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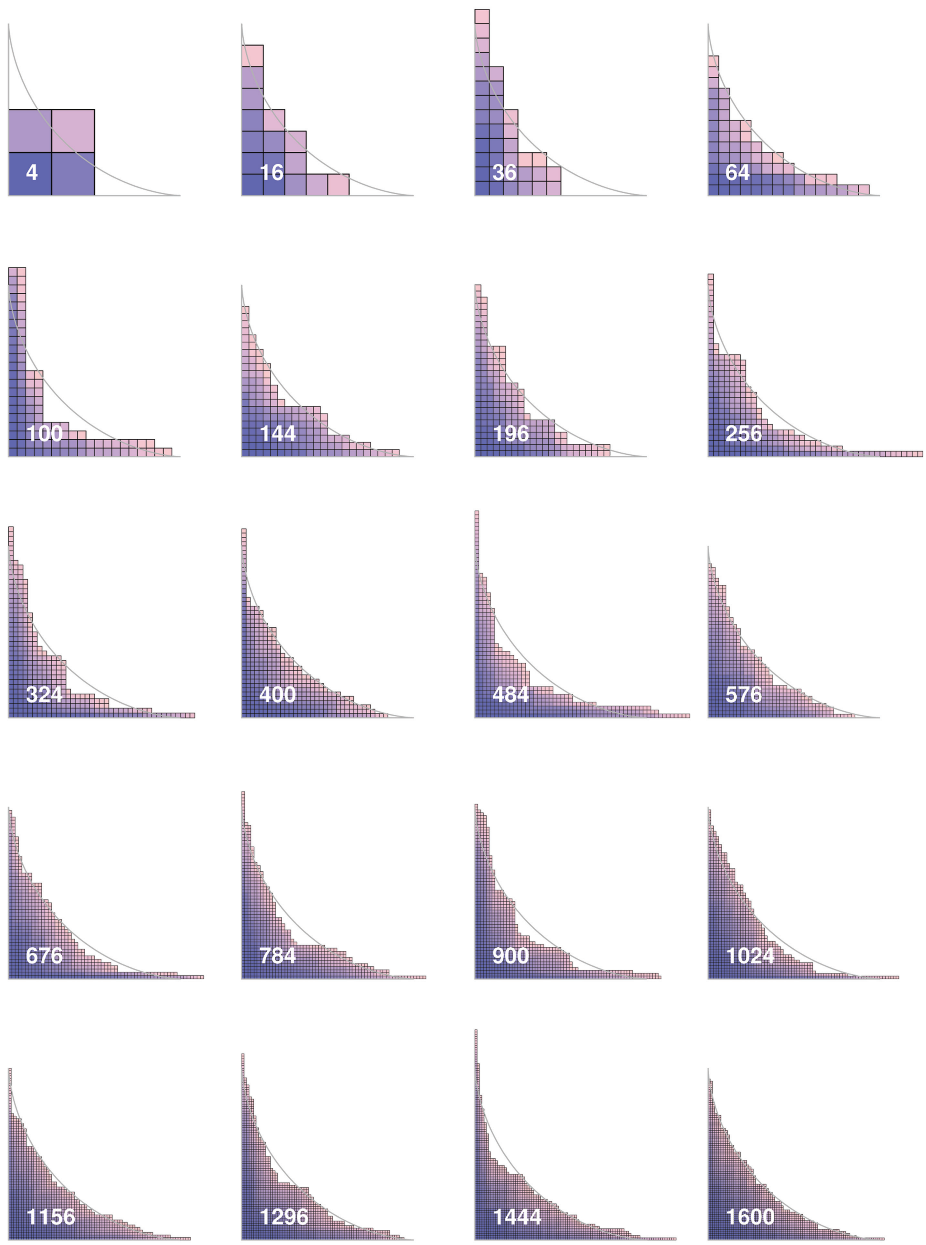
Notices

of the American Mathematical Society

February 2011

Volume 58, Number 2



About the Cover:
Random Young diagrams
(see page 323)

American Mathematical Society

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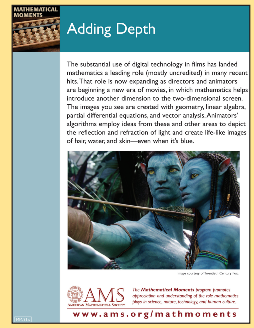

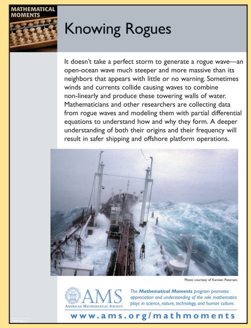

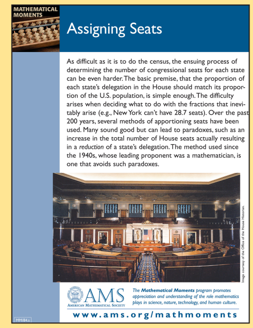

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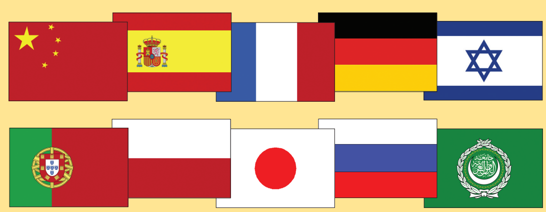

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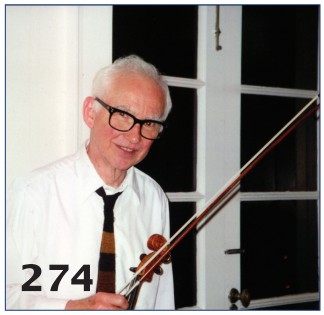
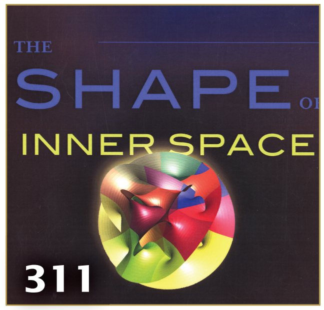
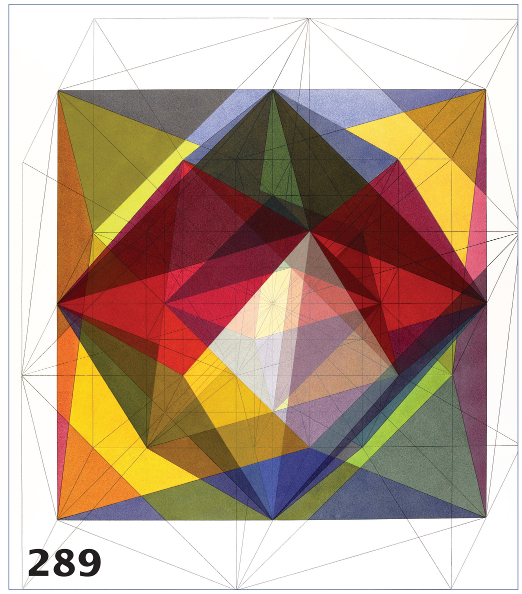
Communications

Commentary







Features

This month features some aspects of algebra and of history. We explore the quest of Enriques for an important completeness theorem. [Fortuitously, one of the authors of those articles has been honored with the National Medal of Science.] And we examine the role of trinomials in modern aspects of the subject. On the historical side, we examine the role of the Parker Fellowships in the development of mathematics at Harvard, and in the United States as a whole. Finally, a memorial article for Henry Helson recalls the life and career of an important and influential mathematical analyst.

—Steven G. Krantz, Editor

Notices

of the American Mathematical Society

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Departments

From the AMS Secretary

I thank Randi D. Ruden for her splendid editorial work, and for helping to assemble this issue. She is essential to everything that I do.

—Steven G. Krantz
Editor

Classroom Assessment vs. Student Satisfaction

This is an article much in agreement with, but taking a different angle from, a recent Opinion piece, "Evaluation of our courses", by Steven Zucker, which appeared in the August 2010 issue of the *Notices* (page 821).

Zucker raises an issue I have cared greatly about: our so-called course evaluations are not evaluations at all. Administrators interpret (and often use) the scores as measuring instructor effectiveness. Yet such scores cannot reflect how much students have learned. The scores encourage us to become popular teachers rather than good educators.

Testing course effectiveness requires deeper ideas than asking students how they liked the course near its end. No matter how you phrase the questions or what you ask, such a survey cannot provide reliable information about course effectiveness. By an effective course I mean one that enables students to use the material later on.

Fifteen years ago, I got extraordinarily high survey scores. Yet my vector calculus students did very poorly on final exams. In 1995, backed by a Sloan Foundation grant, I developed an Internet-based student assessment tool: "Interactive Questionnaires" (IQs). An IQ is an enhanced interactive quiz.

Here's how it worked. I sent IQs to students in email. By applying one instruction, a student could open the IQ as a UNIX program. It would guide the student on problems he or she likely couldn't have done independently. When the student was satisfied, the program automatically mailed me the student's responses. My system produced formatted "interaction portfolios" from these responses. I could evaluate them electronically to assess the entire class on a finely parsed topic or an individual's progress in the steps to topic mastery.

I could do this without losing class time, long before final exam failure. From this I found when and why my class's modest early mastery could mysteriously disappear—rather than grow—as the final approached.

My published paper, at the URL `http://www.math.uci.edu/~mfried/edlist-tech/gold02-08-98.pdf`, gives example questions. It shows how I used interaction portfolios to produce IQ reports. You can see how the email system—"Problem of the day"—coordinated daily classes with IQ intercessions. IQs got students to write mathematics in a controlled setting. For IQ pieces that stretched students beyond a one-word answer, I could electronically batch-grade: an activity with no resemblance to the onerous task of inspecting cryptic scribbles on paper.

I designed IQs to engender step-by-step thinking. I added nuance to what most UC administrators knew of a particular minority student (see `http://www.math.uci.edu/~mfried/edlist-tech/gold02-08-98.html`). My story informed administrators that he was the only exception to vector calculus totally wiping out minority students from participating in mathematics, science, and engineering.

I have visited prestigious universities. Their classes included some students whose mental energies allowed them to digest and analyze at impressive rates. Such students may quickly imitate the instructor's "analyzing" with less training than I typically saw at UC Irvine. Zucker apparently sees that phenomenon at Johns Hopkins.

Responding to classes at different levels is part of the teacher skill. IQ questions must change according to class level, especially to take advantage of the additional skills students acquire using them. Yet, one change baffled me.

Despite documentable student progress on critical topics, my survey scores dropped from 6.2–6.7 to 3.5–4.5 (on a scale of 1 to 7). That shook me at first. (I am told that carefully controlled studies have shown negative correlation of survey scores to learning.) My demeanor and classroom presentation, even my graphic illustrations, did not change from pre- to post-IQs. Yet, post-IQ students saw me as less sympathetic to their trials with a difficult course.

Compatible with Zucker's experience, the IQ reports led students to see they could work harder. Many of my students (certainly not all) interpreted that as a negative. Students at elite institutions might feel that using a tool like IQs is not necessary—a little like spoon-feeding.

Less necessary maybe, but it isn't like spoon-feeding. As I became confident in the response to IQs, I saw students develop confidence that classes made sense, something many told me they doubted before this experience.

Having instructors at other institutions use IQs would allow comparing how the tool works for students with differing initial aptitudes. Teachers would get different results, but the improvement would be in student accomplishment, not in instructor likability.

My tools should go through a Web interface with security controls. Higher administration money backing such a development would take the onus off each instructor. I see that changing how students respond to working harder.

IQs provide, as could other assessments, a rethinking of how we document what our classes accomplish. As an alternative to polling students at the end of the semester, they can measure the ongoing effectiveness of a long course.

—Mike Fried
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at Irvine
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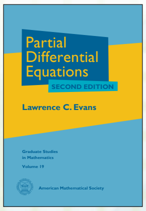
American Mathematical Society

A Selection of the Best-Selling AMS Titles in 2010



Partial Differential Equations

Second Edition

Lawrence C. Evans, *University of California, Berkeley, CA*

This book would be invaluable for a graduate student preparing to do research in PDEs; I wish I had a copy in graduate school.

—MAA Reviews

Graduate Studies in Mathematics, Volume 19; 2010; 749 pages; Hardcover; ISBN: 978-0-8218-4974-3; List US\$93; AMS members US\$74.40; Order code GSM/19.R

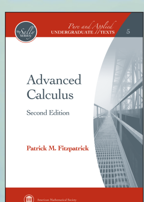
Introduction to Analysis

Fifth Edition

Edward D. Gaughan, *New Mexico State University, Las Cruces, NM*

Basic topics of analysis presented to develop in the reader an accurate intuitive feeling for the subject

Pure and Applied Undergraduate Texts, Volume 1; 1998; 240 pages; Hardcover; ISBN: 978-0-8218-4787-9; List US\$62; AMS members US\$49.60; Order code AMSTEXT/1



Advanced Calculus

Second Edition

Patrick M. Fitzpatrick, *University of Maryland, College Park, MD*

A rigorous presentation of the fundamental concepts of mathematical analysis, intended to encourage an appreciation of the subject's coherence

Pure and Applied Undergraduate Texts, Volume 5; 2006; 590 pages; Hardcover; ISBN: 978-0-8218-4791-6; List US\$82; AMS members US\$65.60; Order code AMSTEXT/5

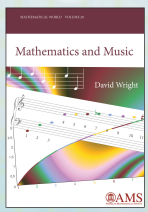
Computational Topology

An Introduction

Herbert Edelsbrunner, *Duke University, Durham, NC, and Geomagic, Research Triangle Park, NC*, and **John L. Harer**, *Duke University, Durham, NC*

An introduction to the often difficult concepts of algebraic topology in a way motivated by applications

2010; 241 pages; Hardcover; ISBN: 978-0-8218-4925-5; List US\$59; AMS members US\$47.20; Order code MBK/69



Mathematics and Music

David Wright, *Washington University, St. Louis, MO*

In classroom use, this polished book gives mathematics students a demystification of music theory's prerequisites and music students a motivated selective review of high school mathematics.

—CHOICE Magazine

Mathematical World, Volume 28; 2009; 161 pages; Softcover; ISBN: 978-0-8218-4873-9; List US\$35; AMS members US\$28; Order code MAWRLD/28

Famous Puzzles of Great Mathematicians

Miodrag S. Petković, *University of Nis, Serbia*

The author has done an admirably accurate and thorough job in presenting his material... The problems are here, their histories are here, the mathematics needed to solve them is here.

—MAA Reviews

2009; 325 pages; Softcover; ISBN: 978-0-8218-4814-2; List US\$36; AMS members US\$28.80; Order code MBK/63

Probability: The Science of Uncertainty

with Applications to Investments, Insurance, and Engineering

Michael A. Bean

A review of the basic probability of distributions, emphasizing the many applications arising in investments, insurance and engineering

Pure and Applied Undergraduate Texts, Volume 6; 2001; 448 pages; Hardcover; ISBN: 978-0-8218-4792-3; List US\$72; AMS members US\$57.60; Order code AMSTEXT/6

The Knot Book

An Elementary Introduction to the Mathematical Theory of Knots

Colin C. Adams, *Williams College, Williamstown, MA*

Amazingly understandable … After reading it twice, I still pick it up and scan it … this book belongs in every mathematical library.

—Charles Ashbacher, Book Reviews Editor, Journal of Recreational Mathematics

2004; 307 pages; Softcover; ISBN: 978-0-8218-3678-1; List US\$30; AMS members US\$24; Order code KNOT

A Decade of the Berkeley Math Circle

The American Experience, Volume I

Zvezdelina Stankova, *Mills College, Oakland, CA, and University of California, Berkeley, CA*, and **Tom Rike**, *Oakland High School, CA*, Editors

An engaging account of an American adaptation of mathematical circles designed to inspire a new generation of mathematical leaders

Titles in this series are co-published with the Mathematical Sciences Research Institute (MSRI).

MSRI Mathematical Circles Library, Volume 1; 2008; 326 pages; Softcover; ISBN: 978-0-8218-4683-4; List US\$49; AMS members US\$39.20; Order code MCL/1

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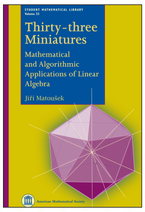


Thirty-three Miniatures

Mathematical and Algorithmic Applications of Linear Algebra

Jiří Matoušek, *Charles University, Prague, Czech Republic*

A collection of clever mathematical applications of linear algebra, each designed to be covered conveniently in a 90-minute lecture

Student Mathematical Library, Volume 53; 2010; 182 pages; Softcover; ISBN: 978-0-8218-4977-4; List US$36; AMS members US$28.80; Order code STML/53

Low-Dimensional Geometry

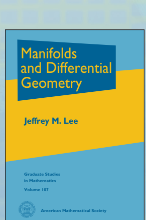
From Euclidean Surfaces to Hyperbolic Knots

Francis Bonahon, *University of Southern California, Los Angeles, CA*

...if just because no other yet competes for this crucial niche, but also happily excellent in every respect—passionately told, expertly rendered, exquisitely organized, and sumptuously illustrated. ... Essential.

—*CHOICE Magazine*

Student Mathematical Library, Volume 49; 2009; 384 pages; Softcover; ISBN: 978-0-8218-4816-6; List US$54; AMS members US$43.20; Order code STML/49



Manifolds and Differential Geometry

Jeffrey M. Lee, *Texas Tech University, Lubbock, TX*

A thorough, modern introduction to smooth manifolds and differential geometry, helping the student see the material from several perspectives

Graduate Studies in Mathematics, Volume 107; 2009; 671 pages; Hardcover; ISBN: 978-0-8218-4815-9; List US$89; AMS members US$71.20; Order code GSM/107

Ricci Flow and the Sphere Theorem

Simon Brendle, *Stanford University, CA*

An alternative approach to Ricci flow techniques, from a rising young star in geometric analysis

Graduate Studies in Mathematics, Volume 111; 2010; 176 pages; Hardcover; ISBN: 978-0-8218-4938-5; List US$47; AMS members US$37.60; Order code GSM/111

An Introductory Course on Mathematical Game Theory

Julio González-Díaz, *Universidade de Santiago de Compostela, Spain*, **Ignacio García-Jurado**, *Universidad de Coruña, Spain*, and **M. Gloria Fiestras-Janeiro**, *Universidade de Vigo, Spain*

A substantial treatment of game theory that includes the subtle and often overlooked topic of cooperative games

Graduate Studies in Mathematics, Volume 115; 2010; 324 pages; Hardcover; ISBN: 978-0-8218-5151-7; List US$62; AMS members US$49.60; Order code GSM/115

Continuous Time Markov Processes

An Introduction

Thomas M. Liggett, *University of California, Los Angeles, CA*

General theory of Markov processes combined with applications to special examples, addressing important topics such as interacting particle systems

Graduate Studies in Mathematics, Volume 113; 2010; 271 pages; Hardcover; ISBN: 978-0-8218-4949-1; List US$55; AMS members US$44; Order code GSM/113

What's Happening in the Mathematical Sciences, Volume 7

Dana Mackenzie

Nine current research topics that illustrate the beauty and liveliness of today's mathematics

What's Happening in the Mathematical Sciences, Volume 7; 2009; 127 pages; Softcover; ISBN: 978-0-8218-4478-6; List US$19.95; AMS members US$15.96; Order code HAPPENING/7

Lectures on Fractal Geometry and Dynamical Systems

Yakov Pesin and **Vaughn Climenhaga**, *Pennsylvania State University, University Park, PA*

An introduction to fractal geometry and dynamical systems that emphasizes their interrelationship, and their interaction with the theory of chaos

Student Mathematical Library, Volume 52; 2009; 314 pages; Softcover; ISBN: 978-0-8218-4889-0; List US$51; AMS members US$40.80; Order code STML/52

Mathematics Everywhere

Martin Aigner and **Ehrhard Behrends**, *Freie Universität Berlin, Germany*, Editors

Translated by Philip G. Spain

Everyday examples of how mathematics permeates the real world, in far-ranging areas from compact discs to climate change

2010; 330 pages; Softcover; ISBN: 978-0-8218-4349-9; List US$49; AMS members US$39.20; Order code MBK/72

Ricci Flow and Geometrization of 3-Manifolds

John W. Morgan, *Stony Brook University, NY*, and **Frederick Tsz-Ho Fong**, *Stanford University, CA*

An overview of how to use Ricci flow to establish the Poincaré Conjecture and the more general Geometrization Conjecture for all 3-manifolds

University Lecture Series, Volume 53; 2010; 150 pages; Softcover; ISBN: 978-0-8218-4963-7; List US$41; AMS members US$32.80; Order code ULECT/53

TEXTBOOKS FROM THE AMS

Applied Mathematics

Letters to the Editor

Math and Poetry as Exemplified in Ernst Schröder's Work

I had occasion elsewhere to note the artistic facets in the work of Ernst Schröder. I am referring not only to his logical texts, but also to his mathematical ones, such as *On the iteration of functions*, where Schröder formulated the equation now named after him. This German mathematician gave great significance to the visual layout of his formalism. For example, he defines the dual form of a theorem, its conjugate, and its dual conjugate. So every sentence admits four forms, which are arranged in a square: in the left upper corner the theorem, in the right upper corner the dual theorem, in the the left lower corner the conjugate theorem, and in the right lower corner the dual conjugate theorem. Why all this structure? Once introduced and explained, the concepts of dual and conjugate are *useless*, every new theorem is stated, and we see that the result has many possible forms.

But this graphic effort is not at all useless if we regard the matter from another point of view—that is, if we are interested not in the theorem alone but in its symmetries and the possible connections among its forms. It is not pedantry. Schröder was often lazy, self-contradictory, and not as accurate as one would expect (especially from a mathematician). The raison d'être for this array of formulas abides in the realm of art, in the plasticity of the medium. I think of Beethoven's first movement of his piano Sonata Opus 78, where the composer inserted a double *ritornello*. Did he want to hear repeatedly the same material? Of course not. Artistic reason drove Beethoven to polish his sonata in this way.

More than other mathematicians, for Schröder, art is in the foreground, revealing a search for poetry and elegance.

—*Davide Bondoni*
Independent scholar
`davidebond@yahoo.it`

(Received November 3, 2010

More on Course Surveys

The August 2010 issue of the *Notices* carried an Opinion column I wrote entitled "Evaluation of Our Courses". My main point is that we math profs *do not* get fairly evaluated. Something gets done, as we know; it is only what is easy to carry out. By doing only that, we tacitly assert our acceptance of the students' point of view on education in college.

The survey outcomes are used in some places as professional evaluation, and that is a serious problem. This is independent of the sporadic statements that students make in the written commentary. (I agree with what Martin Scharlemann wrote in his Letter to the Editor in the November 2010 issue of the *Notices*, that the more sober comments can be helpful.)

I would like to draw the reader's attention to the study that can be found at the URL: `http://www.journals.uchicago.edu/doi/pdf/10.1086/653808`.

It is a carefully controlled study that demonstrates the negative correlation between good survey scores for the instructor in one semester and good learning by the students in the next.

