

Gessen diligently tracks down nearly anyone known to have had any contact with Perelman. The closest that Gessen gets to Perelman, however, is through Rukshin, and this is to the adolescent. Ryzhik is clear that he never managed to form a close bond with Perelman. She interviews Perelman's undergraduate geometry teacher and thesis adviser, Zalgaller, now in Israel, who says that he had nothing to teach Perelman but who saw his contribution as directing unsolved problems to Perelman and seeing that Perelman's solutions were published. The precise role that Alexander Danilovich Alexandrov, the former president of Leningrad University who had returned to the classroom, played in Perelman's intellectual development is even less clear, although there can be no doubt that he must have exerted a decisive influence. Paradoxically, the nearer in time Gessen gets to the adult Perelman, the further he seems to recede. Perelman's Russian postgraduate collaborators, Burago and Gromov, and those he met in the United States tell Gessen little.

Writing about someone who refuses to be interviewed may, as Gessen allows, be easier than writing about a cooperating subject, but it also imposes an obligation to exercise caution, as it is harder to check facts and hypotheses. It is difficult to escape the disquieting impression that Gessen's narrative strays a little too far in places from the actual evidence at hand. In the book's penultimate chapter, entitled "The Madness", Gessen writes, "The more Perelman talked about his disappointment with the mathematical establishment, and the more his acquaintances decorated his stories with demonizing details, the more Perelman's sense of betrayal deepened. His world, which had begun narrowing in his first university year and then broadened slightly both times he had traveled to the United States, was now headed for its final disastrous narrowing." How, if she hasn't talked with Perelman, can she talk about a sense of betrayal? In what ways was Perelman's world narrowing in university? He clearly was undergoing tremendous mathematical growth. Likewise, the notion that Perelman was disastrously disappointed with Hamilton seems to be extrapolated solely from a quote from the *New Yorker* article. Gessen is far too good a journalist to misquote a primary source, but no fair-minded reader can fail to note the distressing number of places in which she impugns her sources' testimony. She breathlessly recounts, for example, secondhand reports of the alleged screaming match Perelman had with the accountant at Steklov. However, the accountant, whom Gessen interviewed, denied that there was yelling. Gessen's dismissal of the eyewitness account ("though over her years at the Steklov, she [the accountant] may have grown accustomed to extreme and unexpected expressions of human emotion") seems a stretch, at best.

Gessen argues that the people who surrounded Perelman sheltered him from ordinary reality, allowing him to mistakenly believe that the world is as he thinks it should be. This elaborate narrative is totally conjectural—Gessen has no evidence about what Perelman believes. Undaunted, she goes on to diagnose Perelman with a full-blown case of Asperger's syndrome. I simply don't know enough to evaluate these claims and am entirely unconvinced. Everyone agrees that Perelman lives simply, so why not make the simpler assumption that he wants privacy and does not want to be encumbered with fame or money? Perelman's recent refusal of the million-dollar Clay Millennium award suggests this, particularly since the Clay Institute made it clear that Perelman would not have to participate in any public ceremony.

Even putting aside the evidentiary questions, I found the second half of the book offensive. I felt uncomfortable reading about a living individual who wishes to remain out of public sight. Publicly diagnosing someone with a serious psychological disorder without consultation seems ethically questionable, not to mention presumptuous. Doing any sort of mathematics requires precision, careful attention to meaning, and concentration. Gessen's account of British psychiatrist Simon Baron-Cohen's autism-spectrum quotient test, and the purported strong correlation between high-functioning autism and mathematical ability in a test population, runs dangerously close to medicalizing precisely these traits. Gessen's presumption does not end with psychiatric expertise. She opines freely on Perelman's work, characterizing it as solving the "very, very complicated olympiad problem" into which she has Hamilton casting Thurston's geometrization conjecture. She cavalierly ranks top mathematicians in descending order from those who open new fields by posing questions no one has thought to ask (such as Poincaré and Thurston) to those who devise ways to answer those questions (such as Hamilton) to the bottom of the top, those poor souls (such as Perelman) who take the last steps in completing proofs. Mathematicians will easily discern the depth of Gessen's mathematical ignorance, but others will not, and it is depressing to see Perelman's inspiring achievement and powerful new ideas reduced to psychobabble: "Speaking of the imaginary four-dimensional space, he referred to things that could and could not occur 'in nature'. In essence, he [Perelman] was able to do in mathematics what he had tried to do in life: grasp at once all the possibilities of nature and annihilate everything that fell outside that realm—castrati voices, cars, anti-Semitism, and any other uncomfortable singularity."