As for myself, I get so-so survey scores, but it has been long reported that my students are better prepared. I elect to put a lot of the burden of learning on them, and I find that appropriate at JHU. Maybe I should be less demanding? Once I said to my class, probably near the midpoint of the semester, "I'll run the lectures any way you want." Many hands went up. "But it won't change the scope of the course nor the nature of the exams." All hands went down.

Michael Fried, in his Opinion column in this issue of the *Notices*, reports on his experience in the opposite direction. When he started paying attention to improving his students' learning, his survey ratings suffered greatly.

—*Steven Zucker*
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`sz@math.jhu.edu`

(Received November 3, 2010)

Correction

The November 2010 issue of the *Notices* carried a book review, by Michèle Audin, of *Mathematicians fleeing from Nazi Germany* by Reinhard Siegmund-Schultze. In the reference list in the review, the authors of reference [2] are given as J-P Kahane, K. Krickeberg, and L. Lee. The third name should be instead L. Lorch. The *Notices* thanks Lee Lorch for pointing out this error.

—*Allyn Jackson*

Submitting Letters to the Editor

The *Notices* invites readers to submit letters and opinion pieces on topics related to mathematics. Electronic submissions are preferred (`notices-letters@ams.org`); see the masthead for postal mail addresses. Opinion pieces are usually one printed page in length (about 800 words). Letters are normally less than one page long, and shorter letters are preferred.

The Great Trinomial Hunt

Richard P. Brent and Paul Zimmermann

A *trinomial* is a polynomial in one variable with three nonzero terms, for example $P = 6x^7 + 3x^3 - 5$. If the coefficients of a polynomial P (in this case $6, 3, -5$) are in some ring or field F, we say that P is a polynomial over F, and write $P \in F[x]$. The operations of addition and multiplication of polynomials in $F[x]$ are defined in the usual way, with the operations on coefficients performed in F.

Classically the most common cases are $F = \mathbf{Z}, \mathbf{Q}, \mathbf{R}$, or $\mathbf{C}$, respectively the integers, rationals, reals, or complex numbers. However, polynomials over finite fields are also important in applications. We restrict our attention to polynomials over the simplest finite field: the field GF(2) of two elements, usually written as 0 and 1. The field operations of addition and multiplication are defined as for integers modulo 2, so $0+1 = 1$, $1+1 = 0$, $0 \times 1 = 0$, $1 \times 1 = 1$, etc.

An important consequence of the definitions is that, for polynomials $P, Q \in \mathrm{GF}(2)[x]$, we have

$$(P + Q)^2 = P^2 + Q^2$$

because the "cross term" $2PQ$ vanishes. High school algebra would have been much easier if we had used polynomials over GF(2) instead of over $\mathbf{R}$!

Trinomials over GF(2) are important in cryptography and random number generation. To illustrate why this might be true, consider a sequence $(z_0, z_1, z_2, \ldots)$ satisfying the recurrence

$$z_n = z_{n-s} + z_{n-r} \bmod 2, \tag{1}$$

where r and s are given positive integers, $r > s > 0$, and the initial values $z_0, z_1, \ldots, z_{r-1}$ are also given. The recurrence then defines all the remaining terms $z_r, z_{r+1}, \ldots$ in the sequence.

It is easy to build hardware to implement the recurrence (1). All we need is a shift register capable of storing r bits and a circuit capable of computing the addition mod 2 (equivalently, the "exclusive or") of two bits separated by $r - s$ positions in the shift register and feeding the output back into the shift register. This is illustrated in Figure 1 for $r = 7$, $s = 3$.

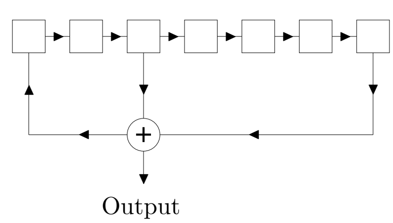

Figure 1. Hardware implementation of $z_n = z_{n-3} + z_{n-7}$ mod 2.

The recurrence (1) looks similar to the well-known Fibonacci recurrence

$$F_n = F_{n-1} + F_{n-2};$$

indeed, the Fibonacci numbers mod 2 satisfy our recurrence with $r = 2$, $s = 1$. This gives a sequence $(0, 1, 1, 0, 1, 1, \ldots)$ with period 3: not very interesting. However, if we take r larger, we can get much longer periods.

Richard Brent is professor of mathematics at the Mathematical Sciences Institute of the Australian National University, Canberra. His email address is `AMS@rpbrent.com`.

Paul Zimmermann is directeur de recherche at INRIA Nancy. His email address is `Paul.Zimmermann@loria.fr`.

The period can be as large as $2^r - 1$, which makes such sequences interesting as components in pseudo-random number generators or stream ciphers. In fact, the period is $2^r - 1$ if the initial values are not all zero and the associated trinomial

$$x^r + x^s + 1,$$

regarded as a polynomial over GF(2), is *primitive*. A primitive polynomial is one that is irreducible (it has no nontrivial factors) and satisfies an additional condition given in the "Mathematical Foundations" section below.

A *Mersenne prime* is a prime of the form $2^r - 1$. Such primes are named after Marin Mersenne (1588–1648), who corresponded with many of the scholars of his day, and in 1644 gave a list (not quite correct) of the Mersenne primes with $r \le 257$.

A *Mersenne exponent* is the exponent r of a Mersenne prime $2^r - 1$. A Mersenne exponent is necessarily prime, but not conversely. For example, 11 is not a Mersenne exponent because $2^{11} - 1 = 23 \cdot 89$ is not prime.

The topic of this article is a search for primitive trinomials of large degree r, and its interplay with a search for large Mersenne primes. First, we need to explain the connection between these two topics and briefly describe the GIMPS project. Then we describe the algorithms used in our search, which can be split into two distinct periods, "classical" and "modern". Finally, we describe the results obtained in the modern period.

Mathematical Foundations

As stated above, we consider polynomials over the finite field GF(2). An *irreducible polynomial* is a polynomial that is not divisible by any nontrivial polynomial other than itself. For example, $x^5 + x^2 + 1$ is irreducible, but $x^5 + x + 1$ is not, since $x^5 + x + 1 = (x^2 + x + 1)(x^3 + x^2 + 1)$ in $\mathrm{GF}(2)[x]$. We do not consider binomials $x^r + 1$, because they are divisible by $x + 1$, and thus reducible for $r > 1$.

An irreducible polynomial P of degree $r > 1$ yields a representation of the finite field $\mathrm{GF}(2^r)$ of 2^r elements: any polynomial of degree less than r represents an element, the addition is polynomial addition, whose result still has degree less than r, and the multiplication is defined modulo P: one first multiplies both inputs and then reduces their product modulo P. Thus $\mathrm{GF}(2^r) \simeq \mathrm{GF}(2)[x]/P(x)$.

An irreducible polynomial P of degree $r > 0$ over GF(2) is said to be *primitive* iff $P(x) \neq x$ and the residue classes $x^k \bmod P$, $0 \le k < 2^r - 1$, are distinct. In order to check primitivity of an irreducible polynomial P, it is only necessary to check that $x^k \neq 1 \bmod P$ for those k that are maximal nontrivial divisors of $2^r - 1$. For example, $x^5 + x^2 + 1$ is primitive; $x^6 + x^3 + 1$ is irreducible but not primitive, since $x^9 = 1 \bmod (x^6 + x^3 + 1)$. Here 9 divides $2^6 - 1 = 63$ and is a maximal divisor as $63/9 = 7$ is prime.

We are interested in primitive polynomials because x is a generator of the multiplicative group of the finite field $\mathrm{GF}(2)[x]/P(x)$ if $P(x)$ is primitive.

If r is large and $2^r - 1$ is not prime, it can be difficult to test primitivity of a polynomial of degree r, because we need to know the prime factors of $2^r - 1$. Thanks to the Cunningham project [20], these are known for all $r < 929$, but not in general for larger r. On the other hand, if $2^r - 1$ is prime, then all irreducible polynomials of degree r are primitive. This is the reason that we consider degrees r that are Mersenne exponents.

Starting the Search

In the year 2000 the authors were communicating by email with each other and with Samuli Larvala when the topic of efficient algorithms for testing irreducibility or primitivity of trinomials over GF(2) arose. The first author had been interested in this topic for many years because of the application to pseudo-random number generators. Publication of a paper by Kumada et al. [12], describing a search for primitive trinomials of degree 859 433 (a Mersenne exponent), prompted the three of us to embark on a search for primitive trinomials of degree r, for r ranging over all known Mersenne exponents. At that time, the largest known Mersenne exponents were 3 021 377 and 6 972 593. The existing programs took time proportional to r^3. Since $(6972593/859433)^3 \approx 534$, and the computation by Kumada et al. had taken three months on nineteen processors, it was quite a challenge.

The GIMPS Project

GIMPS stands for *Great Internet Mersenne Prime Search*. It is a distributed computing project started by George Woltman, with home page `http://www.mersenne.org`. The goal of GIMPS is to find new Mersenne primes. As of December 2010, GIMPS has found thirteen new Mersenne primes in fourteen years and has held the record for the largest known prime since the discovery of M_{35} in 1996. Mersenne primes are usually numbered in increasing order of size: $M_1 = 2^2 - 1 = 3$, $M_2 = 2^3 - 1 = 7$, $M_3 = 2^5 - 1 = 31$, $M_4 = 2^7 - 1 = 127$, …, $M_{38} = 2^{6972593} - 1$, etc.

Since GIMPS does not always find Mersenne primes in order, there can be some uncertainty in numbering the largest known Mersenne primes. We write M'_n for the nth Mersenne prime in order of discovery. There are gaps in the search above $M_{39} = 2^{13466917} - 1$. Thus we can have $M'_n > M'_{n+1}$ for $n > 39$. For example, $M'_{45} = 2^{43112609} - 1$ was found before $M'_{46} = 2^{37156667} - 1$ and $M'_{47} = 2^{42643801} - 1$. At the time of writing this article, forty-seven Mersenne primes are known, and the largest is $M'_{45} = 2^{43112609} - 1$.

It is convenient to write r_n for the exponent of M_n, and r'_n for the exponent of M'_n. For example, $r'_{45} = 43\,112\,609$.

Swan's Theorem

We state a useful theorem known as Swan's theorem, although the result was found much earlier by Pellet [14] and Stickelberger [18]. In fact, there are several theorems in Swan's paper [19]. We state a simplified version of Swan's Corollary 5.

Theorem 1. *Let $r > s > 0$, and assume $r + s$ is odd. Then $T_{r,s}(x) = x^r + x^s + 1$ has an even number of irreducible factors over* GF(2) *in the following cases:*

a) r even, $r \neq 2s$, $rs/2 = 0$ or 1 mod 4.

b) r odd, s not a divisor of $2r$, $r = \pm 3$ mod 8.

c) r odd, s a divisor of $2r$, $r = \pm 1$ mod 8.

In all other cases $x^r + x^s + 1$ has an odd number of irreducible factors.

If both r and s are even, then $T_{r,s}$ is a square and has an even number of irreducible factors. If both r and s are odd, we can apply the theorem to the "reciprocal polynomial" $T_{r,r-s}(x) = x^r T(1/x) = x^r + x^{r-s} + 1$, since $T_{r,s}(x)$ and $T_{r,r-s}(x)$ have the same number of irreducible factors.

For r an odd prime and excluding the easily checked cases $s = 2$ or $r - 2$, case (b) says that the trinomial has an even number of irreducible factors, and hence must be reducible, if $r = \pm 3 \bmod 8$. Thus we only need to consider those Mersenne exponents with $r = \pm 1 \bmod 8$. Of the fourteen known Mersenne exponents $r > 10^6$, only eight satisfy this condition.

Cost of the Basic Operations

The basic operations that we need are squarings modulo the trinomial $T = x^r + x^s + 1$, multiplications modulo T, and greatest common divisors (GCDs) between T and a polynomial of degree less than r. We measure the cost of these operations in terms of the number of bit or word operations required to implement them. In $\mathrm{GF}(2)[x]$, squarings cost $O(r)$, due to the fact that the square of $x^i + x^j$ is $x^{2i} + x^{2j}$. The reduction modulo T of a polynomial of degree less than $2r$ costs $O(r)$, due to the sparsity of T; thus modular squarings cost $O(r)$.

Modular multiplications cost $O(M(r))$, where $M(r)$ is the cost of multiplication of two polynomials of degree less than r over GF(2); the reduction modulo T costs $O(r)$, so the multiplication cost dominates the reduction cost. The "classical" polynomial multiplication algorithm has $M(r) = O(r^2)$, but an algorithm[1] due to Schönhage has $M(r) = O(r \log r \log\log r)$ [16].

[1]*This algorithm differs from the Schönhage-Strassen integer-multiplication algorithm, which does not work over* GF(2). *For details see* [2, 16].

A GCD computation for polynomials of degree bounded by r costs $O(M(r)\log r)$ using a "divide and conquer" approach combined with Schönhage's fast polynomial multiplication. The costs are summarized in Table 1.

Table 1: Cost of the basic operations.

modular squaring	$O(r)$
modular product	$O(M(r))$
GCD	$O(M(r)\log r)$

Testing Irreducibility

Let $\mathbf{P}_r(x) = x^{2^r} - x$. As was known to Gauss, $\mathbf{P}_r(x)$ is the product of all irreducible polynomials of degree d, where d runs over the divisors of r. For example,

$$\mathbf{P}_3(x) = x(x+1)(x^3+x+1)(x^3+x^2+1)$$

in $\mathrm{GF}(2)[x]$. Here x and $x+1$ are the irreducible polynomials of degree 1, and the other factors are the irreducible polynomials of degree 3. Note that we can always write "+" instead of "−" when working over GF(2), since $1 = -1$ (or, equivalently, $1 + 1 = 0$).

In particular, if r is an odd prime, then a polynomial $P(x) \in \mathrm{GF}(2)[x]$ with degree r is irreducible iff

$$x^{2^r} = x \bmod P(x)\,. \tag{2}$$

(If r is not prime, then (2) is necessary but not sufficient: we have to check a further condition to guarantee irreducibility; see [8].)

When r is prime, equation (2) gives a simple test for irreducibility (or primitivity, in the case that r is a Mersenne exponent): just perform r modular squarings, starting from x, and check whether the result is x. Since the cost of each squaring is $O(r)$, the cost of the irreducibility test is $O(r^2)$.

There are more sophisticated algorithms for testing irreducibility based on modular composition [11] and fast matrix multiplication [3]. However, these algorithms are actually slower than the classical algorithm when applied to trinomials of degree less than about 10^7.

When searching for irreducible trinomials of degree r, we can assume that $s \le r/2$, since $x^r + x^s + 1$ is irreducible iff the reciprocal polynomial $x^r + x^{r-s} + 1$ is irreducible. This simple observation saves a factor of 2. In the following, we always assume that $s \le r/2$.

Degrees of Factors

In order to predict the expected behavior of our algorithm, we need to know the expected distribution of degrees of irreducible factors. Our complexity estimates are based on the assumption that trinomials of degree r behave like the set of all polynomials of the same degree, up to a constant factor:

Assumption 1. *Over all trinomials $x^r + x^s + 1$ of degree r over* $\mathrm{GF}(2)$*, the probability π_d that a trinomial has no nontrivial factor of degree $\le d$ is at most c/d, where c is an absolute constant and $1 < d \le r/\ln r$.*

This assumption is plausible and in agreement with experiments, though not proven. It is not critical, because the correctness of our algorithms does not depend on the assumption—only the predicted running time depends on it. The upper bound $r/\ln r$ on d is large enough for our application to predict the running time. An upper bound of r on d would probably be incorrect, since it would imply at most c irreducible trinomials of degree r, but we expect this number to be unbounded.

Some evidence for the assumption, in the case $r = r_{38}$, is presented in Table 2. The maximum value of $d\pi_d$ is 2.08, occurring at $d = 226\,887$. It would be interesting to try to explain the exact values of $d\pi_d$ for small d, but this would lead us too far afield.

Table 2: Statistics for $r = r_{38}$

d	$d\pi_d$
1	1.00
2	1.33
3	1.43
4	1.52
5	1.54
6	1.60
7	1.60
8	1.67
9	1.64
10	1.65
100	1.77
1000	1.76
10000	1.88
226887	2.08

Sieving

When testing a large integer N for primality, it is sensible to check whether it has any small factors before applying a primality test such as the AKS, ECPP, or (if we are willing to accept a small probability of error) Rabin-Miller test. Similarly, when testing a high-degree polynomial for irreducibility, it is wise to check whether it has any small factors before applying the $O(r^2)$ test.

Since the irreducible polynomials of degree d divide $\mathbf{P}_d(x)$, we can check whether a trinomial T has a factor of degree d (or some divisor of d) by computing

$$\gcd(T, \mathbf{P}_d).$$

If $T = x^r + x^s + 1$ and $2^d < r$, we can reduce this to the computation of a GCD of polynomials of degree less than 2^d. Let $d' = 2^d - 1$, $r' = r \bmod d'$, $s' = s \bmod d'$. Then $\mathbf{P}_d = x(x^{d'} - 1)$,

$$T = x^{r'} + x^{s'} + 1 \bmod (x^{d'} - 1),$$

so we only need to compute

$$\gcd(x^{r'} + x^{s'} + 1, x^{d'} - 1).$$

We call this process "sieving" by analogy with the process of sieving out small prime factors of integers, even though it is performed using GCD computations.

If the trinomials that have factors of degree less than $\log_2(r)$ are excluded by sieving, then by Assumption 1 we are left with $O(r/\log r)$ trinomials to test. The cost of sieving is negligible. Thus the overall search has cost $O(r^3/\log r)$.

The Importance of Certificates

Primitive trinomials of degree $r < r_{32} = 756\,839$ are listed in Heringa et al. [10]. Kumada et al. [12] reported a search for primitive trinomials of degree $r_{33} = 859\,433$ (they did not consider r_{32}). They found one primitive trinomial; however, they missed the trinomial $x^{859433} + x^{170340} + 1$, because of a bug in their sieving routine. We discovered the missing trinomial in June 2000 while testing our program on the known cases.

This motivated us to produce *certificates* of reducibility for all the trinomials that we tested (excluding, of course, the small number that turned out to be irreducible). A certificate of reducibility is, ideally, a nontrivial factor. If a trinomial T is found by sieving to have a small factor, then it is easy to keep a record of this factor. If we do not know a factor, but the trinomial fails the irreducibility test (2), then we can record the residue $R(x) = x^{2^r} - x \bmod T$. Because the residue can be large, we might choose to record only part of it, e.g., $R(x) \bmod x^{32}$.

The Classical Period

The period 2000–2003 could be called the *classical* period. We used efficient implementations of the classical algorithms outlined above. Since different trinomials could be tested on different computers, it was easy to conduct a search in parallel, using as many processors as were available. For example, we often made use of PCs in an undergraduate teaching laboratory during the vacation, when the students were away.

In this way, we found three primitive trinomials of degree $r_{32} = 756\,839$ (in June 2000), two of degree $r_{37} = 3\,021\,377$ (August and December 2000), and one of degree $r_{38} = 6\,972\,593$ (in

August 2002).[2] The computation for degree r_{38} was completed and double-checked by July 2003.

For degree $r_{38} = 6\,972\,593$, there turned out to be only one primitive trinomial $x^r + x^s + 1$ (assuming, as usual, that $s \le r/2$).[3] How can we be sure that we did not miss any? For each nonprimitive trinomial we had a certificate, and these certificates were checked in an independent computation. In fact, we found a small number of discrepancies, possibly due to memory parity errors in some of the older PCs that were used. This is a risk in any long computation—we should not assume that computers are infallible. The same phenomenon was observed by Nicely [13] in his computation of Brun's constant (which also uncovered the infamous "Pentium bug").

Since we had caught up with the GIMPS project, we thought (not for the last time) that this game had finished and published our results in [4, 5]. However, GIMPS soon overtook us by finding several larger Mersenne primes with exponents ± 1 mod 8: $r'_{41} = 24\,036\,583, \ldots, r'_{44} = 32\,582\,657$.

The search for degree $r_{38} = 6\,972\,593$ had taken more than two years (February 2001 to July 2003), so it did not seem feasible to tackle the new Mersenne exponents $r'_{41}, \ldots, r'_{44}$.

The Modern Period

We realized that, in order to extend the computation, we had to find more efficient algorithms. The expensive part of the computation was testing irreducibility using equation (2). If we could sieve much further, we could avoid most of the irreducibility tests. From Assumption 1, if we could sieve to degree $r/\ln r$, then we would expect only $O(\log r)$ irreducibility tests.

What we needed was an algorithm that would find the smallest factor of a sparse polynomial (specifically, a trinomial) in a time that was fast *on average.*

There are many algorithms for factoring polynomials over finite fields; see, for example, [8]. The cost of most of them is dominated by GCD computations. However, it is possible to replace most GCD computations by modular multiplications, using a process called *blocking* (introduced by Pollard [15] in the context of integer factorization and by von zur Gathen and Shoup [9] for polynomial factorization). The idea is simple: instead of computing $\gcd(T, P_1), \ldots, \gcd(T, P_k)$ in the hope of finding a nontrivial GCD (and hence a factor of T), we compute $\gcd(T, P_1 P_2 \cdots P_k \bmod T)$ and backtrack if necessary to split factors if they are not irreducible. Since a GCD typically takes about 40 times as long as a modular multiplication for $r \approx r'_{41}$, blocking can give a large speedup.

During a visit by the second author to the first author in February 2007, we realized that a second level of blocking could be used to replace most modular multiplications by squarings. Since a modular multiplication might take 400 times as long as a squaring (for $r \approx r'_{41}$), this second level of blocking can provide another large speedup. The details are described in [6]. Here we merely note that m multiplications and m squarings can be replaced by one multiplication and m^2 squarings. The optimal value of m is $m_0 \approx \sqrt{M(r)/S(r)}$, where $M(r)$ is the cost of a modular multiplication and $S(r)$ is the cost of a modular squaring, and the resulting speedup is about $m_0/2$. If $M(r)/S(r) = 400$, then $m_0 \approx 20$ and the speedup over single-level blocking is roughly a factor of ten.

Using these ideas, combined with a fast implementation of polynomial multiplication (for details, see [2]) and a subquadratic GCD algorithm, we were able to find ten primitive trinomials of degrees $r'_{41}, \ldots, r'_{44}$ by January 2008. Once again, we thought we were finished and published our results [7], only to have GIMPS leap ahead again by discovering M'_{45} in August 2008 and M'_{46} and M'_{47} shortly afterward. The exponent r'_{46} was ruled out by Swan's theorem, but we had to set to work on degrees $r'_{45} = 43\,112\,609$ and (later) the slightly smaller $r'_{47} = 42\,643\,801$.

The search for degree r'_{45} ran from September 2008 to May 2009, with assistance from Dan Bernstein and Tanja Lange, who kindly allowed us to use their computing resources in Eindhoven, and resulted in four primitive trinomials of record degree.

The search for degree r'_{47} ran from June 2009 to August 2009 and found five primitive trinomials. In this case we were lucky to have access to a new computing cluster with 224 processors at the Australian National University, so the computation took less time than the earlier searches.

The results of our computations in the "modern period" are given in Table 3. There does not seem to be any predictable pattern in the s values. The number of primitive trinomials for a given Mersenne exponent $r = \pm 1$ mod 8 appears to follow a Poisson distribution with mean about 3.2 (and hence it is unlikely to be bounded by an absolute constant—see the discussion of Assumption 1 above).

The Modern Algorithm—Some Details

To summarize the "modern" algorithm for finding primitive trinomials, we improve on the classical algorithm by sieving much further to find a factor of smallest degree, using a factoring algorithm

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[2] *Primitive trinomials of degree r_{34}, r_{35}, and r_{36} were ruled out by Swan's theorem, as were r_{39} and r'_{40}.*

[3] *The unique primitive trinomial of degree* 6 972 593 *is* $x^{6972593} + x^{3037958} + 1$. *It was named* Bibury *after the village that the three authors of* [5] *were visiting on the day that it was discovered.*

Table 3: Primitive trinomials $x^r + x^s + 1$ whose degree r is a Mersenne exponent, for $s \le r/2$.

r	s
24 036 583	8 412 642, 8 785 528
25 964 951	880 890, 4 627 670, 4 830 131, 6 383 880
30 402 457	2 162 059
32 582 657	5 110 722, 5 552 421, 7 545 455
42 643 801	55 981, 3 706 066, 3 896 488, 12 899 278, 20 150 445
43 112 609	3 569 337, 4 463 337, 17 212 521, 21 078 848

based on fast multiplication and two levels of blocking. In the following paragraphs we give some details of the modern algorithm and compare it with the classical algorithms.

Given a trinomial $T = x^r + x^s + 1$, we search for a factor of smallest degree $d \le r/2$. (In fact, using Swan's theorem, we can usually restrict the search to $d \le r/3$, because we know that the trinomial has an odd number of irreducible factors.) If such a factor is found, we know that T is reducible, so the program outputs "reducible" and saves the factor for a certificate of reducibility. The factor can be found by taking the GCD of T and $x^{2^d} + x$; if this GCD is nontrivial, then T has at least one factor of degree dividing d. If factors of degree smaller than d have already been ruled out, then the GCD only contains factors of degree d (possibly a product of several such factors). This is known as *distinct degree factorization* (DDF).

If the GCD has degree λd for $\lambda > 1$, and one wants to split the product into λ factors of degree d, then an *equal degree factorization* algorithm (EDF) is used. If the EDF is necessary it is usually cheap, since the total degree λd is usually small if $\lambda > 1$.

In this way we produce certificates of reducibility that consist just of a nontrivial factor of smallest possible degree, and the lexicographically least such factor if there are several.[4] The certificates can be checked, for example with an independent program using NTL [17], much faster than the original computation (typically in less than one hour for any of the degrees listed in Table 3).

For large d, when $2^d \gg r$, we do not compute $x^{2^d} + x$ itself, but its remainder, say h, modulo T. Indeed, $\gcd(T, x^{2^d} + x) = \gcd(T, h)$. To compute h, we start from x, perform d modular squarings, and add x. In this way, we work with polynomials of degree less than $2r$. Checking for factors of degree d costs d modular squarings and one GCD. Since we check potential degrees d in ascending order, $x^{2^d} \bmod T$ is computed from $x^{2^{d-1}} \bmod T$, which was obtained at the previous step, with one extra modular squaring. Thus, from Table 1, the cost per value of d is $O(M(r) \log r)$. However, this does not take into account the speedup due to blocking, discussed above.

The critical fact is that most trinomials have a small factor, so the search runs fast on average.

After searching unsuccessfully for factors of degree $d < 10^6$, say, we could switch to the classical irreducibility test (2), which is faster than factoring if the factor has degree greater than about 10^6. However, in that case our list of certificates would be incomplete. Since it is rare to find a factor of degree greater than 10^6, we let the program run until it finds a factor or outputs "irreducible". In the latter case, of course, we can verify the result using the classical test. Of the certificates (smallest irreducible factors) found during our searches, the largest is a factor $P(x) = x^{10199457} + x^{10199450} + \cdots + x^4 + x + 1$ of the trinomial $x^{42643801} + x^{3562191} + 1$. Note that, although the trinomial is sparse and has a compact representation, the factor is dense and hence too large to present here in full.

Classical versus Modern

For simplicity we use the $\widetilde{O}$ notation that ignores log factors. The "classical" algorithm takes an expected time $\widetilde{O}(r^2)$ per trinomial, or $\widetilde{O}(r^3)$ to cover all trinomials of degree r.

The "modern" algorithm takes expected time $\widetilde{O}(r)$ per trinomial, or $\widetilde{O}(r^2)$ to cover all trinomials of degree r.

In practice, the modern algorithm is faster by a factor of about 160 for $r = r_{38} = 6\,972\,593$, and by a factor of about 1000 for $r = r'_{45} = 43\,112\,609$.

Thus, comparing the computation for $r = r'_{45}$ with that for $r = r_{38}$: using the classical algorithm would take about 240 times longer (impractical), but using the modern algorithm saves a factor of 1000.

How to Speed Up the Search

The key ideas are summarized here. Points (1)–(4) apply to both the classical and modern algorithms; points (5)–(6) apply only to the modern algorithm.

(1) Since the computations for each trinomial can be performed independently, it is easy to conduct a search in parallel, using as many computers as are available.

(2) Because the coefficients of polynomials over GF(2) are just 0 or 1, there is a one-one correspondence between polynomials of degree $< d$ and binary numbers with d bits. Thus, on a 64-bit computer, we can encode a polynomial of degree d in $\lceil (d+1)/64 \rceil$ computer words. If we take care writing the programs, we can operate on such polynomials using full-word computer

[4] *It is worth going to the trouble to find the lexicographically least factor, since this makes the certificate unique and allows us to compare different versions of the program and locate bugs more easily than would otherwise be the case.*

operations, thus doing 64 operations in parallel.

(3) Squaring of polynomials over GF(2) can be done in *linear time* (linear in the degree of the polynomial), because the cross terms in the square vanish:

$$\left(\sum_k a_k x^k\right)^2 = \sum_k a_k x^{2k}.$$

(4) Reduction of a polynomial of degree $2(r-1)$ modulo a trinomial $T = x^r + x^s + 1$ of degree r can also be done in linear time. Simply use the identity $x^n = x^{n+s-r} + x^{n-r} \bmod T$ for $n = 2r-2, 2r-3, \ldots, r$ to replace the terms of degree $\geq r$ by lower-degree terms.

(5) Most GCD computations involving polynomials can be replaced by multiplication of polynomials, using a technique known as "blocking" (described above).

(6) Most multiplications of polynomials can be replaced by squarings, using another level of blocking, as described in [6].

Conclusion

The combination of these six ideas makes it feasible to find primitive trinomials of very large degree. In fact, the current record degree is the same as the largest known Mersenne exponent, $r = r'_{45} = 43\,112\,609$. We are ready to find more primitive trinomials as soon as GIMPS finds another Mersenne prime that is not ruled out by Swan's theorem. Our task is easier than that of GIMPS, because finding a primitive trinomial of degree r, and verifying that a single value of r is a Mersenne exponent, both cost about the same: $\widetilde{O}(r^2)$.

The trinomial hunt has resulted in improved software for operations on polynomials over GF(2), and has shown that the best algorithms in theory are not always the best in practice. It has also provided a large database of factors of trinomials over GF(2), leading to several interesting conjectures that are a topic for future research.

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Federigo Enriques's Quest to Prove the "Completeness Theorem"

Donald Babbitt and Judith Goodstein

The golden age of the Italian school of algebraic geometry began with Antonio Luigi Gaudenzio Giuseppe Cremona and included among its main contributors Enrico Castelnuovo, Federigo Enriques, and Francesco Severi. The Italian school spanned nearly a century, from the unification of Italy in 1861 to Enriques's posthumously published post-World War II monograph on algebraic surfaces [Enrq 49]. In the 1890s Enriques, a mathematician who once quipped that "intuition is the aristocratic way of discovery, rigour the plebian way" [Hodge 48], and his colleague and future brother-in-law Castelnuovo began their monumental work on the birational theory of algebraic surfaces over the complex numbers **C**.[1] Severi joined them in this effort a few years later.

Broadly speaking, the aspect of algebraic surfaces that will concern us here can be traced back to Rudolf Friedrich Alfred Clebsch, Arthur Cayley, and Max Noether, who in 1868–1875 introduced two different genera for characterizing an algebraic surface F. The first (proposed by Clebsch) was the dimension of the space of algebraic regular (i.e., without poles or, over **C**, holomorphic) 2-forms on F, a direct analog of *the genus of an algebraic curve* C, i.e., the dimension of the space of algebraic regular 1-forms on C. Shortly after, Arthur Cayley, seeking an easier way to calculate Clebsch's genus, introduced an expression in the degree and characteristics of the singularities of a generic projection of the surface to $\mathbf{P}^3$ which he hoped gave the same number. But, in fact, it turned out that the second could be negative, unlike the first. The first number is now called the *geometric genus* and is denoted by p_g, and the second one is called the *arithmetic genus* and is denoted by p_a. The difference $p_g - p_a$ is called the *irregularity* of F and is always ≥ 0. If $p_g = p_a$ we say the surface is *regular*. Otherwise it is said to be *irregular*. In modern language: $p_a := (a_F - 1)$ where a_F is the Euler characteristic of the structure sheaf or the constant term of the Hilbert polynomial of F [Dieu 74].[2]

In 1885 Émile Picard initiated the study of "Picard integrals of the first kind" on an algebraic surface F, i.e., integrals whose integrands are closed algebraic regular 1-forms on an algebraic surface F. Subsequently he showed that there are no Picard integrals of the first kind on smooth surfaces $F \subset \mathbf{P}^3(\mathbf{C})$ which were known to be regular. In 1901, Enriques showed that when there are such integrals, the irregularity of F is > 0.

These results indicate that there is an interesting relation between the irregularity of F and the space of Picard integrals of the first kind on F.

The work of these mathematicians paved the way for the Fundamental Theorem of Irregular Surfaces—one of the triumphs of early-twentieth-century algebraic geometry. The theorem states that:

$$p_g - p_a = \dim P(F) = b_1/2$$

where $\dim P(F)$ is the dimension of the linear space of Picard integrals of the first kind on F and b_1 is the first Betti number of F.[3] The Fundamental Theorem and its proof came about, in turn, through the combined efforts of Picard, Enriques, Castelnuovo, Severi, and Henri Poincaré. In the modern language going back to Hodge, this is the assertion that $h^{0,1} = h^{1,0} = b_1/2$.

In his landmark paper, "Sulla proprietà caratteristica delle superficie algebriche irregolari",

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[1] *See Appendix A for the definitions of some of the key terms of birational algebraic geometry used in this paper.*

[2] *See [Mum 76], Ch. 6.C, for details on the Hilbert polynomial of F. Additional discussion of the arithmetic genus and its history can be found in [Kl 05], p. 243.*

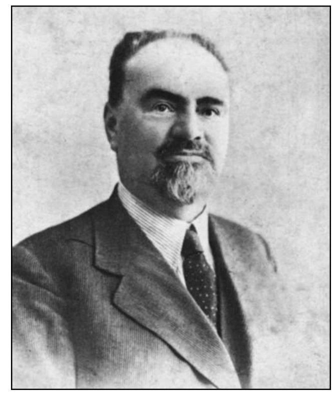
[3] *More of this story can be found in [Kl 05].*

published in 1905 in the *Rendiconti dell'Accademia delle Scienze dell'Istituto di Bologna*, Enriques provided a key piece of the proof of the Fundamental Theorem [Enrq 05]. Generally referred to as the Completeness Theorem, Enriques's result and proof were of an algebro-geometric nature—i.e., not involving transcendental techniques such as complex analysis and topology. Francesco Severi offered his own algebro-geometric proof in 1905 also [Sev 05]. Five years later, in 1910, Poincaré gave an independent transcendental proof of both the Fundamental Theorem and the Completeness Theorem [Poin 10]. Although both Enriques's and Severi's proofs turned out to be fatally flawed, as Severi himself showed in 1921 [Sev 21],[4] neither mathematician gave up trying to find a satisfactory algebro- geometric proof of the Completeness Theorem. From 1921 on, Enriques and Severi carried on an open feud, which spilled over into other areas of their professional lives. By the time the English geometer Leonard Roth arrived in Rome in 1930 to study with them and Castelnuovo, relations between Enriques and Severi had completely soured. "Either he [Severi] had just taken offence or else he was in the process of giving it," Roth recalled [Roth63]. Perhaps testifying to their pugnacity, both ultimately joined the Fascist Party, Severi in 1932, Enriques in 1933.

The University of Rome geometer Fabio Conforto, who served as Castelnuovo's assistant in the early 1930s, later organized and edited Enriques's lectures on rational surfaces for publication, an exercise that gave him some insight into Enriques's character. At a gathering of mathematicians to commemorate Enriques, held in Rome in February of 1947, he noted: "The problem of conferring integrity upon the theorem of the characteristic series of a continuous complete system tormented Enriques for his entire life" [Conf 47]. Unfortunately, no mathematician would find such a proof during Enriques's lifetime. For reasons possibly aesthetic, the Italians chafed at Poincaré's transcendental proof.[5] In 1945, nine months before his death, Enriques sent a plaintive letter to the Italian algebraic geometer Beniamino Segre, in which he lamented the lack of a proof and suggested that his idea of infinitely close curves of higher order—an idea he had advanced in earlier lectures and elaborated on in several papers between 1936 and 1938—might lead to one. Enriques was right (!), although he did not have the mathematical tools to convert his idea into an acceptable algebro-geometric proof. It is the story of Enriques's relentless quest that we would like to tell here.

Who Was Federigo Enriques?

Expelled by Spain in 1492, many Sephardic Jews found a safe haven in Tuscany, including the ancestors of Abramo Giulio Umberto Federigo Enriques, who was born in Livorno on January 5, 1871, and died in Rome on June 14, 1946. One of three children of Giacomo Enriques and Matilde Coriat, he moved to Pisa with his family when he was seven, displaying at an early age a taste for logic and numbers. Bored by a homework assignment set by his tutor that involved computing the squares of numbers from 1 to 30, Federigo, then age eleven, figured out that he could generate the squares by adding successive odd numbers: 1, $1 + 3$, $1 + 3 + 5$, and so on. Buoyed by his discovery, he went on to calculate the squares of numbers from 1 to 1,000, publishing his results in a small pamphlet, which cost him his entire savings (seven lira). When Enriques's daughter asked him, many years later, whether his parents had been pleased with his enterprise, he flashed a smile and replied, "They never knew about it." The eleven-year-old had shown even then a streak of independence he never lost [Enrq 47].



Federigo Enriques, 1930s. ***Photo courtesy of Lorenzo Enriques.***

Enriques's mother, a Tunisian by birth, lavished attention on the education of her children. The family was well off, thanks in part to her substantial dowry, which allowed her husband, a wealthy rug merchant, to stop working altogether after their marriage. Federigo and his siblings were raised and remained in academic circles: Paolo Enriques, his younger brother, became a biologist; Elbina, their older sister, married Guido Castelnuovo, who wrote and published two papers on the geometry of algebraic surfaces in 1891, the first two papers on the subject in Italy.

When he turned thirteen Federigo Enriques discovered geometry, although his mother assumed he would lose interest quickly. "You know how this boy is," she once wrote to her sister, "every day his head has a new idea that lasts for about the space of a morning" [Enrq 47]. Despite his mother's skepticism, at the Liceo di Pisa Enriques honed his taste not just for mathematics but also for a number of other subjects—logic, epistemology, pedagogy, and the history of science—displaying an intellectual curiosity that he later told a colleague could be traced back to "a philosophical infection" contracted in high school [ScoDra 53].

[4] *Severi [Sev 41] claimed to have discovered a gap in both proofs while reconstructing and simplifying Poincaré's proof of the Completeness Theorem.*

[5] *See the discussion at the end of Chapter 2.4 in [Brig-Cil 95] on this point.*

In the summer of 1887 Enriques took and passed the *esame di licenza liceale*, the final high school examination. That fall, he entered the University of Pisa and the Scuola Normale Superiore, the elite teacher-training school connected to the university, graduating with a doctor's degree in mathematics, with honors, in June 1891. His mathematics teachers there included Enrico Betti, Luigi Bianchi, Vito Volterra, and Riccardo De Paolis, professor of higher geometry, who also served as his thesis advisor.

Enriques's postgraduate training began with a year of studies and teaching at the Scuola Normale (1891–1892) and a second year (1892–1893) in Rome, where Guido Castelnuovo, that university's newly minted associate professor of analytic and projective geometry, took him under his wing. "Enriques was a mediocre reader," Castelnuovo later recalled, a defect that providentially led to their celebrated "peripatetic conversations"; the future direction of the Italian school of algebraic geometry evolved as the two young mathematicians crisscrossed the streets of the capital [Cast 47]. In June 1893, Enriques published his first memoir dealing with the theory of algebraic surfaces, *Ricerche di geometria sulle superficie algebriche* [Enrq 93], followed by a more comprehensive general theory three years later [Enrq 96]. In spring 1894, after a failed bid for a professorial appointment at the University of Turin, Enriques petitioned the University of Pisa for a *libera docenza*, which would allow him to teach projective geometry there. In its report granting the request, the committee of Pisan mathematicians hailed his 1893 memoir as an example of a keen mind and an expansive talent—although noting that "the precision and the rigor of the exposition leave something to be desired" [UPisa 94]. Shortly afterward, Enriques passed his *Abilitazione* exams with honors, another prerequisite for scaling Italy's academic ladder.

After the liberation of Rome, Guido Castelnuovo played a key role in the rebirth of the Lincei, whose president he became in 1945. ***Photograph courtesy of the Accademia Nazionale dei Lincei.***

In 1894 Enriques joined the Bologna faculty as a temporary instructor for the projective and descriptive geometry courses, thanks in part to Vito Volterra's intervention on his behalf. Two years later, he took first place in a national competition for the vacant chair at Bologna; that university remained his home until 1922, when he was called to Rome to teach complementary mathematics, a new course designed for high school mathematics teachers. In 1923 Enriques became professor of higher geometry at Rome, a position he maintained until the 1938 anti-Jewish racial laws deprived him—despite his membership in the Fascist Party, which was summarily rescinded—of his university chair. The Fascist Severi, who had been Enriques's assistant in Bologna, took it over; Enriques would not regain his chair until 1944, following the liberation of Rome. Castelnuovo, who was also Jewish, had retired from teaching at Rome in 1935, but, like his other Jewish university colleagues, was barred from using the department's mathematics library.

Unlike Castelnuovo, who had switched fields after World War I and lectured, wrote, and published about relativity theory and the theory of probability, Enriques continued to work actively in algebraic geometry until the end of his life, often collaborating with colleagues, students, and assistants in publishing a steady stream of papers (nine with Castelnuovo, four with Severi[6]), school textbooks, voluminous university-level treatises, lecture notes, and monographs on that subject alone.

However, Enriques's interests, as they had at the Liceo di Pisa, ranged far and wide—from the foundations of mathematics and physiological psychology to the philosophy and history of science and Einstein's theory of relativity. He founded journals (*Scientia* and *Mathesis*) and organizations (the Italian Philosophical Society and the National Institute for the History of Science); edited, annotated, and published in Italian Euclid's *Elements*; and contributed many articles to the *Enciclopedia Italiana*, a massive multivolume reference set.

As a teacher, Enriques loved nothing better than to engage in his own leisurely peripatetic conversations with students, in the public gardens in Bologna or under its arcades after class. When he moved to Rome, the labyrinthine network of paths in the Villa Borghese became his favorite destination; he would stop there every so often, one student at that time recalls, "to trace mysterious figures on the ground, with the tip of his inseparable walking stick" [Camp 47]. In remarks made shortly after Enriques's death, Fabio Conforto described his colleague and coauthor's "powerful intuitive spirit" and unalterable belief in "an algebraic world that exists in and of itself, independent and outside of us"—a world in which "seeing" was the most important implement in a mathematician's toolbox: Enriques "did not feel the need of a logical demonstration of some property, because he 'saw'; and that provided the assurance about the truth of the proposition in question and satisfied him completely." Conforto

[6] *In 1907, Enriques and Severi won the prestigious Bordin Prize of the French Academy of Sciences for their work on hyperelliptical surfaces.*

recalled that on one memorable occasion, when he had failed

> to see the truth of a statement that [Enriques] considered obvious but that we had tried in vain to demonstrate logically, he stopped suddenly (we were in the course of one of the habitual walks), and instead of trying one final demonstration he flourished his walking stick and, pointing to a puppy on a windowsill, said, "You don't see it? For me, it is as if I said to myself, 'I don't see that puppy!'" [Conf 47].

Their disagreement didn't stop Enriques and Conforto from inserting that particular unproved statement in their 1939 book, *Le superficie razionali*. Because the racial laws banned Italian Jews from publishing, Conforto's name is the only one on the title page.

In 1941 Castelnuovo organized a clandestine university for Jewish students in Rome, operated in conjunction with the Fribourg Institut Technique Supérieure, a private school in Switzerland. Enriques taught a course there in the history of mathematics. Impeccably dressed (one student recalled that he invariably wore gloves), soft-spoken, and regal in his carriage and gestures, Enriques apparently kept his students on the edge of their seats. The Nazi occupation of Rome in fall 1943 forced the school's closure, and both Castelnuovo and Enriques went into hiding.

Severi, on the other hand, was well regarded by the Fascist regime and had been elected to its Academy of Italy in 1929 (Enriques was in the running for the position until the last moment). He was also the founding president of the Italian National Institute for Higher Mathematics during World War II. Briefly suspended from his university post after the liberation of Rome in June 1944, Severi was subsequently absolved of any criminal activity, and he resumed teaching, retiring in 1950. Beniamino Segre then moved from Bologna to Rome to take Severi's chair of geometry.

Although the Fascist racial laws had forced Enriques to step down as the longtime editor of *Periodico di Matematiche* (an influential and well-thumbed magazine that published historical and didactic articles for secondary school teachers), banished him from the university, and denied him the right to publish under his own name in Italy, they did not prevent him from publishing in Italy under the pseudonym "Adriano Giovannini" (derived from the names of his two children, Adriana and Giovanni). In 1942 he published an article in *Periodico di Matematiche* on errors in mathematics and a piece in *Archivio della cultura italiana* on the ideas of Galileo. After he went into hiding, however, even this publishing ceased.

Federigo Enriques with family and friends, circa 1907. *Photo courtesy of Lorenzo Enriques.*

Publishing abroad proved equally problematic. In 1940, he had submitted an article to the Madrid Academy of Sciences, of which he was a member, but at the end of the war, because of disruptions in the mail service, he still didn't know whether or not it had appeared.