The incoherence and ugliness of such assessments contrast with the clarity and grace of the laudations read at the recent conference in Paris cosponsored by the Clay Mathematics Institute and the Institut Henri Poincaré celebrating Perelman's proof. Bill Thurston, for example, closes his by remarking that "in our modern society most of us reflexively and relentlessly pursue wealth, consumer goods, and admiration. We have learned from Perelman's mathematics. Perhaps we should also pause to reflect on ourselves and learn from Perelman's attitude toward life." (See `http://www.claymath.org/poincare/laudations.html`.) One cannot expect Gessen to understand the mathematics, but one wishes for some sense of her own limitations, some caution, some generosity, and some openness to difference. Her cheeky self-confidence and willingness to trample on what she does not understand, so typical of popular culture, wears thin. Perelman and his work deserve better. So, too, do the discipline and the profession.

Book Review

Numbers Rule: The Vexing Mathematics of Democracy, from Plato to the Present

Reviewed by Jonathan K. Hodge

Numbers Rule: The Vexing Mathematics of Democracy, from Plato to the Present
George G. Szpiro
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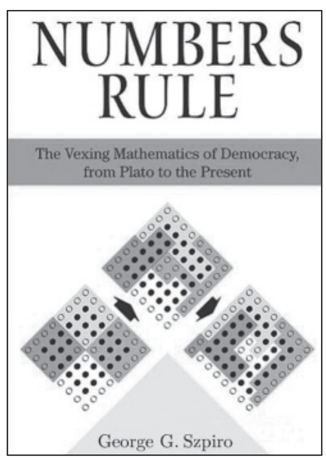
In recent years, the mathematical social sciences—particularly voting and social choice theory—have become a hot topic in both academia and popular culture. Liberal arts mathematics courses now often include sections on fair division and voting theory, and a number of recent textbooks and monographs devote themselves entirely to these topics.

Szpiro's book focuses on one such topic, namely, how mathematics—and mathematicians—have impacted both the theory and practice of democracy. It is an excellent addition to a growing body of literature that aims to convey ideas from the mathematical sciences to general audiences. Moreover, Szpiro's book is unique among other offerings in



the mathematical social sciences in that it focuses on the historical development of the field. The narrative is engaging, witty, and easy to read.

As the book's title would suggest, the story begins in Athens, in the fifth century BC. Szpiro describes how the Athenian government was set up so that "everybody who had any sort of interest in running the city could either participate in the Assembly as he pleased, or was selected by lot, as in the Council, the Court, or the civil service" (p. 7). Plato was highly critical of this form of unrestricted democracy. In fact, it was a majority decision (280 of 501 jurors) by a randomly selected jury that had condemned his beloved teacher, Socrates, to death. Plato concluded, as Szpiro puts it, that "regular folks were not fit to rule and to dispense

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justice" (p. 2). He came to despise democracy and was "chastised as the worst anti-democrat by his detractors" (p. 1).

Plato's counterproposal, as conveyed by the Athenian stranger in the unfinished manuscript, *Laws*, involved multistage elections, significant privilege to the wealthy and well educated (who, to Plato, were one and the same), and other suggestions that, in Szpiro's words, "seem pulled out of a hat like a magician's rabbit" (p. 17). Plato's earlier work, *The Republic*, largely ignored elections and votes, which were viewed as superfluous because, as Szpiro summarizes, "the qualities necessary to become philosopher-king…do not often grow together" and "individuals who possess all these qualities are so rare that the state will hardly ever find more than one who fits the job description" (p. 20). In *Laws*, however, Plato dives in head first, setting the stage for centuries of future debate about the merits and implementation of democracy.

What we see in the story of Plato, Socrates, and Athenian democracy is a fundamental tension between competing social values. How is the ideal of full and equal participation in government to be balanced against the fact that majorities can, and sometimes do, endorse erroneous or unjust propositions? How does one ensure that government officials are qualified for the tasks under their charge when they are selected based solely on popular opinion? These are difficult questions, and in many ways they set the tone for the remainder of the book, which Szpiro accurately describes in the preface as "an elucidation and a historical account of the problems and dangers that are inherent in the most cherished instruments of democracies" (p. x).