In 1942 Enriques finished writing up his lectures at Rome on the theory of algebraic surfaces and set the manuscript aside until the Germans surrendered to the Allies in northern Italy in 1945. That September, Enriques handed the eleven chapters of *Le Superficie Algebriche* to the typesetters, reserving the right to make any necessary additions or changes while the book was in press. However, nine months later he was dead from a heart attack, leaving *Le Superficie Algebriche's* fate up in the air. Giuseppe Pompilj and Alfredo Franchetta, Enriques's last students, and Castelnuovo came to the rescue, volunteering to read the galleys, figuring that it would take only a few months of work. It took much longer, because of "several obscure points", Castelnuovo recalled—particularly in Chapter 9, which covered continuous systems of algebraic curves on irregular surfaces, including an exhaustive discussion of the status of the Completeness Theorem, the author's famous 1905 proof. Indeed, more than a hint of the difficulty the trio faced can be gleaned from the following comment, inserted as a footnote in the chapter:

> The part that follows [that is, the status of Enriques's theorem] has been left incomplete by the author, and the arguments developed there present many gaps; above all, it seems opportune to bring them back, because they contain ideas that perhaps, suitably completed, could furnish the start of a systematic account of the theory.[7]

[7] *Chapter 9, p. 333.*

Several pages of Chapter 9 are then devoted to infinitely close higher-order curves, an idea that first appeared in *Sulla classificazione delle superficie algebriche particolarmente di genere zero*, a volume of lectures on a particular category of surfaces that Enriques compiled and published in collaboration with Luigi Campedelli in 1934. Two years later, Enriques published a paper on infinitely close curves of higher order in the *Rendiconti dell'Accademia Nazionale dei Lincei* [Enrq 36-1] and a second paper on this theme in the pages of the *Rendiconti del Seminario matematico dell'Università di Roma* [Enrq 36-2], which later appeared in abridged form in the Lincei's journal [Enrq 37], presumably so that it might reach a wider audience. Beniamino Segre challenged Enriques's new proof in [Seg 38], which brought a quick response in June 1938 from Enriques [Enrq 38], along with a different proof.[8] And although even Castelnuovo came close to doubting, in the end, the theorem's provability, Enriques's "life's work", as Patrick DuVal later characterized it [DuVal 49], finally appeared between covers three years after he died.

With his appointment to Mussolini's new Academy of Italy in 1929, Francesco Severi became the regime's spokesman for Italian mathematics. ***Photograph courtesy of the Lincei.***

After Enriques's death, Severi recalled [Sev 57] that the personality characteristics they shared ("vivacious, pugnacious, and sometimes impulsive") were what had driven them apart. Their colleague, Tullio Levi-Civita, on the other hand, claimed that their quarrel stemmed from competing textbooks aimed at the same scholastic market. Castelnuovo, the eldest of the trio (born in 1865), and by nature austere and level-headed, assiduously avoided taking sides, insisting instead that "it was the good fortune of the Italian school of algebraic geometry to have this disinterested collaboration between 1890 and 1910" [B-G 09]. In private, Severi lashed out at his Jewish colleagues, claiming that his work was underrated and theirs overrated. However, Beniamino Segre, Severi's protégé and assistant in 1927—and himself a victim of the 1938 racial laws—vigorously defended Severi against charges of anti-Semitism after World War II.

[8] *The anti-Jewish decrees in September 1938 ended publishing in Italian journals for Enriques and Segre, but Enriques and Severi would continue to battle it out in 1942 and 1943 in the pages of the* Swiss Commentarii Mathematici Helvetici.

A towering figure in the world of algebraic geometry from the end of the nineteenth century until his death in 1946, Enriques made a number of major contributions to the field. To put it another way, his quest to rescue the proof of his Completeness Theorem represents a small (though important) fraction of his mathematical activity during his lifetime. Early in his career (1893), he constructed an (unexpected) example of an algebraic surface F for which $p_g = p_a = 0$ but F was not birationally equivalent to $\mathbf{P}^2(\mathbf{C})$. The existence of such a surface indicated at the very minimum that more birational invariants than just p_g and p_a would be needed to classify algebraic surfaces in the same way that mathematicians employed the geometric genus to classify algebraic curves. (See Appendix A for the definition of a birational invariant.) It also sent a signal that the classification itself would be more complicated. Three years later, in 1896, Enriques defined what he called the *plurigenera* of an algebraic surface, an infinite sequence of birational invariants associated with F. Over the next eighteen years, he and Castelnuovo, often together, used the *plurigenera* to distinguish various birational classes of algebraic surfaces. In 1914 their investigations culminated in the classification of algebraic surfaces into four natural classes defined in terms of the behavior of their plurigenera [Enrq 14], [Cast-Enrq 14].[9] Like the Fundamental Theorem of Irregular Surfaces, the classification of algebraic surfaces remains one of the great accomplishments of early-twentieth-century algebraic geometry.

The Completeness Theorem and the Fundamental Theorem

The 1905 Completeness Theorem of Enriques, as has been noted in the first part of this article, furnished one of the key ingredients in the proof of the Fundamental Theorem of Irregular Surfaces. Simply put, it stated that: The characteristic series of a "good" complete continuous system of curves on a smooth algebraic surface F is complete. (Here "good" in the old Italian terminology means not superabundant, or, in modern terms, the first cohomology group of divisor class should be zero.[10] See Appendix A for the definition of the terms used in the statement of the theorem. Interestingly, Severi had given an appropriate condition in 1944 [Sev44].)

The completion of the proof of the Fundamental Theorem followed quickly on the heels of that of the Completeness Theorem. In 1905, with $q := \dim P(F)$, the dimension of the Picard integrals

[9] *See [Gray 99] for an accessible account of plurigenera and the classification of algebraic surfaces.*

[10] *It was always clear to both Enriques and Severi that the system should be sufficiently ample in some sense. Enriques mentions this several times in his 1936 article.*

of the first kind, Severi proved that $p_g - p_a \geq q$ and $p_g - p_a = b_1 - q$; the latter equality was also proved shortly after by Picard. A few months later, using the Completeness Theorem in an essential way, Castelnuovo carried out the final step in the proof by establishing that $p_g - p_a \leq q$. These two results established the Fundamental Theorem, i.e., $p_g - p_a = q = b/2$.[11]

In 1921, as noted, Severi showed that both his and Enriques's 1905 algebro-geometric proofs of this theorem were fatally flawed. Forty years later, Alexander Grothendieck [Groth 61] introduced the use of higher order nilpotents in the structure sheaf to define nonreduced schemes and, using them, proved a very strong existence theorem for the Picard "scheme", the space of divisors mod linear equivalence. When he visited Harvard, Mumford pointed out to him that, as an immediate corollary, this gave the long-sought algebro-geometric proof of a conjecture he had never heard of!

The opening salvo of the last round of papers by Enriques and Severi on their attempted proofs of the Completeness Theorem took place against the backdrop of World War II. On November 23, 1940, on the occasion of the fortieth anniversary of the publication of his first original scientific paper, Severi submitted a ninety-three-page paper to Mussolini's Academy of Italy (the Lincei had ceased to exist in 1939, following its formal annexation by the Fascist Academy) on the general theory of continuous systems of curves on an algebraic surface. The paper, which included a fresh attempt by Severi to impose his own proof of Enriques's 1905 theorem [Sev 41], appeared in *Memorie della R. Accademia d'Italia* a year later. Now enjoying an enforced retirement, Enriques read it and wrote a brief rejoinder pointing out an error in Severi's proof, which he submitted in March 1942 to the Swiss journal *Commentarii Mathematici Helvetici*. The journal published it [Enrq 43] in 1943, but not before sending it on to Severi, who quickly countered with an article of his own [Sev 43]. Much to Enriques's annoyance, the journal's editors (belying Swiss neutrality) denied his request to rebut Severi's arguments.[12] In his 1943 paper, Severi once more shot down Enriques's proof and this time claimed to have proved an even more general theorem. As it happens, Severi's effort turned out to be wishful thinking, which brings us to Enriques's letter to Beniamino Segre, composed several months after German troops abandoned their hold on northern Italy and surrendered to the Allies.

[11] *Here we are using the notation introduced earlier. See also [Kl 05], p. 244.*

[12] *Efforts by one of us (JG) to examine relevant documents in the journal's archives failed.*

The first two mathematicians (l. to r.) are Beniamino Segre and Beppo Levi. The others are unidentified Italian mathematicians. *Courtesy of Sergio Segre.*

The Remarkable Letter from Federigo Enriques to Beniamino Segre in 1945

On September 11, 1945, writing from Rome, Enriques sent a letter to Beniamino Segre in response to a letter Segre had sent to Enriques from Manchester, U.K., on August 2nd. Most of Enriques's letter deals with his ongoing quest to vindicate his 1905 theorem.

Rome, September 11, 1945

> Dear Segre,
> I thank you for your nice letter of August 2 (I received it the other day), and I congratulate you on your scientific activity.... [After a brief discussion on the work of another Italian mathematician, Beppo Levi, Enriques comes straight to the point.] Personally, I am especially interested in the problem of a continuous system on irregular surfaces which was discussed in my 1942 lectures on surfaces, which is now in press [Enrq 49]. This question is extremely delicate. I was unable at that time to reconstruct the proof that you had indicated on the basis of the information that you had given me before my departure for Paris [1937?]. Severi, with whom you have had more interaction, believed he had finally succeeded in giving a proof [Sev 41]. His exposition seems obscure to me and therefore dubious; I believed (in the *Commentarii Helvetici* paper [Enrq 43]) to have overcome the difficulty: In reality, my proof

> was erroneous, but this realization also pointed out the error in Severi's proof. At that time, I was not allowed to add anything to my paper in the *Commentarii Helvetici* although Severi was allowed to write a note to my paper [Sev 43] in which he says that he has obtained a more general theorem…. But shortly afterward, Severi himself, who was expounding that theory in his lectures at the Institute of Higher Mathematics, realized that his proposed proof was flawed due to a radical error. *I really wish that this thing could be settled. I have reexamined my earlier proof based on infinitely close curves of various orders and I believe it is substantially right, even if it is not rigorously complete* [our italics].

In closing, Enriques asks Segre, whose August 2nd letter must have been written in English, a language he became proficient in while living in England during the war, to give him more details in Italian.[13] In particular, he asks Segre to limit himself to a very specific type of surface and continuous system, where Enriques apparently expects there will be the need for infinitesimally close curves of higher order. [See Appendix B for a copy of the original letter.]

Later that year, Castelnuovo also urged Segre to put his mind to the problem. He wrote many letters to Segre in 1945, mainly dealing with conditions in postwar Italy and urging him not to turn his back on his homeland, but on this occasion he tried to enlist Segre's help in bringing closure to Enriques's 1905 theorem. "Given the delicacy of the matter, a work aiming *exclusively* at settling the interesting question about the characteristic series of a continuous system of curves would seem appropriate," Castelnuovo wrote [his italics]. "Under what conditions," he asked,

> can one use the theorem of the completeness of the characteristic series of a complete continuous system? The theorem has been used in fundamental investigations on surfaces, and we would need to have a fully satisfying geometrical demonstration [Cast 45].

It doesn't appear that Segre, who turned from algebraic geometry to combinatorial geometry within a few years, took the bait.

[13] *While organizing the Beniamino Segre papers, JG found Enriques's letter. There is no copy of Segre's reply in his papers.*

Conclusion

Contrary to the received wisdom on the subject, Enriques came tantalizingly close to realizing his quest. In the final paragraph of the last paper he would publish in Italy on the Completeness Theorem, Enriques summed up his novel approach, reiterated his deep-felt belief that the history of failed attempts in the history of science is instructive for future researchers, and issued a gentle rebuke to his naysayers, declaring

> But the use of infinitely close points and curves and the corresponding *language*, even if not rigorous, is fruitful and even succeeds in giving correct and coherent results for those who can learn to understand and adopt them as I have succeeded in doing in a clear and definitive manner in other parts of my work. A skeptical attitude toward these ideas is easy to have but is not very productive. Instead, those who are more trusting of what these concepts can yield, will, I am sure, discover new results in other fields. [Enrq 38]

Indeed, Enriques himself realized that he lacked the tools to make his infinitely close points and curves "rigorously complete". For Enriques, the important thing was always to make discoveries; the necessary rigorous proofs, he used to tell students, would follow in due time. It is a pity that he did not live to see that he had had the right intuitive idea in the 1930s—an idea that eventually would lead to a rigorous proof of his Completeness Theorem.

In an unpublished paper prepared in 2008 about Grothendieck's work and how it affected his own understanding of the algebraic geometry of Enriques's generation, the American mathematician David Mumford wrote:

> Although Enriques's 1905 paper on the Completeness Theorem missed the key issue, this paper [Enrq 38] does have the right idea. He speaks of the exponential map in the Picard variety and asserts that analogously higher order infinitely near curves can be generated from first order infinitely near ones. Unfortunately, he possesses no tools whatsoever for going beyond an intuitive description of why the method of higher order infinitesimals should work: the theory of schemes is what he lacked. [Mum 08]

See the accompanying article by David Mumford, "Intuition and Rigor and Enriques's Quests", for the end of this story.

Appendix A

Birational Equivalence and Birational Invariants[14]

Much of Enriques's work dealt with the birational geometry of projective algebraic surfaces $F \subset \mathbf{P}^n(\mathbf{C})$, where $\mathbf{P}^n(\mathbf{C})$ is complex n-dimensional projective space. By birational geometry we mean the study of birational mappings $T : F$ "$\to$" F' of projective algebraic surfaces and the corresponding notions of birational equivalence and birational invariants.

Perhaps the most familiar smooth algebraic surfaces to nonalgebraic geometers are those given by the locus of zeros of an irreducible homogeneous polynomial f on $\mathbf{P}^3(\mathbf{C})$, i.e.,

$$F_f := \{(z_0 : z_1 : z_2 : z_3) \in \mathbf{P}^3(\mathbf{C}) \mid f(z_0 : z_1 : z_2 : z_3) = 0\},$$

where $(z_0 : z_1 : z_2 : z_3)$ denotes the standard homogeneous coordinates in $\mathbf{P}^3(\mathbf{C})$ and f is an irreducible homogeneous polynomial whose differential never vanishes at points of F_f. It turns out that these surfaces are always *regular*. However, it is the smooth *irregular* surfaces, i.e., when $p_g - p_a > 0$, that are most relevant for us. Irregular surfaces will thus, of necessity, be embedded in $\mathbf{P}^n(\mathbf{C})$, $n > 3$. They will be given by the locus of zeros of some set of homogeneous polynomials on $\mathbf{P}^n(\mathbf{C})$ (1) with conditions on the polynomials that make sure they define an irreducible algebraic set of dimension 2 and (2) that are locally transverse intersections of hypersurfaces with locally independent differentials. From now on, we will suppress the reference to "complex smooth algebraic" and refer simply to curves and surfaces.

To define the notion of a birational mapping between surfaces F and F', we first need to define the notions of a Zariski open subset in F and rational mapping from F to F'. A *Zariski open set* in F is just the complement $F - \bigcup_i W_i$ of a finite set of subvarieties $W_i \subset F$. A *rational mapping* of F to F' consists of a Zariski open subset F_1 of F and a mapping $T : F_1 \to F'$, given by rational functions f of the coordinates of F. T is *birational* if T is injective and T^{-1} is rational. F and F' are *birationally equivalent* if there exists a birational mapping between them.

Example. Let $F = F' = \mathbf{P}^2(\mathbf{C})$[15] and $F_1 = \mathbf{P}^2(\mathbf{C}) - \{z_0z_1z_3 = 0\}$ and define T by $(z_0 : z_1 : z_2) \to (1/z_0 : 1/z_1 : 1/z_2)(= (y_0 : y_1 : y_2))$. Note that the domain of T^{-1} is $\mathbf{P}^2(\mathbf{C}) - \{y_0y_1y_3 = 0\}$ and $T^{-1}((y_0 : y_1 : y_2)) = (1/y_0 : 1/y_1 : 1/y_2)$.

A *birational invariant* is an integer-valued function $\mathbf{I}$ from the set (category) of algebraic surfaces to the integers $\mathbf{Z}$ such that $\mathbf{I}(F) = \mathbf{I}(F)$ if F and F' are birationally equivalent. (There are similar definitions for algebraic curves.)

In 1870 M. Noether proved that the geometric genus p_g was a birational invariant, while Zeuthen, in 1871, proved that the arithmetic genus p_a was also a birational invariant. The *plurigenera* of Enriques are also birational invariants.

Remark. Birational invariants that are also topological invariants: An algebraic surface F over $\mathbf{C}$ is also a topological space. As a topological space p_g and p_a are also topological invariants, i.e., if algebraic surfaces F and F' are homeomorphic, then $p_g = p_g'$ and $p_a = p_a'$.

The Definition of the Terms in the Completeness Theorem

A *good algebraic curve C on F* usually means it is sufficiently ample, that is, it is a sufficiently high multiple of a hyperplane section in some projective embedding of F. In the case of the Completeness Theorem, it is enough that it has no higher cohomology groups. In simplest terms, this means $\dim L(D)$ equals its self-intersection number (D^2) minus the genus of D plus the arithmetic genus of F plus two and that there are no regular 2-forms on F zero along D.

A *complete algebraic (sometimes called a continuous or nonlinear) system of curves C_a, $a \in S$ on F*, where S is an algebraic variety, is maximal in the set of algebraic systems of curves on F.[16]

The algebraic system is *good* if it consists of good algebraic curves.

The *characteristic linear system at a curve C_0 of an algebraic system, C_a, $a \in S$ on F* is the set of divisors on C_0 which are the limits of the intersections $C_0 \cap C_a$ as a tends to 0. It is not hard to show that this defines a linear system on C_0.

The linear system on C_0 is "complete" as a linear system if it is maximal in the set of linear systems on C_0.[17]

Acknowledgments

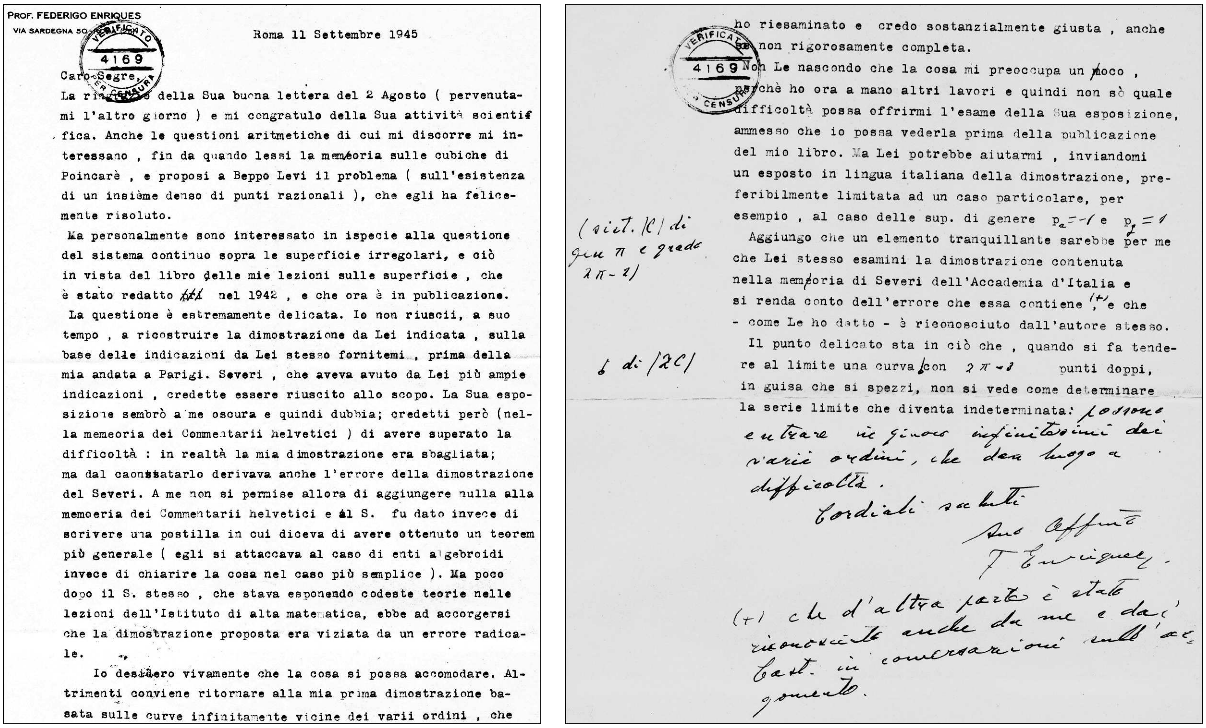
We thank David Mumford, who put his deep knowledge of the history and mathematics related to the Completeness Theorem at our disposal; Steven Kleiman for suggesting many improvements, both historical and mathematical, to the original text; Lorenzo Enriques for permission

[14] *A very nice discussion of birational algebraic geometry can be found in the paper by A. Grassi in the* Bulletin of the AMS *(2009), pp. 99–123.*

[15] *Here we think of* $\mathbf{P}^2(\mathbf{C})$ *embedded in* $\mathbf{P}^3(\mathbf{C})$ *by* $(z_0 : z_1 : z_2) \to (z_0 : z_1 : z_2 : 0)$.

[16] *See [Shaf 74], Ch. I.3.5, for the definition of an algebraic system of curves on a surface.*

[17] *See [Mum 76], Ch. 6A, for a discussion of linear systems.*



PROF. FEDERIGO ENRIQUES
VIA SARDEGNA 50 - ROMA (25)

VERIFICATO PER CENSURA 4169

Roma 11 Settembre 1945

Caro Segre,
La ringrazio della Sua buona lettera del 2 Agosto (pervenutami l'altro giorno) e mi congratulo della Sua attività scientifica. Anche le questioni aritmetiche di cui mi discorre mi interessano , fin da quando lessi la memoria sulle cubiche di Poincarè , e proposi a Beppo Levi il problema (sull'esistenza di un insième denso di punti razionali), che egli ha felicemente risoluto.
Ma personalmente sono interessato in ispecie alla questione del sistema continuo sopra le superficie irregolari, e ciò in vista del libro delle mie lezioni sulle superficie , che è stato redatto nel 1942 , e che ora è in publicazione.
La questione è estremamente delicata. Io non riuscii, a suo tempo , a ricostruire la dimostrazione da Lei indicata , sulla base delle indicazioni da Lei stesso fornitemi , prima della mia andata a Parigi. Severi , che aveva avuto da Lei più ampie indicazioni , credette essere riuscito allo scopo. La Sua esposizione sembrò a me oscura e quindi dubbia; credetti però (nella memeoria dei Commentarii helvetici) di avere superato la difficoltà : in realtà la mia dimostrazione era sbagliata; ma dal caonstatarlo derivava anche l'errore della dimostrazione del Severi. A me non si permise allora di aggiungere nulla alla memoeria dei Commentarii helvetici e al S. fu dato invece di scrivere una postilla in cui diceva di avere ottenuto un teorem più generale (egli si attaccava al caso di enti algebroidi invece di chiarire la cosa nel caso più semplice). Ma poco dopo il S. stesso , che stava esponendo codeste teorie nelle lezioni dell'Istituto di alta matematica, ebbe ad accorgersi che la dimostrazione proposta era viziata da un errore radicale.
Io desidero vivamente che la cosa si possa accomodare. Altrimenti conviene ritornare alla mia prima dimostrazione basata sulle curve infinitamente vicine dei varii ordini , che ho riesaminato e credo sostanzialmente giusta , anche se non rigorosamente completa.
Non Le nascondo che la cosa mi preoccupa un poco , perchè ho ora a mano altri lavori e quindi non sò quale difficoltà possa offrirmi l'esame della Sua esposizione, ammesso che io possa vederla prima della publicazione del mio libro. Ma Lei potrebbe aiutarmi , inviandomi un esposto in lingua italiana della dimostrazione, preferibilmente limitata ad un caso particolare, per esempio , al caso delle sup. di genere $p_a = -1$ e $p_g = 1$ (sist. |C| di gen π e grado $2\pi - 2$)
Aggiungo che un elemento tranquillante sarebbe per me che Lei stesso esamini la dimostrazione contenuta nella memoria di Severi dell'Accademia d'Italia e si renda conto dell'errore che essa contiene(+), e che - come Le ho detto - è riconosciuto dall'autore stesso.
Il punto delicato sta in ciò che , quando si fa tendere al limite una curva b di |2C| con $2\pi - 3$ punti doppi, in guisa che si spezzi, non si vede come determinare la serie limite che diventa indeterminata: possono entrare in giuoco infinitesimi dei varii ordini, che dan luogo a difficoltà.
Cordiali saluti
Suo Affmo
F. Enriques.