As the account unfolds, the reader is given the opportunity to turn back the clock and embark on a journey that spans thousands of years and includes both well-known figures and their lesser-known counterparts. For instance, anyone who has studied voting theory is undoubtedly familiar with the work of eighteenth-century French contemporaries Borda and Condorcet. But readers might be surprised to learn that both the Borda count and Condorcet's method of pairwise comparisons were proposed centuries earlier, the former by a German cardinal named Nikolaus von Cusanus and the latter by a Catalan monk named Ramon Llull. In fact, Szpiro notes that "until quite recently most researchers believed that interest in the theory of voting and elections had started toward the end of the eighteenth century, at the time of the French Revolution. But toward the middle of the twentieth century, medievalists were surprised to discover manuscripts in the Vatican library and elsewhere that showed that sophisticated ideas had already been around half a millennium earlier" (p. 33). Szpiro has clearly done his research with this book, and the result is a strikingly thorough and engaging read.

One of the things I like most about the book is that it reveals mathematics to be a decidedly human endeavor, fraught with controversy and able to both expose and help solve real problems. Both the characters and the plot of the story defy the one-dimensional stereotypes that students sometimes associate with mathematicians and the study of mathematics. The historical figures surveyed include economists, lawyers, theologians, military officers, philosophers, artists, politicians, scientists, and yes, mathematicians. The personal and professional lives of each are explicated in detail both within the text and in biographical appendices at the end of each chapter. The effect is to add meaning to the intellectual contributions explored and to place them in the broader context of human experience.

Part of that experience includes spirited debate and a fair dose of name calling. Recall our friends Borda and Condorcet. As it turns out, they weren't friends at all. In fact, Condorcet was a fairly vocal critic of Borda. As Szpiro notes, "Condorcet did not think highly of Borda. In fact he did not even consider him a very capable mathematician… Condorcet wrote that Borda likes to talk a lot and wastes his time tinkering with childish experiments" (p. 89).

But the rift between Borda and Condorcet was nothing compared to that between Edward V. Huntington, a professor of mathematics at Harvard, and Walter F. Willcox, a professor of social science and statistics at Cornell. Their rivalry, which spanned decades in the early 1900s, involved polarizing rhetoric, highly publicized personal attacks, and more than a hint of deception. The substance of Huntington and Willcox's debate was the problem of apportionment—that is, how to allocate the appropriate number of seats to each state in the U.S. House of Representatives. Once again, the difference of opinion between the two professors was ultimately one of competing values. Willcox was a proponent of Webster's method of major fractions, which shows no bias to either large or small states, whereas Huntington supported Hill's method of equal proportions, which minimizes the relative differences in representation between the states. (Incidentally, Huntington could be viewed as the winner in this battle, as the method of equal proportions was adopted in 1941 and is, to this day, the method used to apportion the U.S. House. On the other hand, Michel Balinski and Peyton Young would later provide some vindication for Willcox, stating: "It seems amazing therefore that Hill's method could have been chosen in 1941… and that Webster's method was discarded. A peculiar combination of professional rivalry, scientific error, and political accident seems to have decided the issue" (pp. 195–196).)

The stories of Borda and Condorcet and of Willcox and Huntington serve to illustrate another valuable takeaway from Szpiro's book—namely, it dispels the myth that mathematics is a value-free endeavor, a matter of black and white, of finding the one right answer. If there is anything to be learned from the mathematical study of voting and elections, it is that sometimes there are no universally correct answers. Paradoxes abound, and the correct procedure often depends on the values and beliefs of those using it. Arrow proved this for voting systems, and Balinski and Young did the same for apportionment methods. These examples and many others support Bradley and Schaefer's [1] assertion that as the mathematization of the social sciences continues, "norms, values, and purpose need to become part of the common discourse of researchers." Szpiro notes that even the great mathematician Pierre-Simon de Laplace "did not find it beneath himself to bend the rigorous rules of mathematics somewhat when needed" (p. 97).

In summary, Szpiro's book fills a unique role in an increasingly popular field. Being written for general audiences, it suffers from some oversimplifications (such as stating that "misrepresenting one's preferences brings no advantage" (p. 212) in approval voting; in fact, approval voting is not completely immune to strategic voting, although it is less vulnerable than other nonranked systems) and minor imprecisions in language (for instance, the occasional conflation of the words *plurality* and *majority*). My only other substantive complaint is that the book feels like it ends too soon. The final chapter mentions single transferrable vote and approval voting, but only briefly. In addition, although works by some of today's leading mathematical voting theorists, such as Donald Saari and Alan Taylor, are included in the bibliography, they are not discussed at all within the text. Of course, every author must make choices about what to include and what to omit. Szpiro, in general, has chosen well. The result is a readable, engaging, and intellectually stimulating book that accomplishes its goal of "[introducing] readers to the subject matter in an entertaining way" (p. x).

Reference

[1] W. J. Bradley and K. C. Schaefer, *The Uses and Misuses of Data and Models: The Mathematization of the Human Sciences*, Sage Publications, 1998.