(+) che d'altra parte è stato riconosciuto anche da me e dai Cast. in conversazioni sull'argomento.

Appendix B. Text of a letter from F. Enriques to Beniamino Segre, 11 September 1945, recounting the recent history of his efforts to prove the "Completeness Theorem". The war had officially ended. But the censor's stamp in the upper left-hand corner reminds us that postal authorities still opened and read letters. ***Courtesy of California Institute of Technology.***

to publish his grandfather's letter and information about his family; Sergio Segre for opening Beniamino Segre's papers to scholars; Elisa Piccio, Michele Vallisneri, and Annalisa Capristo for discussions; Francesca Rosa for archival research in Rome and Pisa; and Sara Lippincott for editorial suggestions.

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Intuition and Rigor and Enriques's Quest

David Mumford

In the preceding article we have seen that Enriques and, indeed, the whole Italian school of algebraic geometry in the first half of the twentieth century were frustrated by one glaring gap in their theory of algebraic surfaces. This magnificent theory answered essentially all the basic questions about algebraic surfaces and had been constructed using purely geometric tools. But one of its central theorems seemed to defy all their attempts to give it a geometric proof. It had been proven by analytic means by Poincaré with his theory of "normal functions",[1] so the theory was sound—but this approach was alien to their intuitions. It was much like the need for analysis in proving the prime number theorem before Selberg found his elementary proof. In my own education, I had assumed they were irrevocably stuck, and it was not until I learned of Grothendieck's theory of schemes and his strong existence theorems for the Picard scheme that I saw that a purely algebro-geometric proof was indeed possible. I say here "algebro-geometric", not "geometric", because the first requirement in moving ahead had been the introduction of new algebraic tools into the subject first by Zariski and Weil and subsequently by Serre and Grothendieck.

When Professors Babbitt and Goodstein wrote me about Enriques's work in the 1930s, I realized that the full story was more complex. As I see it now, Enriques must be credited with a nearly complete geometric proof *using, as did Grothendieck, higher order infinitesimal deformations.* In other words, he anticipated Grothendieck in understanding that the key to unlocking the Fundamental Theorem was understanding and manipulating geometrically higher order deformations. Let's be careful: he certainly had the correct ideas about infinitesimal geometry, though he had no idea at all how to make precise definitions. If you compare his ideas here with, for example, the way Leibniz described his calculus, the level of rigor is about the same. To use a fashionable word, his "yoga" of infinitesimal neighborhoods was correct, but basic parts of it needed some nontrivial algebra before they could ever be made into a proper mathematical theory.

Enriques himself realized that he did not have a clear definition of higher order infinitesimal deformations, and so he was uncertain what sort of arguments were permissible for "curves in a higher order neighborhood" of an actual curve on a surface. As we will see below, he had two lines of reasoning. One depended on being able to *add* infinitesimal points on a group, forming as it were $(n+m)\epsilon$ by adding $n\epsilon$ and $m\epsilon$. Another was based on an infinitesimal analog of Poincaré's normal function construction, which uses the Jacobian varieties of the level curves of a suitably general rational function f on the surface F (a so-called *Lefschetz pencil* on F). In this paper, I will use Enriques's 1936 paper in the Mathematical Seminar of Rome[2] to explain these two ideas. I will translate the key parts of this paper and add my commentary so the reader can see exactly what Enriques did and how it can be made into a fully rigorous argument with modern technique.

There is no exact way of assigning a percentage to the degree of completeness of an argument. Certainly Grothendieck's tools were needed, and

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[1]Sur les courbes tracées sur les surfaces algébriques, *in two parts:* Annales Écoles Normale Sup. ***29****, 1910, and* Sitzungsberichte Berlin Math. Gesellschaft ***10****, 1911. An excellent presentation can be found in O. Zariski's book* Algebraic Surfaces*, Chapter VII, section 5, published by Springer, 1934, and reprinted by Chelsea, 1948.*

[2]Rendiconti del seminario matematico della universita di Roma, *series 4, vol. 1, 1936, pp. 1-9.*

First definitions: It may be useful for nonalgebraic geometers to set the stage with the basic definitions. A *projective algebraic variety* V is the locus of zeros of a finite set of homogeneous polynomials in some projective space, which, moreover, is irreducible, i.e., not the union of two or more proper subsets of the same type. A *Zariski open set* in V is just the complement $V - \bigcup_i W_i$ of a set of subvarieties $W_i \subset V$. Such varieties are linked by *rational maps*, maps $f : V_1' \to V_2, V_1' \subset V_1$ a Zariski open set, given by rational functions f of the coordinates of V_1. (Think of x/y, which defines a rational map $(\mathbb{P}^2)' \to \mathbb{P}^1$ which cannot be defined at $x = y = 0$.) The main players are the *nonsingular* or *smooth* varieties, those which are locally transverse intersections of hypersurfaces with locally independent differentials.

Linear systems: The key structures that were studied on smooth varieties were their *linear systems*. If $H \subset V$ is the intersection of V with the hyperplane at infinity, think of the vector space L of affine coordinate functions, the span of $\{1, x_1, \ldots, x_n\}$. Note that almost all the functions in L have a simple pole along H. In general, we start with a *divisor* D on V which is just a linear combination $\sum_i n_i D_i$ of codimension 1 subvarieties $D_i \subset V$. Then we have the vector space $L(D)$ of rational functions f on V such that f has at most an n_i-fold pole at D_i if $n_i > 0$ and at least a $-n_i$-fold zero on D_i if $n_i < 0$. Alternately, one defines (f) to be the divisor of zeros and poles of a rational function f, assigning positive coefficients equal to the order of vanishing at all its zeros and similarly negative coefficients at poles. Using this language, we see that $L(D) = \{f \mid (f) + D \geq 0\}$. These $L(D)$ are the *complete linear systems*, and any of their linear subspaces are called *linear systems*. Computing the dimension of $L(D)$ is the concern of the class of theorems called "Riemann-Roch". The Italian geometers preferred to deal with the nonnegative divisors $(f) + D$ themselves. The set of these is written as $|D|$ (or $|D|_V$ if we need to make the ambient space explicit) and is just the projectivization of the vector space $L(D)$ (because $(f) + D = (g) + D$ only if f/g is a constant). To complete this list of standard definitions, we say two divisors D_1, D_2 are *linearly equivalent* (written $D_1 \equiv D_2$) if $D_1 - D_2 = (f)$ for some f and the *Picard group* $\mathrm{Pic}(F)$ is the group of divisors mod linear equivalence, called *divisor classes*.

they make the argument cleaner and more elegant. But Enriques's approach needs relatively few additional arguments to make it into a complete proof. He himself, with customary optimism, called it *pienamente rigorosa* ("fully rigorous")! Castelnuovo, on the other hand, when he comes to this argument in his edition of Enriques's posthumous book summarizing much of his lifetime's work, says more conservatively: "The section which follows has been left incomplete by the author and thus the argument which is developed has many gaps; however, it was thought appropriate to reproduce it because it contains ideas that perhaps, appropriately completed, will furnish the starting point for a systematization of the theory." We leave it to the reader to form his or her own judgment.

What Is This Fundamental Theorem?

The easiest way to explain the Fundamental Theorem from First Principles is to recall some basic facts from the theory of algebraic curves. If C is a curve of genus g, you consider the set S_n of all unordered n-tuples of points on C (also called the symmetric nth power of C, or $C^n/(\text{symm.gp.})$). It is the set of positive divisors of degree n on C. Then if $n > 2g - 2$, S_n is very elegant space: it has a fibered structure with fibers that are the complete linear systems of divisors of degree n and hence projective spaces and base space an abelian variety, that is, a complex torus of dimension g, independent of n and called the Jacobian of C. Two n-tuples are in the same fiber if and only if they are linearly equivalent. This generalizes to a surface F (and to any higher dimensional variety) if we replace n-tuples of points by divisors D on F drawn from any "sufficiently ample" cohomology class.[3] Consider the set $\mathcal{D}$ of all such D, which is called a *complete continuous system* of divisors on F. It has a similar fibered structure. The fibers are again linear systems of divisors (as before, two divisors being in the same linear system if they are the zeros and poles of a rational function), and the base space is an abelian variety, the Picard variety of F, which is again independent of the choice of the divisors.

The hard question was: what is the dimension of the Picard variety? and, specifically, was it equal to the irregularity q? One way to approach this is to fix one member $C \in \mathcal{D}$ (which we can take to be an irreducible curve) and look at the intersections $A = D.C, D \in \mathcal{D}$. There is a number n such that all these are n-tuples of points on C, n being the self-intersection number $(C.C)$. Linearly equivalent divisors D will give linearly equivalent n-tuples on C, but inequivalent divisors will usually remain inequivalent. However, if we let an inequivalent divisor D approach C along a one-dimensional family of divisors D_t, then

[3] *E.g., a high enough multiple of a hyperplane section for some projective embedding of F is sufficiently ample.*

Invariants of surfaces: The main numerical invariant of an algebraic curve is its *genus* g, introduced by Riemann. This can be defined as the dimension of the vector space of rational 1-forms with no poles on a curve C, that is, a differential that can be written locally as $f\,dx$, where x is a local coordinate in $U \subset C$ and f is a rational function with no poles in U. One always defines the *canonical divisor class* K (up to linear equivalence) to be the divisor of zeros and poles of any rational 1-form, so we get $g = \dim L(K)$. In modern sheaf-theoretic terminology, $L(K) = \Gamma(\Omega^1_C)$. Topologically, g is also the number of handles in the surface defined by the set of complex points on the curve. A central question in the theory of algebraic surfaces was how to generalize the genus to higher dimension. Clebsch[4] defined the *geometric genus* p_g of a surface F as the dimension of the vector space of rational 2-forms with no poles, i.e., locally given as $f\,dx \wedge dy$ with f having no poles locally. Again this equals $\dim L(K)$ or $\Gamma(\Omega^2_F)$. This can be hard to compute, and soon after Cayley[5] found that the dimension of the space of rational 2-forms with mild poles was much easier to compute. Somewhat simplifying the history, it was found that there was a number p_a, the *arithmetic genus*, such that for sufficiently ample curves $C \subset F$ with genus $g(C)$, the dimension of the space of 2-forms with simple poles at C was $p_a + g(C)$. Now looking at the leading term in the pole of such a 2-form, one finds that it is naturally a 1-form on C called its *residue*. Since there are p_g independent 2-forms with no poles, Cayley's dimension $p_a + g(C)$ less p_g is the dimension of the space of residues and that is at most $g(C)$. So $p_a \le p_g$. The first surfaces investigated had $p_a = p_g$, and these were called *regular*. Then $q = p_g - p_a$ was defined as the *irregularity* of F. In cohomological terms, we now define p_a by $p_a + 1 = \chi(\mathcal{O}_F)$ and, knowing $p_g = h^2(\mathcal{O}_F)$, we have $q = h^1(\mathcal{O}_F)$.

the limit of the intersections $C.D_t$ will be an n-tuple, which always belongs to one and the same linear equivalence class, also denoted by $(C.C)$. Put formally, this means that if we let $\mathbb{P}$ = projectivization of the tangent space $T_C\mathcal{D}$ to $\mathcal{D}$ at its point defined by C, then $\mathbb{P}$ maps to a linear system of n-tuples on C, called the *characteristic linear system* of $\mathcal{D}$. The hypothesis was that this linear system was the complete linear system $|(C.C)|$ on C, and it was always referred to as *the completeness of the characteristic linear system of a complete continuous system.*

Once this was established, a small argument shows that the dimension of the Picard variety is indeed equal to the irregularity q of F. Let K_F be the canonical divisor on the surface F. Using residues, one finds that the divisor class $(C.C+K_F)$ is the canonical class K_C on C. Writing $|(C.C)|_C$ for the linear system of n-tuples on C given by the self-intersection, we use the Riemann-Roch theorem on C, which shows that

$$\dim|(C.C)|_C = (C.C) - g(C) + 1 + \dim|(K_F.C)|_C.$$

But $2g(C) - 2 = \deg(K_C) = (C.C + K_F)$, and a theorem of Severi shows that $\dim|(K_F.C)|_C = \dim|K_F|_F = p_g$ if C is sufficiently ample. Putting this together, we find $\dim|(C.C)|_C = \frac{(C.C-K_F)}{2} + p_g$. On the other hand, a slight generalization of Cayley's definition of p_a shows that for all sufficiently ample curves $C \subset F$, $\dim|C|_F = \frac{(C.C-K_F)}{2} + p_a$, so q is exactly the codimension of the trace of the linear system $|C|_F$ on F inside the linear system $|(C.C)|_C$ on C. But if the characteristic linear system of $\mathcal{D}$ is complete, this codimension equals the codimension of $|C|_F$ in $\mathcal{D}$, and this is the dimension of the Picard variety. We can, of course, rewrite this using cohomology exact sequences.[6]

Enriques's 1936 Paper

To the best of my knowledge, Enriques's first paper on his most successful approach to the Fundamental Theorem is the one with the title "Curve infinitamente vicine sopra una superficie algebrica", published in the *Rendiconti del Seminario Matematico della R. Universita di Roma*, 1936.[7] The basic idea in this paper is repeated in his note *Sulla proprietà caratteristica delle superficie algebriche irregolari* in the *Rendiconti della Accademia Nazionale dei Lincei*, 1938, and in Chapter 9 of his posthumous book *Le Superficie Algebriche*, published in 1949 with the editorial help of Castelnuovo (whose conservative evaluation we have quoted above).

As the title says, the paper is all about curves on an algebraic surface, but not actual curves. If $C \subset F$ is a curve in the usual sense, i.e., it is a subset defined by zeros of polynomials, then the paper concerns other "curves" $C_1, C_2, C_3, \ldots$ which he calls infinitely close to C in the neighborhood of order $1, 2, 3, \ldots$ (*" infinitamente vicine ad una C*

[4] Comptes Rendus de l'Acad. Fr.*, vol. 67, 1868, p. 1238.*

[5] *Footnote on page 333 in* Le Superficie Algebriche.

[6] *Incidentally, I'd like to dispel the misconception that Italian algebraic geometry had nothing to do with cohomology of sheaves. Higher cohomology groups were implicit in most of their work but were always treated indirectly with geometric tools. For instance, the result that H^2 of the sheaf of 2-forms on the projective plane was one-dimensional is equivalent to the classical Cayley-Bacharach theorem.*

[7] *I am very grateful to Dr. Pier Vittorio Ceccherini for his help in obtaining a copy of this article, which is hard to find in the United States, and to Francine Laporte for a great deal of help with the translation.*

nell'intorno del 1^o *ordine, del* 2^o *ordine e così via"*). He also calls these curves $C_1, C_2, \ldots$ "successive" to C, and we follow this for lack of a better English word.

To make sense of this paper we have to have some idea of what he meant by "infinitely close". The problem is that Enriques was so thoroughly a geometer that he avoided ever using an equation of any sort, and it is hard to make sense of infinitely close things without some equations. Most easily, one can put a parameter t into the defining equations of the curve, replacing $f(x_1, \ldots, x_m) = 0$ or simply as $f(x) = 0$, by $f(x) + tf_1(x) + t^2 f_2(x) + \cdots$ and calculate mod t^n for some n. But you can read his papers and never know that algebraic varieties had to be defined by polynomials! However, in two places in small asides he gives some clues about what he meant. These are places where he refers to auxiliary varieties parameterizing divisors. Here is the first from §2 of his paper dealing with infinitely close "groups" of points on curves, that is, positive 0-cycles of some degree m, $G_m = \sum_{i=1}^{m} P_i \in S_m$ (P_i not necessarily distinct):

> And first of all, let us note that the groups of points G_m (made up of m points), infinitely close to a given group on a curve, can be properly defined using differential expressions and conditions, as elements or "points" of the variety representing the groups of m points of the curve.
> *E anzitutto osserviamo che i gruppi di punti* G_m *(costituti di m punti) infinitamente vicini ad un gruppo dato, sopra una curva, riusciranno bene definiti mediante espressioni e condizioni differenziali, siccome elementi o ⟨⟨punti⟩⟩ della varietà rappresentiva dei gruppi di m punti della curva.*

This variety "representing groups" is nothing but the mth symmetric power of the curve C. He is reducing the study of infinitely close complex objects on the curve C to the study of infinitely close points on the auxiliary parameterizing space. And again from §3 he says:

> For our aims it suffices to consider curves successive to C on linear branches (within the space that has C, C_1, C_2, etc., for "points").
> *E pel nostro scopo basta limitarsi a considerare curve successive alle C su rami lineari (entro l'ente che ha per ⟨⟨punti⟩⟩ le* C, C_1, C_2 *ecc.)*

Here he restricts himself to successive infinitely close points along a one-dimensional arc in an auxiliary space, one that parametrizes effective divisors on a surface—which he assumes is readily constructed. We are left with the issue: what does he mean by the "point" infinitely close of order n to a real point on a curve or on an analytic branch?

Algebraic geometry had introduced "infinitely near" points on all varieties as points on any variety obtained by blowing up the original variety a certain number of times. Given an analytic branch on the variety X, one can indeed blow up one point $P \in X$. At a point P on a surface where x, y are local coordinates, the blowup will be the closure of the graph of x/y. This will replace P by a projective line whose points correspond to tangent directions, the projectivization of $T_P F$. Then we can blow up the new limit point of the branch after "lifting" it to the first blowup, and continue to do this n times. The resulting infinitely near points on some auxiliary parameterizing variety is one approach to making sense of Enriques's concept of infinitely close curves on a surface. There was also a tradition of studying Puiseux series, power series in fractional exponents to describe curves, and then the terms of order n give another approach to higher order neighborhoods. But after Grothendieck's work, I think the best approach is clearly to define an infinitely close point of order n on a branch to be its unique subscheme whose ring of functions is isomorphic to $(k[t]/(t^{n+1}))$. On the whole variety, the infinitely close points of order n are then the set of all such subschemes on different branches. In my notes below, I will use this approach. Such an infinitely close point is essentially the same as what in differential geometry are called n-jets on a manifold.

Enriques, I believe, thought of a sequence of true points $P_0, P_1, \ldots, P_n$, *equally spaced* on an analytic arc, and then imagined (as Leibniz might) that P_1 and hence all the rest approached P_0. Then the "limit" of P_n is the point of order n successive to P_0. I think we all recognize that there is a common intuitive meaning to the idea that on a linear branch, there is something like a unique infinitely close point of order n at each usual point. However, as we'll see, it gets sticky when you begin to play games with this concept and don't actually have a precise definition.

The Translation, Part I, and Enriques's First Argument

Now I will translate the key sections of this paper interspersed with commentary to make the article more accessible. I show Enriques's words in italics and my commentary in sans-serif typeface to distinguish the two voices.

In the demonstration of the characteristic property of irregular surfaces with geometric and numerical (=arithmetic) *genera* p_g *and* p_a*, that is that they contain continuous systems of curves formed of* $\infty^{p_g - p_a}$ *inequivalent* (that is, not linearly equivalent) *linear systems, one needs to count the number of curves of the continuous system that are* *infinitely close to a given curve K and to show that they cut on K the complete characteristic system. But to the demonstration that I gave in my note in the Academy of Bologna in 1904 and that at first was accepted by all, or rather restated with slight modifications, the objection was raised that curves infinitely close to K (whose existence result from compatibility of superabundance conditions) do not necessarily lead to the existence of continuous series of curves containing them.*

The "complete characteristic system" is the linear system that I denoted by $L_K((K.K))$ or, in sheaf-theoretic terms, by $\Gamma(\mathcal{O}_K(K^2))$ or written projectively by $|(K.K)|_K$. In this translation, I have retained his symbol K, although now K is usually reserved for the canonical divisor class. This is the best place to describe what he had done in 1904, namely, he showed that for every positive divisor $G \in |(K.K)|_K$, he could construct a curve K_1 infinitely close to K of first order which intersects K in G. His method was to take a second sufficiently ample divisor $\widetilde{K}$ and look at the curves E in $|K+\widetilde{K}|_F$ that pass through the intersection points $K \cap \widetilde{K}$. He first showed that these curves E intersect K, after discarding the points $K \cap \widetilde{K}$, in the complete characteristic system. He then argued that if such an E is infinitely close to $K+\widetilde{K}$ in the first order, E effectively has a double point infinitely close to each point of $K \cap \widetilde{K}$ and then, by a Riemann surface argument, this E must continue to split into two pieces $K_1 + \widetilde{K}_1$. This K_1 is the infinitely close curve to K that we want. This argument is correct. For instance, one can use the exact sequence:

$$0 \longrightarrow \mathcal{O}_F(\widetilde{K}) \longrightarrow m_{K\cap\widetilde{K}}\mathcal{O}(K+\widetilde{K}) \longrightarrow \mathcal{O}_K((K.K)) \longrightarrow 0$$

to verify the first point. The second point follows because if you deform a double point $u.v = 0$ by a curve through $u = v = 0$, the result is $u.v + t.(au + bv) = 0$ and modulo t^2, this equals $(u+tb).(v+ta) = 0$. And finally, if resolving a set of double points disconnects a curve, the same holds for any deformation in which the double points persist.

Now, however, we would argue directly that $\mathcal{O}_K((K.K))$ was the normal sheaf to K and thus its sections define first-order deformations. But Enriques's approach is perfectly correct.

Next he addresses the reasons that his old argument was incomplete by making some very general observations about the pitfalls of drawing global conclusions from infinitesimal facts.

To explain this doubt: if one defines, for example, a curve as the intersection of two or more surfaces, one cannot say that on the curve so defined one always has (one) *single point infinitely close to a given point; one will have instead ∞^1 if the given curve is a line of contact of the defining surfaces: in this case there are no further successive points infinitely close a point taken in this neighborhood of a proper point of the curve but outside its tangent.*

What he is saying is that if the surfaces meet transversely along the curve, at each point of the curve there would be only one infinitesimal direction in which you can move, namely along the curve; on the other hand, if the surfaces are tangent, you can start to move infinitesimally away from the curve in many directions, but unless they are tangent to an even higher order, one cannot extend this path to second order and stay inside the intersection unless you moved along the curve. In the language of schemes, the intersection of surfaces tangent along a curve is an everywhere nonreduced scheme, generically with the square of its nilpotent ideal equal to (0). The "Zariski tangent space" has dimension two, but there are higher order infinitely close points only for the one direction tangent to the reduced curve.

Re-examining the same question in my "Lessons on the classification of surface", edited by L. Campedelli, I observed, however, that the conclusions of my treatment would keep their validity if one admitted that curves infinitely close to a given curve on a surface had an effective existence and that one could operate on them as on finite curves, by adding and subtracting. Thus, letting C_1 be a curve infinitely close to C and inequivalent to it, the operation $+C_1 - C$, successively repeated, serves to define, in the neighborhood of any curve K whatsoever, a series of infinitely close curves $K_1, K_2, K_3, \dots$ belonging to a suitably high order neighborhood and this leads to the conclusion that this K should belong to a continuous nonlinear series in which that K_1 would be close to K.

It remained however to justify the intuitive truth: that one can effectively operate on infinitely close curves on a surface by addition and subtraction. And this is precisely the aim of the present note.

A Modern Version of This Intuitive Argument

The argument immediately above is the core of Enriques's first argument for the theorem using infinitely close curves (but, as he says, not of the more refined "proof" that follows). When I read this, it sounded a bit far-out. But then, putting it carefully into the language of schemes, I found it could be given quite a complete and elegant modern formulation.

Let me paraphrase his idea like this: suppose $t \mapsto \phi(t) \in A$ is a 1-parameter subgroup of a Lie group A (such as the Picard variety of the surface F). Then start with a point $\phi(\epsilon) \in A$ where ϵ is first taken to be *positive*, not infinitesimal. Then simply adding, using the group law in A, gets you back $\phi(n\epsilon)$ for all n. Now suppose ϵ is infinitesimal, passing from finite numbers to infinitesimals as in Leibniz's treatment of calculus. Then $\phi(\epsilon)$ lies in what Enriques is calling the first-order neighborhood of the identity $e \in A$. Operating with the group law on A, he wants to automatically generate something like $\phi(n\epsilon)$ which is to live in the nth-order neighborhood of e. In particular, his difference $C_1 - C$ is like an infinitely close point of first order $\phi(\epsilon)$ in the Picard variety of F and "adding", he wants to generate the higher order "points" of the Picard variety, hence the higher order infinitesimal deformations K_n of K.

Nowadays we can use the language of schemes to make this precise. A point on A in the nth-order neighborhood of e is just a morphism of $\mathrm{Spec}(k[t]/(t^{n+1}))$ to A whose set-theoretic image is the point e. So does it make sense that from a $k[t]/(t^2)$-valued point, we can use the group law alone to get a $k[t]/(t^{n+1})$-valued point?

In fact, this is true in characteristic 0, and a simple algebraic argument provides rigorous support for Enriques's Leibnizian treatment of infinitesimals. The key is this purely algebraic fact:

Proposition. *If the characteristic of k is 0, then the subring of*

$$k[t_1,\dots,t_n]/(t_1^2,\dots,t_n^2)$$

of elements invariant under permutations of the t_i is isomorphic to $k[s]/(s^{n+1})$ where $s = t_1 + \cdots + t_n$.

The reason is that on the one hand the invariant subring is generated, as usual, by the elementary symmetric polynomials in the t_i and on the other hand:

$$(t_1+\cdots+t_n)^k \equiv k!\cdot k^{th}\,\text{elem.symm.polyn.}(t_1,\dots,t_n) \mod (t_1^2,\dots,t_n^2).$$

In characteristic zero, we can divide by $k!$; hence the proposition follows.

Now given an infinitely close point of first order $\phi : \mathrm{Spec}(k[t]/(t^2)) \longrightarrow A$, we get the following n-fold summation by adding via the group law on A:

$$\phi\circ p_1 + \cdots + \phi\circ p_n : \mathrm{Spec}(k[t_1,\dots,t_n]/(t_1^2,\dots,t_n^2)) \longrightarrow A$$

and, by commutativity, the pullback of all functions on A are permutation invariant; hence this map factors through $\mathrm{Spec}(k[s]/(s^{n+1}))$. This is the infinitely close point of order n.

In his example, we would say that he wants to *add* a divisor class on $F \times \mathrm{Spec}(k[t]/(t^2))$, trivial on F, to itself to get a divisor class on $F\times\mathrm{Spec}(k[s]/(s^{n+1}))$. Suppose the divisor class is defined by a 1-cocycle $\{1 + tf_{\alpha,\beta}\}$. Then adding this to itself n-times gives the divisor class on $F\times\mathrm{Spec}(k[t_1,\dots,t_n]/(t_1^2,\dots,t_n^2))$ given by

$$\begin{aligned}\prod_{i=1}^{n}&(1+t_i f_{\alpha,\beta})\\ &= \sum_{k=0}^{n}\left(k^{th}\,\text{elem.symm.polyn.in } t_i\right)\cdot f_{\alpha,\beta}^k\\ &= \sum_{k=0}^{n}\frac{s^k}{k!}f_{\alpha,\beta}^k.\end{aligned}$$

Thus this n-fold added divisor class is defined over the subring $k[s]/(s^{n+1})$ and the divisor class is defined by the 1-cocycle given by the truncated exponential. This, I think, is the precise meaning behind Enriques's assertion that *adding* first-order deformations defines higher deformations—of course only in characteristic zero. Enriques certainly did not know such an argument, but this at least confirms that his intuition was completely sound—and, as Hartshorne remarked to me, was based on a characteristic zero world without his knowing it.

The Translation, Part II, Analysis of Divisors on Curves

Let us take the steps of the argument in the case of groups of points and linear series (linear series on a curve are the same as linear systems) *on a curve.*

One will observe firstly that the ordinary manner of adding and subtracting series on a curve falls apart when one treats infinitely close groups or series. Thus if, among the curves of a certain order that pass through a certain group G (of points on some curve), *one considers those that are tangent to the basic curve at the points of G, one then subtracts from the series cut out not a group infinitely close to G but the group G itself.*

From that one should not conclude that as a result infinitely close groups of points or linear systems on a given curve do not have a real existence and that—in every respect—one cannot operate on them by addition and subtraction: the law of continuity operates in the field of algebra and so one must rather admit a priori that, with appropriate considerations, these entities and the operations we are dealing with will succeed in being properly justified. And first of all, we note that groups of points G_m (made up of m points), infinitely close to a given group on a curve, will be properly defined using differential expressions and conditions, as elements or "points" of the variety representing the groups of m points of the curve. (The variety representing m-tuples is just the mth symmetric power of the curve which we called S_m above. I think that when he talks here of differential conditions, Enriques foreshadows the basic idea behind nonreduced schemes.)

Let us suppose that the curve has genus p and let us consider, in particular, the Jacobian variety

V that represents the groups of p points, G, of that curve. (As Enriques knew well, the pth symmetric power of the curve is not exactly the Jacobian but it is birational to it, that is, they are the same on Zariski open sets. In fact, on the Zariski open set of "nonspecial" p-tuples, those which do not move in positive dimensional linear systems $|G_p|$, the two are the same and for special p-tuples, there is still a map from S_p to the Jacobian V but it is many-to-one, blowing down linear systems to points. Using such p-tuples on C to describe almost all of the Jacobian was used by Poincaré and will be used by Enriques in a very crucial way below.)

It is known that this V has a continuous group ∞^p of commuting birational transformations that one defines on the curve using the sum of a difference $G_1 - G$ of two groups. (He is just saying that the group of divisor classes, that is, divisors mod linear equivalence, is a *group*, hence the Jacobian is a group as well as a projective variety, that is, an abelian variety. But he prefers to use the birationally equivalent model given by the pth symmetric power, and here the group law is only defined on a Zariski open set and, more precisely, if G', G, and G_1 are three groups of p points, then for almost all G', $G' + G_1 - G$ will be linearly equivalent to a unique p-tuple G'', hence $G' \to G''$ defines a birational map on p-tuples but not an everywhere defined morphism.)

Now, if the second group G_1 comes infinitely close to the first, the operation $+G_1 - G$ does not cease describing a transformation of V, more precisely (it is) an infinitesimal transformation that is a generator of the group in the sense of Sophus Lie. By means of this infinitesimal transformation, the group of points G gives rise to an analytic series of transformations by which will be properly defined the groups $G_2, G_3, \ldots$, successive to G_1 which fall in the second, then in the third neighborhood of G and so forth. (Enriques's "analytic series" are what we call 1-parameter subgroups $\phi(t)$ in the Jacobian. In the complex torus representation of the Jacobian, they are just straight lines through the origin. $C_1 - C$ defines a tangent vector at the origin, hence a specific ϕ, which, considered to order n, are his higher order infinitesimals on the Jacobian and define what he calls $G_2, G_3, \ldots$.).

After that it is clear how one treats infinitely close complete linear systems g^r_m on a curve (The notation g^r_m simply means a complete linear system of m-tuples of dimension r. It is "nonspecial" if $r = n - p$, the generic case whenever $n \geq p$.)*: at least in the simplest case of a nonspecial series, their sum and difference can be reduced to the sum and difference of the groups of p points that one gets from them as residuals of $m - p$ fixed points. If one wants to operate on special series, it is convenient to extend them by adding fixed points so they become nonspecial.*

But it's enough to restrict ourselves to this: given on a curve K a nonspecial series $g^r_n = g$ and a series $g_1 = (g^r_m)$ infinitely close to it (defined, as we've said, on the representing variety (the representing variety being the symmetric power. In fact, all he's going to use is the basic case of nonspecial p-tuples, or g^0_p which, as we said, gives a Zariski open subset of the Jacobian.)*, there is determined a continuous (analytic) series of series and in it the series successive to g_1, namely $g_2, g_3, \ldots$ neighboring g in the neighborhoods of second, third, etc., order.*

In place of the operation $+G_1 - G$, one can equally carry out on groups of p points of the curve (or on its nonspecial series of a given order) the inverse operation: $+G - G_1$. This defines, starting from G, a continuous (analytic) series of groups of p points complementary to $G_1, G_2, G_3, \ldots$ that we can denote $\overline{G}_1, \overline{G}_2, \overline{G}_3, \ldots$ where in general one has

$$\overline{G}_i \equiv 2G - G_i.$$

To make this as concrete as possible, I think what Enriques has in mind is that you start at some nonspecial p-tuple of points $G = \sum_{i=1}^p P_i$ on C (so that the pth symmetric power and the Jacobian are locally isomorphic near G). You can take any p-tuple G_1 in its first-order neighborhood, and this defines a tangent vector to the Jacobian at the point defined by G. Then, knowing that the Jacobian is a complex torus, you get a 1-parameter subgroup of the Jacobian. In the pth symmetric power this gives power series $P_i(t)$ with $P_i = P_i(0)$ so that

$$t \mapsto G(t) = P_1(t) + \cdots + P_p(t)$$

represents a 1-parameter subgroup of the Jacobian. This was classical stuff, the then century-old theory of abelian functions if you take as its beginning Abel's 1829 paper in Crelle's Journal. One could write down these power series using abelian functions. Then his G_ℓ and $\overline{G}_\ell$ are "points" on this analytic branch of order ℓ for $t > 0$ and $t < 0$, respectively.

The "Pencil" Construction of Poincaré and Enriques

Instead of beginning with a translation of Enriques's second argument for the Fundamental Theorem, which follows and certainly has some gaps, it seems better to first present a modern variant that will show that his argument is fundamentally sound and that will make explicit its links to the normal function technique of Poincaré. Then the reader can see why Enriques chose a somewhat different, more classical route but one that unfortunately ran into some difficulties. The core of Enriques's second argument introduces an idea that originated in Poincaré's analytic approach: the use of a suitably general *pencil* on the surface

F so as to construct curves on F by sweeping out finite groups on the curves of the pencil.

What is a pencil? Simply put, one takes any nonconstant rational function f on F (e.g., one of its coordinate functions x_i) and considers the one-dimensional family of level curves $f = t$ (including $f = \infty$, the poles of f): call this C_t. All the curves C_t are linearly equivalent and form a projective line in the linear system $|C_0|$ (which, as we said, is a projective space). The function f will not be defined at finite set points $\{P_1, \ldots, P_d\}$ where all the curves C_t intersect: these are called the *base points* of the pencil. One can also consider the closure of the graph of f in $F \times \mathbb{P}^1$: this will be a surface F^* mapping to F by a birational map in which the base points P_i have each been replaced by projective lines called the exceptional curves E_i. In the generic case, the curves C_t meet transversely at each P_i and F^* is smooth and is the standard surface obtained by blowing up each P_i. Now f becomes an everywhere defined map $f^* : F^* \to \mathbb{P}^1$ with disjoint fibers C_t.

Poincaré and after him Enriques used the Jacobian varieties J_t of each curve C_t. (When C_t is singular, one uses the so-called "generalized Jacobian" of C_t.) The dimension of J_t is the genus of C_t, which we denote by p. We may set up a one-to-one correspondence between the points of the Jacobian J_t and the divisor classes of degree 0 on the curve C_t, and this correspondence will be set up by a "universal" divisor D_t on $C_t \times J_t$, i.e., $D_t.(C_t \times \{a\})$ represents the divisor class corresponding to a. In the classical approach, D_t is readily defined using abelian functions. It is then convenient to glue the J_t together, forming a variety $\mathcal{J}$ of dimension $p + 1$, which maps to $\mathbb{P}^1$ with fibers the individual Jacobians J_t. Then the union of all the products $C_t \times J_t$ forms a variety $F^* \times_{\mathbb{P}^1} \mathcal{J}$ of dimension $p + 2$ and the D_t's glue together to one big divisor $\mathcal{D}$ on this product.[8]

Taking some sufficiently ample curve D on F, Enriques's old 1904 argument constructed for him $q = p_g - p_a$ independent infinitely close curves D_1 in the first-order neighborhood of D. More precisely, he took the complete characteristic series $|(D.D)|_D$ and for any $a \in L_D(D.D)$, he defines an infinitely close curve $D_1^{(a)}$ in the first-order neighborhood of D. He wants to prolong these to infinitely close curves of higher order as the key step in showing that dim(Pic)= q. It is more natural today to consider the difference $D_1^{(a)} - D$ as defining a divisor on the scheme $F \times \mathrm{Spec}(k[t]/(t^2))$, which is trivial on F itself. (I think it is correct to say that although the Italian school was well aware that one could form divisors with negative coefficients, they strongly preferred to deal with positive divisors, and hence were averse to this step.)

We can intersect both D and $D_1^{(a)}$ with all the members of the pencil C_t. If m is the intersection number $(D.C_t)$, we get groups of m points $G_t^{(m)} = D.C_t$ and infinitely close groups $G_t^{(a,m)} = D_1^{(a)}.C_t$ on each C_t. As Enriques pointed out above, quoting Lie, the difference $G_t^{(a,m)} - G_t^{(m)}$ defines a tangent vector $v_t^{(a)}$ at 0 to the Jacobian variety J_t. All the tangent vectors $v_t^{(a)}$ together form a vector field to $\mathcal{J}$ along the zero-section of $\mathcal{J}$ over $\mathbb{P}^1$. In the language of schemes, such a vector field is the same as a morphism $f_1^{(a)} : \mathrm{Spec}(k[t]/(t^2)) \times \mathbb{P}^1 \to \mathcal{J}$ with $f_1^{(a)}(\mathrm{Spec}(k[t]/(t^2)) \times \{t\}) \subset J_t$. Note that $D_1^{(a)} - D$ can be recovered from the universal divisor $\mathcal{D}$ using the morphism $f_1^{(a)}$, that is, $D_1^{(a)} - D$ (lifted to F^*) is just the "pullback" of $\mathcal{D}$ via the morphism $1_{F^*} \times f_1^{(a)} : F^* \times \mathrm{Spec}(k[t]/(t^2)) \to F^* \times_{\mathbb{P}^1} \mathcal{J}$ (possibly up to adding some multiple of C_t).

Now J_t is known to be a complex torus. So through any tangent vector at the origin such as $v_t^{(a)}$, there is a straight line, that is, a one-parameter subgroup of J_t. Truncating this at nth order, we get canonical morphisms $\mathrm{Spec}(k[t]/(t^{n+1})) \to J_t$ for each t extending the vector $v_t^{(a)}$. It is clear that these vary at least analytically as t varies. But in fact, they fit together into an *algebraic* map $f_n^{(a)} : \mathrm{Spec}(k[t]/(t^{n+1})) \times \mathbb{P}^1 \to \mathcal{J}$ with $f_n(\mathrm{Spec}(k[t]/(t^{n+1})) \times \{t\}) \subset J_t$. This is an easy consequence of the elementary fact that meromorphic functions on $\mathbb{P}^1$ are all rational,[9] using crucially the fact that the construction works for all t with no exceptions. Finally we can define the sought-for infinitely close $D_n^{(a)}$ of higher order (as a divisor class) as D plus the pullback of $\mathcal{D}$ via $1_{F^*} \times f_n^{(a)}$. Using the Riemann-Roch theorem, we can show it is represented by a positive divisor.

How does this connect to Poincaré's argument? Although he came to it from a completely different route, he used the noninfinitesimal points on the one-parameter subgroups $\exp(s.v_t^{(a)}), s \in \mathbb{R}$, to construct global divisor classes on F. Without tracing all the links, let us just say that he constructed an explicit basis of 1-forms $\{\omega_t^{(k)}\}$ simultaneously on all but one of the curves C_t such that the vector fields $v_t^{(a)}$ have constant inner product zero with all of them, zero with $p - q$, and arbitrary constants with the remaining q. Moreover, fixing one of the base points x_0 of the pencil, D_t is given in the classical way by the divisor

[8] *Technical aside: any of the exceptional curves gives a section of F over $\mathbb{P}^1$ that serves to "rigidify" the relative Picard functor. Anyway, standard abelian functions define it, too.*

[9] *The most general results of this type were given by Serre in his famous "GAGA" paper "Géométrie algébrique et géométrie analytique",* Annales de l'Institut Fourier, *1956.*

$(\sum_{i=1}^{p} x_i) - p.x_0$ on C_t defined over $\{c_k\} \in J_t$ by:

$$\left(\sum_{i=1}^{i=p} \int_{x_0 \in C_t}^{x_i \in C_t} \omega_t^{(k)} \right) = c_k.$$

My guess is that Enriques knew he was dealing with the same approach as Poincaré to a certain extent, though he may not have realized how close his infinitely close curves $D_1^{(a)}$ were to Poincaré's basis of 1-forms.

The Translation, Part III, the Use of a Pencil

Section 3 of his paper deals with representing the surface F as a branched cover of the projective plane $\mathbb{P}^2$ and considering infinitely close curves on F via their images in $\mathbb{P}^2$. We have quoted above one sentence in which he alludes to parameterizing curves (and divisors) on a surface F by some auxiliary variety. Otherwise the section does not seem to add very much, and we omit it.

Section 4 deals with the injectivity of the map from the group of divisors on F mod linear equivalence (the Picard group of F) to the group of divisors mod linear equivalence on a curve K in F, given by intersection with K. In particular, he asserts that for suitable K, a nonzero infinitesimal divisor class $C_1 - C$ should have nonzero intersection with K. This is closely related to the lemma of Enriques-Severi that, in cohomological terms, asserts that $H^1(\mathcal{O}(-K)) = (0)$ if K is a sufficiently ample divisor. It was proven to modern standards by Zariski in 1951 in the *Annals of Math.*, volume 55, and treated cohomologically in Serre's fundamental paper "Faisceaux Algébriques Cohérents", *Annals of Math.*, volume 61, 1955. I omit this section, too.

Below I will translate section 5, in which a pencil of curves on the surface F is introduced so that he can extend a curve $C \subset F$ to infinitesimal neighborhoods by extending the divisors $C.K_t$ on each K_t.[10]

On the surface F we can choose two linear systems of regular (irreducible) curves $|C|$ and $|K|$ in such a way that the curves C cut nonspecial series on the curves K; it suffices to suppose $|C|$ sufficiently ample with respect to $|K|$, as, for example, by assuming it contains a multiple of $|K|$.

This assumed, let C and C_1 be two nonequivalent infinitely close curves, which certainly exist if the surface is irregular ($p_a < p_g$). By the theorem demonstrated in the preceding section, C and C_1 will cut inequivalent groups of m points $G = G^{(m)}$ and $G_1 = G_1^{(m)}$ on K, which will define two different complete nonspecial series g and g_1; consequently, thanks to the operation $+g_1 - g$, one will construct a continuous (analytic) series of inequivalent linear series in which one finds the series (always nonspecial) $g_2, g_3, \ldots$ infinitely close to g in neighborhoods of second, third, etc., order.

One chooses on a K a group G^n of the characteristic series and in this a point A. (From the context in the previous omitted section, it is clear that he is choosing here a *pencil* $\langle K_1, K_2 \rangle$ from the linear system $|K|$ with base points $G^n = K_1 \cap K_2$ plus a specific base point $A \in G^n$. Also the number π introduced below is the genus of K. In what follows, he will make constructions on arbitrary curves K in this pencil and then take their loci as the curve varies in the pencil. To do this he wants canonical groups on each K_t, not just groups given up to linear equivalence. So he constructs next groups of degree π which—if nonspecial—are unique up to linear equivalence.)

The point A counted $m - \pi$ times will determine a group G^m of the series g (This is the unique group in the linear system g of the form:

$$G^m = (m - \pi)A + G^\pi, \; G^\pi = \pi \text{ further points.}$$

This constructs a *canonical* group of points G^π representing the divisor class. Some argument is needed to check that G^π is indeed nonspecial.) *and similarly a group G_1 of m points of the series g_1 and then a group of g_2 and so on. The loci* (as K varies in the pencil) *of the groups G_k^m defined in this way will be curves $L, L_1, L_2, \ldots$ of the same order, passing a certain number i times through the points of G^n and $m - \pi + i$ times through A. In fact, it's easy to check that if L touches a particular K of the pencil with base G^n, so that the corresponding generator G^m has on this K a point coinciding with a point of G^n, the same thing happens for L_1 and for L_2, etc.*

There is a major gap in his argument here as he doesn't make precise in any way what "the loci of the groups G_k^m" means when the groups are infinitely close. The locus L of the groups of ordinary points G^m on the members of the pencil, that is, the case $k = 0$, is clearly a good algebraic curve, but what are the loci when $k \geq 1$?

As an aside, I want to explain the technical point of where the integer i comes from. It is easier to follow if as above we blow up on F the base points G^n, giving a surface F^*, which is now fibered over $\mathbb{P}^1$ by the pencil of curves K_s. We can certainly consider L^*, defined as the locus of G^π on all the fibers K in the blown-up surface. Moreover, if A^* is the exceptional curve that is the blowup of the point A, from the definition of L^*, we will have a linear equivalence

$$L^* + (m - \pi)A^* \equiv C + iK \text{ for some } i.$$

Since $(C.A^*) = 0$, $(L^*.A^*) = i + m - \pi$, which is why he says that L has an $m - \pi + i$-fold point at A.

This is a reasonable construction showing that the locus of the ordinary groups G^m is an ordinary

[10] *I'd like to thank Michael Artin for his help in understanding this argument of Enriques and especially for pointing out the problem of special groups on some members of the pencil.*

curve L. *BUT* Enriques claims without any discussion that this works for the infinitely close groups $G_1, G_2, \ldots$, sweeping them out to infinitely close curves $L_1, L_2, \ldots$. Here he is assuming that an operation that works for ordinary curves also works for the infinitely close ones. But, worse than that, he has defined the infinitely close groups G_k^π on each K_s in the pencil, successive to G^π, by analytic means, and he needs infinitely close algebraic curves L_k successive to L. To be as concrete as possible, if $G^m = \sum_{i=1}^m P_i(s)$ on the curve K_s, then we can imagine that the infinitely close group G_k^m is defined by the kth-order terms in t of points given by power series $P_i(s,t)$, $1 \le i \le m$, for $s \in \mathbb{P}^1 - S$, $|t| < c(s)$ but allowing for some finite set S of possibly "bad" points where either (a) L has a branch point so the P_i's interchange, (b) G^m is special, or (c) the curve K_s is singular. Enriques needs to define G_k^m for all s in order to prove he has an infinitely close *algebraic* locus. This problem was apparently raised by B. Segre. Enriques discusses this criticism in his later 1938 memoir "Sulla proprietà caratteristica delle superficie algebriche irregolari" (*Rendiconti della Accademia dei Lincei*, volume 27, pp. 493-498). He asserts here (p. 497) that this extension is truly algebraic. Actually I don't think (a) or (c) is a real problem, but (b) certainly is. It is not clear (to me) whether for generic pencils there will be curves K_s where G^π is special. Enriques addressed this briefly in his final book *Le Superficie Algebriche*, p. 336, pointing out that the set of special divisor classes of degree π has codimension two but not saying why this locus can be avoided by the curves in a generic pencil. Such points s may mean that the infinitely close curves L_k must be viewed as deformations of L *plus* a sum of special fibers K_{s_i} and showing that the whole mess is algebraic is not simple. What Enriques missed here is that everything is simpler if you use divisor classes of degree zero instead of positive divisors of degree π and use the existence of a universal divisor on the Jacobian as sketched in the previous section.

With the preceding construction we have defined a curve L (belonging to the linear system $|C + iK|$ and curves $L_1, L_2, L_3, \ldots$ infinitely close to it in the neighborhoods of order $1, 2, 3, \ldots$ as far as one wants, whose real existence is thus demonstrated.

That shows that the curve L_1 infinitely close to L is close to L in a continuous ∞^1 series of inequivalent curves; and since L_1 is substantially an arbitrary curve infinitely close to L, inequivalent to it, this proves that the linear system $|L|$ belongs to a continuous system $\{L\}$ that has as characteristic series the complete characteristic series on the curve L.

The Translation, Part IV, "Algebraization"

Now Enriques comes to the final key idea in his argument, the use of reducible curves in order to extract continuous nonlinear systems of curves from within linear systems:

But whoever looks at the demonstration with critical eyes, as is advisable with reasoning of this nature, will ask not only for the explicit proof that truly the L_1 that we constructed is an arbitrary member of the system of inequivalent curves infinitely close to L, but also that L_1 and then $L_2, L_3, \ldots$ are effectively curves infinitely close to L in the sense that we defined in the section "The Translation, Part I", and Enriques's First Argument, since indeed the construction of these curves $L_1, L_2, L_3, \ldots$ appears to be something different from that definition.

(The question is why the series of higher order infinitely close deformations of L is contained in an *algebraic* family of curves. Today we would only need to cite the existence of the Hilbert scheme. If we have deformations of a curve L to arbitrarily high order, there has to be a component of the Hilbert scheme giving not merely infinitesimal deformations but global ones, thus defining a continuous system of curves containing L and L_1. Enriques, however, found an elementary way to do this:)

To respond to the doubt so raised, one considers the linear system of the sum $|L+C|$ and inside it one considers the curves infinitely close to a reducible curve: they cut the L_1 that we have constructed in as many points as they cut L and so containing L_1 as a component imposes the same number of conditions (the dimension of a special series of the same order plus one): one must conclude that among the curves infinitely close to $L + C$ in the given system, there are curves made up of L_1 (defined as always as a curve infinitely close to L in the sense of the section "The Translation, Part I", and Enriques's First Argument) and a $\overline{C}_1$ infinitely close to C. Concerning this $\overline{C}_1$, one can say that it cuts a group $\overline{G}_1$ on K of a series complementary to that defining the group G_1, the section of C_1 (which was an arbitrary inequivalent curve in the neighborhood of C): indeed, designating with G the group $(C.K)$, one has:

$$G_1 + \overline{G}_1 = 2G.$$

As a consequence, $\overline{C}_1$ is, like C_1, an arbitrary curve among those inequivalent neighboring C: since if one takes $\overline{C}_1$ in the place of C_1, one finds C_1 in the place of $\overline{C}_1$.

Now the reasoning which precedes extends to all the curves infinitely close in neighborhoods of higher order. Among the curves of the linear system $|L + C|$, infinitely close $L_1 + \overline{C}_1$ (i.e., in the second-order neighborhood of $L + C$), one will find reducible curves that contain as a component the L_2, constructed above, and another component $\overline{C}_2$ neighboring $\overline{C}_1$ and successive to C. And continuing, one will find curves infinitely close to C belonging to neighborhoods of appropriate heights that extend $\overline{C}_1$, that is—as has been said—to an *arbitrary inequivalent curve infinitely close C. This shows that these infinitely close curves which cut the complete characteristic series on C are curves belonging to an effective continuous series, and thus that the linear system $|C|$ is contained in a continuous system $\{C\}$ that has on C a complete characteristic system and so is made up of $\infty^{p_g - p_a}$ inequivalent systems. q.e.d.*

The key observation is that the family L_k of deformations of L constructed earlier to all infinitesimal orders can be algebraicized to a family of ordinary curves on F by considering the family of *reducible* curves in the linear system $|L + C|$. He argues that this linear system contains curves of the form $L_k + C_k$ and these must lie in an algebraic family of linearly equivalent curve $L_t + C_t$. Taking either the system $\{L_t\}$ or the system $\{C_t\}$, one sees that the dimension of the Picard variety is indeed $p_g - p_a$.

Summary

Where should we place Enriques if we seek to summarize how algebraic geometry developed in the twentieth century? That he built a comprehensive theory of algebraic surfaces and their classification by what we now call Kodaira dimension is clear. But he is also a transition figure between the age of the classical geometry of varieties and linear systems and the modern period of schemes and cohomology. This transition was not marked at first by the discovery of new theorems but rather by the creation of whole new vocabulary and the toolkit that went along with this. Transitions of this kind may often look as if they appear out of nowhere, but this is rarely the case. Many of the ideas that came to full flower in the 1960s were "in the air" before then. The obstacle to their creation was one of naming, of admitting that it's going to be easier to understand some circle of ideas if you make some vague thing you're working with into a tangible object—reifying something dimly seen.

An example from the topic of this paper is that the characteristic linear system of a linear system on a surface F, if it is not enlarged to a complete continuous system, is incomplete. That is, given a curve C on a surface F and a divisor D on F, it was clear to all classical geometers that there were rational functions on C with poles bounded by $(D.C)$ that did not lift to rational functions on F with poles bounded by D. But the idea that you should give a name (i.e., H^1) to the cokernel:

$$\{\text{fcns. on } C, \text{ poles at } (D.C)\} \textit{ mod} \\ \{\text{restrictions of fcns. on } F, \text{ poles at } D\}$$

was simply not the sort of thing they ever considered. Naming such cokernels came out of algebraic topology and was transplanted into algebraic geometry by Serre. This immediately systematized large areas of classical geometry.

Enriques's particular insight, however, was that he saw that there was a calculus of infinitesimal deformations of subvarieties. Although he gave these names, they remained in a limbo, without substance, *because he did not think of what it meant to have a function on them.* Grothendieck realized that functions on such objects should be rings with nilpotent elements, and this gave life to these infinitesimal deformations. He reified them as ringed spaces, and with the word Spec.

But reification never happens in a vacuum: there has to be a clear need for it, an intuition that has leaped ahead of the available tools. This Enriques had. The proof discussed above is a wonderful example of how, before the new system is invented, an ingenious mind can limn out what the new structure should look like.

So far, we have been emphasizing Grothendieck's theory of schemes and his existence theorems for the Hilbert and Picard schemes that make the Fundamental Theorem seem extremely easy. But we also know more today because Zariski and Weil introduced the parallel world of varieties over fields of finite characteristic. In this world, the Fundamental Theorem that $q = p_g - p_a$ is *false*: many varieties in characteristic p have a nonreduced Picard scheme. This allows us to trace the ideas behind the various attempts to prove the Fundamental Theorem to see where they use the essential hypothesis that the characteristic of the field is zero.

If we study Enriques's intuitive proof and try to make sense of it, as we did in the section "The Translation, Part I", and Enriques's First Argument, the characteristic zero hypothesis comes in through the use of the power series for the exponential function, since that requires dividing by $n!$. More generally the exponential function is the key ingredient in the theorem that all group schemes in characteristic zero are reduced. But Enriques's proof above, in the section "The Translation, Part III, the Use of a Pencil", was different. He used instead the well-established theory of the Jacobian variety. Being a complex torus, it had straight lines through the origin that define 1-parameter subgroups in every direction. This also does not hold in characteristic p: for instance, there are no 1-parameter subgroups of the formal two-dimensional multiplicative group with transcendental slope. The idea of using such subgroups is very ingenious.

In short, Enriques was a visionary. And, remarkably, his intuitions never seemed to fail him (unlike those of Severi, whose extrapolations of known theories were sometimes quite wrong). Mathematics needs such people—and perhaps, with string theory, we are again entering another age in which intuitions run ahead of precise theories.

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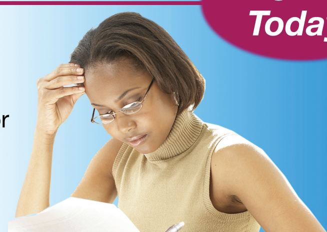

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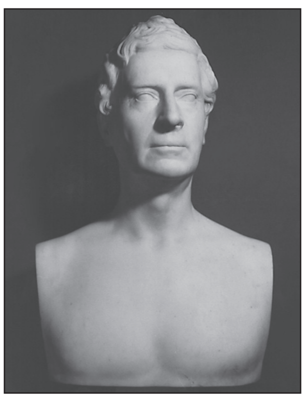
The Contribution of John Parker Jr. to American Mathematics

Steve Batterson

John Parker Jr. was a wealthy Boston merchant who died in 1844. There is no record of Parker having had any training in mathematics. Nor is there any reason to believe that Parker had any special interest in the subject. Yet a bequest written into his will in 1841 played a crucial role in enabling the advanced education of many pioneers of American mathematics.

The rise of American mathematics, begun in the last decade of the nineteenth century, was led by Americans who were profoundly influenced by their study of the subject in Europe [Parshall and Rowe, 1994]. After their return to the United States, these scholars became established at universities, where they launched their own research programs and proselytized the concepts learned in Europe. As higher mathematics diffused across the United States, the country moved toward self-sufficiency and some distinction in the subject.



John Parker Jr.

The first American graduate programs to turn out strong mathematicians were at the University of Chicago and Harvard. Chicago was led by E. H. Moore [Parshall and Rowe, 1994] and Harvard by Maxime Bôcher and W. F. Osgood [Batterson, 2009]. Each of these men had studied in Germany, where the annual expense of about $800 posed a severe challenge to ordinary families. Moore obtained his Ph.D. under H. A. Newton at Yale in 1885. Newton then loaned Moore money to continue his studies in Berlin. Bôcher and Osgood were even more fortunate. As Harvard students, they were eligible to compete for a Harvard traveling fellowship (Htf) that supported study abroad. Both received awards to pursue doctoral degrees at Göttingen.

Bôcher and Osgood returned from Germany to begin distinguished careers at Harvard. Other Htf recipients went on to lead departments at Berkeley, Rice, UCLA, and elsewhere. One striking aspect of the *vitae* of the pioneers of American mathematics is indeed the prevalence there of Htfs. For example, the four nineteenth-century American Mathematical Society (AMS) Colloquium Lecturers included the Htf holders Bôcher, Osgood, and physicist Arthur Webster. Table 1 lists all Htf awardees in mathematics (plus Webster and mathematical physicist B. O. Peirce) through 1910. The last column indicates membership in the National Academy of Sciences (NAS) and the AMS distinctions of colloquium speaker and executive office.

The Htfs were supported by various endowments. The penultimate column of Table 1 indicates the particular Htf held by each individual. Note that the Parker Fellowship was especially important to mathematics. Harvard obtained the funds to endow the Parker Fellowship in 1873 following the death of Anna Parker. The will of Ms. Parker's late husband, John Parker Jr., stipulated that the fellowships were to support the education, at home or abroad, of exceptional students. Harvard programmed the awards for use by their graduates as traveling fellowships. Holders of the Parker Fellowship went on to study under Felix Klein, Sophus Lie, David Hilbert, and Hermann Minkowski. This paper tracks the impact of the Htfs on American mathematics and investigates several underlying questions: Who was John Parker Jr., and what was the motivation behind his munificence to Harvard and his support for advanced study? Why did Harvard, which had just begun its own graduate program in 1872, direct so many resources to study abroad rather than at home?

Forerunners of the Htf

In the time of John Parker Jr., United States scholarship, in all subjects, was inferior to that in Europe. American doctoral education only began in 1860 with the creation of a program at Yale. The absence of indigenous Ph.D. programs made it difficult for early nineteenth-century American universities to obtain competently trained faculty. One solution was to assist promising instructors to make the arduous trip across the Atlantic. In

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Name (began Htf)	Bachelor	Ph.D.	Postdoc	Faculty	Htf	NAS, AMS Position
W. E. Story (1874)	Harv 1871	Leipzig 1875		Hopkins, Clark	Parker	NAS, VP
B. O. Peirce (1877)	Harv 1876	Leipzig 1879	Berlin	Harv	Parker	NAS, VP
W. I. Stringham (1880)	Harv 1877	Hopkins 1880	Leipzig	Berkeley	Parker	VP
F. N. Cole (1882)	Harv 1882	Harv 1886	Göttingen	Mich, Columb	Parker	Sec, VP
M. W. Haskell (1885)	Harv 1883	Göttingen 1889		Berkeley	Parker	VP
A. G. Webster (1886)	Harv 1885	Berlin 1890		Clark	Parker	NAS, Clq
W. F. Osgood (1887)	Harv 1886	Erlangen 1890		Harv	Harris, Parker	NAS, Clq, Pres, VP
M. Bôcher (1888)	Harv 1888	Göttingen 1891		Harv	Harris, Parker	NAS, Clq, Pres, VP
J. B. Chittenden (1891)	Harv 1889	Königsberg 1893		Brook Poly	Kirkland, Parker	
C. L. Bouton (1896)		Leipzig 1898		Harv	Parker	
E. R. Hedrick (1899)	Mich 1896	Göttingen 1891		Misso, UCLA	Parker	Pres, VP
C. N. Haskins (1901)	MIT 1897	Harv 1901	Göttingen	Dartmouth	Harris	VP
J. W. Bradshaw (1902)	Mich 1900	Strasburg 1904		Mich	Parker, Kirkland	
D. R. Curtis (1903)	Berk 1899	Harv 1903	Paris	Northwestern	Parker	VP
E. Swift (1904)	Harv 1903	Göttingen 1907		Princ, Vermont	Parker	
L. A. Howland (1906)	Wesl 1900	Munich 1908		Wesleyan	Parker	
W. A. Hurwitz (1908)	Misso 1906	Göttingen 1910		Cornell	Parker, Harris	
D. Jackson (1909)	Harv 1908	Göttingen 1911		Harv, Minn	Rogers, Hooper	NAS, Clq, VP
G. C. Evans (1910)	Harv 1907	Harv 1910	Rome	Rice, Berkeley	Sheldon	NAS, Clq, Pres, VP

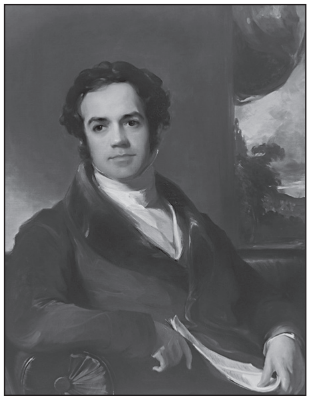
Table 1. Harvard traveling fellowship (Htf) in, or related to, mathematics through 1910.

1805 Yale began these study-abroad opportunities when it supported the education in Edinburgh of its science professor, Benjamin Silliman. Ten years later Harvard first availed itself of the superior educational resources abroad when a new chair in Greek Literature was accepted by Edward Everett [Morison, 1936, 224]. To upgrade the young professor's preparation, the offer came with a special provision for Everett to study in Europe for two years on full salary.

Everett was a gifted scholar who had received his Harvard A.B. in 1811 at the age of seventeen [Frothingham, 1925]. He remained at the university for the next two years, preparing for the ministry and teaching Latin. Everett was then selected to assume the prestigious pulpit of the Brattle Street Church. Despite the security and standing of his position, the opportunity for European study and an academic life held considerable appeal to Everett. He resigned from the Brattle Street ministry.

Joining Everett on the four-week transatlantic voyage was his friend George Ticknor. Ticknor was the son of a prosperous Boston grocer [Tyack, 1967]. George had obtained a degree from his father's alma mater, Dartmouth, and then returned to Boston, where he received tutoring in the classics. A subsequent career in law proved less stimulating for him than literature. With his father's support, in 1815 Ticknor set out to study at Göttingen [Ticknor, 1909, 49].

Everett and Ticknor landed in Liverpool and learned that their continental travel plans were in jeopardy [Frothingham, 1925, 36–38]. Napoleon had just returned to power, and war seemed imminent. The young Americans sat out the uncertainty in London, where they met Lord Byron and other distinguished British figures. After Napoleon was defeated at Waterloo, Everett and Ticknor made their way to Göttingen. The journey transported them into an intellectual environment vastly superior to Harvard and Dartmouth.



George Ticknor

To compare resources, the 200,000 volumes in the Göttingen library were more than ten times the holdings of Harvard [Long, 1935, 12]. The difference in erudition of the faculties was unquantifiable. In a letter to his father on November 15, 1815, Ticknor thus discussed his Greek professor Ernst Schultze:

> Every day I am filled with new astonishment at the variety and accuracy, the minuteness and readiness, of his learning. Every day I feel anew, under the oppressive weight of his admirable acquirements, what a mortifying distance there is between a European and an American scholar! We do not yet know what a Greek scholar is; we do not even know the process by which a man is to be made one. I am sure, if there is any faith to be given to the signs of the times, two or three generations at least must pass before we make the discovery and succeed in the experiment. Dr. Schultze is hardly older than I am…. It never entered into my imagination to conceive that any expense of time or talent could make a man so accomplished in this forgotten language as he is [Ticknor, 1909, 73].

The learned Göttingen faculty, headlined by Carl Friedrich Gauss and Johann Friedrich Blumenbach, covered a broad spectrum of scholarship. Both Everett and Ticknor immersed themselves in study while taking courses and receiving individual instruction. Most of their work involved philology, but the Americans delved into related topics and attended lectures by Blumenbach on natural history [Ticknor, 1909, 79–80], [Long, 1935, 65–66]. On vacations they traveled, carrying letters of introduction from their professors. In Weimar, Everett and Ticknor had discussions with Goethe [Long, 1935, 27].

The descriptions by Everett and Ticknor of their educational experiences made profound impressions on the American recipients of their letters. Among the Harvard correspondents were President John Thornton Kirkland and former Latin tutor Joseph Cogswell. Cogswell, a close friend of Ticknor's, was inspired to travel to Göttingen for study the following year [Ticknor, 1874]. When a new Harvard chair was endowed in French and Spanish, the call went to Ticknor, who was not trained in either subject [Tyack, 1967, 62].

Everett received his Ph.D. in 1817 and was granted two additional years of leave for travel in Europe [Morison, 1936, 226]. He proposed to Kirkland that Harvard cultivate its philology faculty by sending a recent graduate to Göttingen for further study. Kirkland approved the plan, arranging an ad hoc scholarship for George Bancroft [Howe, 1908], who lacked the wherewithal of Ticknor and Cogswell [Howe, 1908, 32–33].

Over the period 1819–1822, Everett, Ticknor, Cogswell, and Bancroft returned to Harvard [Morison, 1936, 226–228]. Each hoped to pass on his newfound learning. The results were at best mixed. As an ambitious instructor of Greek, Bancroft was ridiculed for the European airs he affected. Cogswell fared little better. At Göttingen he had acquired an interest in library science. Harvard hired him as college librarian. Cogswell's efforts to implement German library methods and values met with resistance from the Harvard administration [Ticknor, 1874].

Frustrated by Harvard's intransigence, Cogswell and Bancroft concluded that movement on their agenda required a new institution [Long, 1935, 92–95], [Ticknor, 1874, 134–137]. Together they established a secondary school, the Round Hill School, with the objective of introducing European methods of learning to upper-class New England teenagers. The noble experiment lasted eight years. Then, going their separate ways, Cogswell and Bancroft both went on to noteworthy accomplishments.

Cogswell persuaded John Jacob Astor to fund what became the New York Public Library. As its first librarian, Cogswell acquired its collection and prepared the catalogue. Bancroft entered politics and served in the cabinet of President James Polk and later as minister to Great Britain and to Germany. His books on colonial history were highly regarded in the nineteenth century. Despite his disastrous teaching stint at Harvard, Bancroft would remain forever grateful for the Göttingen study opportunity made possible for him by President Kirkland.

With endowed chairs, Everett and Ticknor were better positioned to bring new ideas to Harvard. Their European experiences animated scholarship never before available on the campus [Morison, 1936, 227–230]. Everett's course on the history of Greek literature impressed at least one young junior in the college. Ralph Waldo Emerson wrote

"There was an influence on young people from the genius of Everett which was almost comparable to that of Pericles in Athens" [Frothingham, 1925, 63].

Everett's brilliance, eloquence, and connections caused his name to arise for other prestigious positions. Something about his makeup made it difficult for Everett to find contentment. He remained a Harvard professor for five years. Subsequent titles included United States congressman, Massachusetts governor, minister to Great Britain, Harvard president, United States secretary of state, and United States senator. In 1860 Everett was the candidate for vice president on the Constitutional Union Party ticket headed by John Bell.

Of the four students who ventured abroad, Ticknor had the longest and greatest impact on Harvard [Morison, 1936, 230–238]. His extensively researched courses on Spanish and French literature brought new areas of learning to the college. Over his sixteen years on the faculty, Ticknor agitated for educational reforms to serve strong students. His success in this endeavor, although limited, advanced the study of modern languages at Harvard.

Both Ticknor and Everett married into families of considerable wealth. Everett's wife, Charlotte, was the daughter of Peter Brooks, the richest person in Boston. Ticknor's father-in-law, Samuel Eliot, endowed the chair that brought Everett to Harvard. In Jacksonian Boston abundant endowments of wealth, learning, and refinement combined to confer lofty status. With personal friendships that included Thomas Jefferson, the Adamses, Daniel Webster, and Lafayette, Ticknor and Everett were two of the most respected figures in the city. Both lived in the fashionable area northeast of the Boston Common. Regular gatherings in Ticknor's home at Park and Beacon Streets were prestigious social events [Tyack, 1967].

The Bequest of John Parker Jr.

The family fortune that made possible John Parker Jr.'s surprising bequest began with his father, wholesale merchant John Parker. Following the Revolutionary War the port of Boston pulsed with commercial opportunities. Peter Brooks, for example, made his fortune in marine insurance. Operating out of a store on Long Wharf, the elder Parker built a lucrative commission business. Early in the nineteenth century he brought his first two sons—John Jr. and Peter—into the business. The next child was a daughter, and three more Parker boys were born in the 1790s. The younger sons attended Harvard.

The Brooks and Parkers were among a few dozen elite families whose privileged economic circumstances gave them considerable control over early nineteenth-century Boston. Amid this deeply religious culture, the first obligations were to God and family. A secondary consideration was to assist the less fortunate. In 1810 a call went out to Boston's "wealthiest and most influential citizens" to contribute funds for a hospital [Bowditch, 1972, 3]. John Parker's $500 donation, while substantial, did not rank in the top twenty-five. By 1818 sufficient money had been raised to begin construction of Massachusetts General Hospital. To provide for annual operating expenses, a novel plan was devised. Leading citizens bought shares to start up an insurance company that committed one-third of its profits to the hospital.

Large investors included John Parker, Peter Brooks, and Josiah Quincy, each purchasing a $10,000 piece of the Massachusetts Hospital Life Insurance Company [White, 1955, 12–13]. Smaller shareholders included George Ticknor and the mathematically learned Nathaniel Bowditch, who served as the company actuary. Parker, Brooks, Quincy, Ticknor, and Bowditch were vice presidents and directors of the corporation. Each of these men had ties to Harvard, where Bowditch served as a fellow and where Quincy became president in 1829. In the interlocking affairs of the Boston upper class, about one-half of the governing Harvard fellows were officers of Massachusetts Hospital Life. Social and intellectual encounters among these men were further facilitated by the Boston Athenaeum library, where the $300 fee restricted membership to an exclusive clientele.

Not much is known about John Parker Jr., the benefactor of Harvard's Parker Fellowship. He was born on June 4, 1783. Growing up, he followed after his father in many respects. The Parkers' religion was Unitarian, and their political allegiance focused on John Quincy Adams. At the age of twenty-two John Jr. became a partner in his father's business. Four years later he married Anna Sargent. Although John Jr. was connected to Harvard through his family, friends, and business associates, there is no evidence that he had direct ties to the university during his lifetime.[1]

John and Anna lived in the heart of the Boston neighborhoods favored by the upper class. Their home was in Colonnade Row on Tremont Street, overlooking the Boston Common. Colonnade Row was a stylish development of nineteen four-story brick town homes designed by the prominent architect Charles Bulfinch. Among the other occupants were members of the Lawrence and Lowell families. From the Parkers' location, the two-block walk along the Common to George Ticknor's house passed the home of Josiah Quincy. Colonnade Row was just a few blocks in a different direction, along Sumner Street, from the residences of Edward Everett, Daniel Webster, and Nathaniel Bowditch.

The John Parkers' mercantile success positioned them to join other wealthy men in various ventures.

[1] *A posthumous connection is that his great-niece, Anna Parker, married Abbott Lawrence Lowell, who became president of Harvard in 1909.*

Both Parkers served as directors of major banks. When the elder John Parker died in 1840, his estate was valued at over $2,200,000. One of the beneficiaries, John Parker Jr., then drafted his own will. Without any children of his own, Parker Jr. generously left up to $5,000 each to numerous relatives, friends, and charities. The largest immediate legatees were his wife, siblings, and minister. Anna was to receive $80,000 and the income from a $100,000 trust fund. Parker Jr. was very explicit as to the ultimate disposition of the money from this fund:

> Also at my wife's decease it is my will that the sum of fifty thousand dollars… shall be paid to the President and Fellows of Harvard College in Cambridge to perform this my will…. To the instruction, education, and maintenance of one or more individuals as they may successively arise, of eminent natural talent or genius for some one or more of the sciences taught in said College… at home or in foreign countries, for his or their most perfect education…whose possessors, whether strictly poor or not, are not blessed with pecuniary means adequate to effecting the high state of improvement and advance in science for which they seem to be destined by nature…[Eliot, 1872–1873, Appendix II].

Out of the dissolved fund for his wife, John Parker Jr. directed that $10,000 remain in a separate trust, with the income to support five new beds in Massachusetts General Hospital. The remaining $40,000 was designated for his heirs. Parker signed the will on February 22, 1841. He and Anna then embarked on eighteen months of travel through Europe [Transcript, 1842, 2].

The Parkers spent the following winter in Italy, enjoying the art during extended stays at Florence, Rome, and Naples. In Florence, Parker commissioned his bust to be sculpted by the American Hiram Powers (see image on page 262). With daguerrotype portraits just emerging in the 1840s, Powers's marble sculptures had found a niche a few years earlier among rich and famous Americans. Among Powers's clients was the wealthy Boston industrialist Abbott Lawrence, whose brothers, Amos and William, were Colonnade Row neighbors of Parker's.

Edward Everett was another recent Powers model. Possibly there was some interaction in Florence between Everett and Parker. One month prior to Parker's sitting, Everett departed from a long Florence residence to assume the post of United States Minister to the Court of St. James. Evidence suggests that Everett and Parker did meet in London prior to the latter's return home. Everett's appointment book recorded, without any further details, a meeting for May 31, 1842, with a "Mr. Parker'' [Everett, 1930, reel 41A]. The Parkers boarded a steamship in Liverpool on August 19. A newspaper reported that John arrived in Boston carrying dispatches for Washington [Transcript, 1842, 2].

John Parker Jr. died on December 29, 1844. Although Harvard did not receive the $50,000 until after Anna's death nearly thirty years later, the bequest deserves analysis in the context of the 1841–1844 period in which it was made. Harvard and Massachusetts General were then the preferred charities of the Boston elite. The donation by Parker was the third largest commitment received by Harvard up to this point (1844) in its history.

To appreciate the contemporary magnitude of $50,000, consider a gift of the same amount made by Abbott Lawrence in 1847. Lawrence's donation was then the biggest by any living person to the university. It led to the consequential hiring of Louis Agassiz and the establishment of what became the Lawrence School of Science. Lawrence, who made subsequent donations, is remembered as a prominent Harvard patron.

One can only speculate on what lay behind the carefully customized stipulations in the will of John Parker Jr. With respect to the provisions for talented Harvard individuals for study in foreign countries, Everett, Ticknor, and/or Bancroft may have had some influence. They were the Boston exemplars of gifted students studying abroad. The Whig politics and family background of John Parker placed him among the small, closely connected group of Bostonians that included Everett and Ticknor. Bancroft, although aligned with the opposition Jacksonian Democrats, served as the Collector of Customs for the Port of Boston from 1837 to 1844. Bancroft would later acknowledge his long-held desire "to requite benefits" he received from President Kirkland that made possible his own study at Göttingen [Bancroft, 1871]. Did the customs collector impress these sentiments upon one of Boston's most prosperous maritime merchants as he was formulating his will? Bancroft's papers contain no mention of John Parker during this period. Still there is a likelihood that their mutual business interests led to personal interactions.

Another area for speculation concerns the provision by Parker that extended eligibility not just to the needy but also to individuals of moderate means. Both Osgood and Bôcher qualified under these liberal terms. Parker may have been influenced by the way educational opportunities had expanded for his younger siblings as their father's wealth increased.

Harvard Establishes Traveling Fellowships

Although studying at European universities had been life-altering experiences for Everett, Ticknor, and Bancroft, few of their countrymen followed them before the last quarter of the century. Meanwhile, graduate-level education did not then exist in the United States, although master's degrees had long been available at Harvard and Yale. The criteria for these degrees followed the long-standing British model: simply pay a fee and wait a few years beyond the bachelor's degree. During this interval the candidate, rather than engaging in scholarship, was expected to demonstrate satisfactory moral behavior.

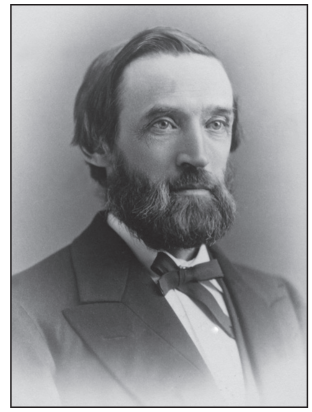
In the late 1840s some advanced-study opportunities opened in the United States. At Harvard, Benjamin Peirce offered analytical and celestial mechanics to the few students who elected to brave his abstruse lectures [Cajori, 1890, 137–138]. Agassiz and others taught substantial courses in the Lawrence School of Science. Yale began a program of higher-level instruction in philology, applied chemistry, and mathematics. The courses were intended for Yale graduates and were not, at first, linked to any degree.

Thirteen years later, in 1860, Yale announced the first Ph.D. degree in the United States. The modest course and thesis requirements for the two-year program compare today to those of a master's or undergraduate honors degree. Anything more ambitious was precluded by the limited personnel at Yale, where no subject was staffed by more than two professors and low-level faculty (tutors).

The sole mathematics professor, H. A. Newton [Batterson, 2008], continued to have responsibility for teaching sophomores and upperclassmen. Newton himself had received his B.A. ten years earlier. He then studied independently and served as a tutor. With his promotion to professor in 1855, Newton received a one-year leave of absence, with salary, to study in Europe. Newton divided his time abroad between travel and auditing classes in Paris. As the Ph.D. program began, Newton was attempting to establish his own research program in two areas. One was a short-lived foray into projective geometry, which he had studied at the Sorbonne. The second was the mechanics of meteor orbits, a subject that fascinated a number of New Haven intellectuals. With somewhat meager attainments in these directions, Newton would be one of four mathematicians in the soon-to-be incorporated National Academy of Sciences.

Three students were awarded Ph.D.s at Yale in 1861, including Arthur W. Wright, under the guidance of Newton [Batterson, 2008, 360]. Although Wright and J. Willard Gibbs, who finished two years later, would go on to become distinguished scientists, both benefited from subsequent study in Europe. As doctoral students trickled out of Yale in the 1860s, other American institutions observed with interest. None was immediately moved to offer its own advanced degree.

Harvard's decision, in 1872, to begin a Ph.D. program generated controversy among its faculty [Morison, 1936, 334–335]. Some professors were unqualified to teach advanced courses. Others worried that resources would be siphoned from the inadequately funded undergraduate college. Finally there were questions as to whether an audience existed for the new program. The young Harvard president, Charles Eliot, adamantly defended the higher degree. The gradual buildup of the Ph.D. program would be one of the several threads that Eliot wove to establish Harvard as a first-class university.



H. A. Newton

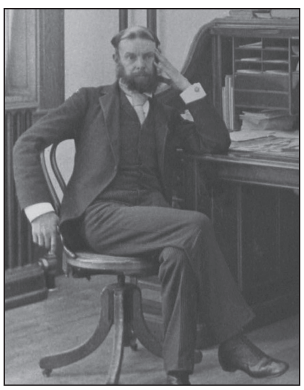
When the Parker bequest became available in 1873, the graduate school was in its infancy. Devoting the funds to graduate scholarships would have given an impetus to the program and been a lure to students. Although no campus Ph.D. had existed during the donor's lifetime, usage for this purpose was fully consistent with his stipulations. A faculty committee was thus charged with formulating procedures for distributing the income. The implementation, as presented to the Harvard fellows by Eliot, established three Parker Fellowships with annual stipends of $1,000 for up to three years. Eligibility was restricted to graduates of the university pursuing nonprofessional studies.

None of the formal provisions addressed whether the work was to be carried out abroad or within the new Harvard graduate school. The silence was not out of a lack of consideration. Eliot was a shrewd planner. He wished not only to grow the graduate school but also to upgrade faculty scholarship. Moreover, other endowments for student support became available at the time the Parker funds did.

In 1868 Harvard received $10,000 from the estate of Henry Harris to aid students in continuing their education after graduation. The following year alumnus Henry Rogers donated $20,000 for a similar purpose. As both the Harris and Rogers endowments were intended for nonprofessional studies in the city of Cambridge, they were ideally suited for graduate school stipends.

And, fortuitously, another donor articulated the concept of a traveling fellowship and completed his gift as arrangements for the Parker Fellowships were being formulated. George Bancroft was then the United States Minister to Germany. Over half a century earlier, President Kirkland had approved an initiative to cultivate Harvard faculty by sending a young graduate to study at Göttingen. Bancroft recalled that "his choice for this travelling scholarship fell upon me" [Bancroft, 1871]. The experiment was in some sense a failure, as Bancroft was not selected for a Harvard professorship. Nevertheless, Bancroft appreciated that his otherwise successful career had been enriched by the opportunity abroad. Now he wanted to give back. "I wish therefore to found a Scholarship on the idea of President Kirkland, that the incumbent should have leave to repair to a foreign country for instruction." Bancroft also suggested that the recipients "may perhaps be afterwards drawn into the corps of professors of the University". Bancroft's endowment for the Kirkland Fellowship amounted to about $11,000.

Eliot adopted the Bancroft-Kirkland vision for the Parker Fellowships. In the president's annual report to the overseers, Eliot wrote "these Fellowships will be attractive prizes, and…, if rightly managed, they will be a means of recruiting the university's body of teachers with young men of good parts and the best possible training" [Eliot, 1872–1873, 23–24]. The four Parker and Kirkland Fellowships were bracketed for study "in this country or in Europe". Despite this flexibility, they were targeted from the beginning at the latter.


William Story

In rapid succession the Parker provisions were ratified, Edward Stevens Sheldon was named the first Parker Fellow, and he departed to study modern languages in Europe [Parker]. Sheldon had graduated third in the class of 1872. He was serving as a proctor and instructor of Italian and Spanish. According to plan, Sheldon returned in 1877 to teach at Harvard. About this time the conditions of the Harris and Rogers Fellowships were adjusted to permit study abroad. Sheldon rose through the ranks and remained a professor to retirement in 1921. Other traveling fellows would follow. What Eliot did not foresee, or felt it immodest to predict, was the impact that the traveling fellowships would have on scholarship beyond Harvard.

The Htf and American Mathematics[2]

The year 1873 marked the beginning of the Htf, as well as the first mathematics Ph.D. at Harvard. That year William Byerly completed the two-year degree under Benjamin Peirce, who never had another doctoral student. Peirce was then the leading mathematician in the United States. The other prominent American scholars in the field were H. A. Newton, George William Hill, and Simon Newcomb. Hill and Newcomb were associated with the Nautical Almanac Office and, like Newton's, their work was in celestial mechanics. Outside of Harvard and Yale, mathematical research barely existed on American campuses.

Byerly was not the most promising mathematician in the Harvard class of 1871. William Story, the only graduate with honors in the subject, was also interested in pursuing his studies further [Cooke and Rickey]. Rather than remaining at Harvard or beginning the more established program at Yale, Story traveled to Germany without a fellowship. In Berlin he attended lectures in mathematics and physics by Karl Weierstrass, Ernst Kummer, and Hermann von Helmholtz. Story's teachers in Leipzig included Carl Neumann and Adolf Mayer. Story returned home without a degree in early 1874. As a Harvard graduate, he was eligible to apply for a traveling fellowship. The two remaining Parker Fellowships were awarded to Story and Ernest Francisco Fenollosa [Parker].

Fenollosa studied philosophy at Cambridge University. He later became an influential authority on East Asian art. Story obtained his Ph.D. in mathematics with another year of study at Leipzig. Returning to a Harvard tutorship in 1875, he became the first traveling fellow to join the faculty.

At this time Harvard's mathematics professors were the sixty-six-year-old Benjamin Peirce and his son James, whose contributions were to teaching and administration. Story was keen to continue his research. Given the high esteem in which he was held by both Peirces, he was positioned to succeed the elder Peirce. Then another opportunity presented itself. The Johns Hopkins University was to open in 1876. Its president, Daniel Coit Gilman, aspired to establish the first American university dedicated to graduate study and research. To lead the program in mathematics Gilman selected Benjamin Peirce's friend, the British mathematician J. J. Sylvester [Parshall and Rowe, 1994]. Sylvester needed a lieutenant to share in the instruction and contribute to scholarship. He sought advice from the Peirces. They recommended Story over Byerly, who was then at Cornell [Cooke and Rickey, 1989, 34–35], [Parshall and Rowe, 1994, 76].

Story moved to Johns Hopkins and was replaced at Harvard by Byerly. The new university marked the beginning of substantial graduate education in the United States. Mathematics was immediately successful. Sylvester inspired an unprecedented number of American students to engage in research. In addition, Johns Hopkins sponsored the first important mathematics journal in the United States. As managing editor of the *American Journal of Mathematics* and instructor in a variety of topics, Story made contributions that, while subordinate to those of Sylvester, were indispensable to this pioneer undertaking.

Johns Hopkins' $500 fellowships added to the desirability of the new program. The Harvard

[2] *The discussion of the development of American mathematics 1876–1900 draws substantially from [Parshall and Rowe].*

graduate school was overshadowed by its younger domestic competitor.

Yet the Htf remained an attraction for the mathematically talented student who, every year or two, graduated from Harvard. In 1876 B. O. Peirce received highest honors in physics. The following year Washington Irving Stringham earned the same distinction in mathematics. Both had taken courses from Benjamin Peirce, a distant relative of B. O. Peirce.

The younger Peirce published experimental physics research as a junior, continuing his laboratory work for a year after graduation. In 1877 one of the Parker Fellowships was renewed, and two were available. There were eighteen applicants for these coveted awards, and B. O. Peirce was one of the winners [Parker].

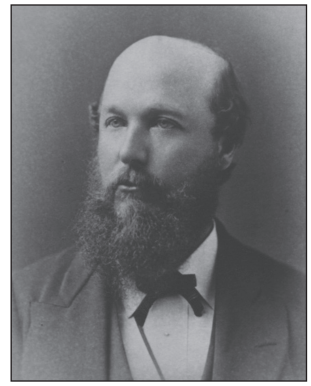
The Harvard administration then reviewed and revised the Parker Fellowship stipend, concluding

> that $800 a year will enable a young man to live comfortably, though plainly, at any foreign university, and pay for the journeys to and from home, provided that he hold the fellowship at least two years. As it is not the policy of the Council to recommend for appointment to fellowships persons who are not likely to hold them for at least two years, it will seldom happen that an incumbent is forced to pay for two voyages across the Atlantic out of the income of a single year. To increase the number of these fellowships to four seemed more desirable than to give three an income which would permit to the incumbents some not indispensable expenditures. For several years past, the Kirkland Fellowship, with an income of only $750 has maintained its incumbents at European universities. Five hundred dollars will support a student at any American university [Eliot, 1877–1878].

B. O. Peirce obtained his Ph.D. at Leipzig and then did postdoctoral work in Helmholtz's Berlin laboratory. Soon after returning to America, he began teaching mathematics and physics at Harvard. In the decade after Benjamin Peirce's death in 1880, Harvard mathematics was dominated by James Peirce, Byerly, and B. O. Peirce, none of whom produced research in the subject. B. O. Peirce later published work in experimental physics. He served as a vice president of the AMS and president of the American Physical Society.

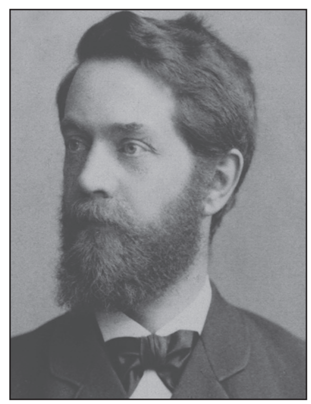
Stringham took a more circuitous route to Europe. Accepting a Hopkins fellowship, he came under the influence of Story in Baltimore [Parshall and Rowe, 1994, 112]. Stringham received his Ph.D. in 1880. With a Harvard A.B., he still met the eligibility requirement for an Htf. Stringham was awarded a Parker Fellowship for postdoctoral study in Leipzig [Parker]. The timing was propitious in that Felix Klein was beginning his Leipzig tenure in the same year. The thirty-one-year-old Klein was a gifted teacher with a profound view of the deep connections in mathematics [Parshall and Rowe, 1994]. By participating in Klein's seminars, Stringham absorbed mathematical developments that were unknown in the United States.

With a Hopkins Ph.D. and two years of postdoctoral study under Klein, Stringham was the best trained American mathematician yet to appear on the job market. The University of California had just fired its professor of mathematics and was looking to upgrade its expertise in the subject [Moore, 2007, 25–40]. In 1882 Stringham became the first mathematics Ph.D. on the Berkeley faculty. He immediately modernized the curriculum and introduced courses beyond calculus. Later, Stringham instituted a Ph.D. program and supervised the first Berkeley mathematics thesis. Beyond his own department, Stringham was a leading figure in university affairs, becoming dean of the faculty.



Washington Irving Stringham

The impetus that Johns Hopkins gave to American mathematical scholarship was largely short-lived [Parshall and Rowe, 1994, 145–146]. Sylvester's students did not go on to distinguish themselves as researchers. In 1883 Sylvester returned to England. Efforts to replace him with Klein and Arthur Cayley were unsuccessful. Eventually, astronomer Simon Newcomb assumed Sylvester's chair on a part–time basis, continuing his work in Washington for the Nautical Almanac Office. Instruction in mathematics devolved to Story and two of Sylvester's Ph.D. students. Without the charisma and mathematical direction of Sylvester, the program lost the special features on which its success had been based [Parshall and Rowe, 1994, 144].

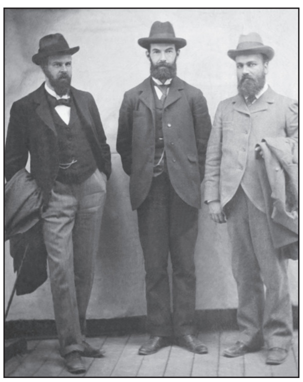


Felix Klein

Nor could Byerly and the Peirces pick up the slack at Harvard. Yale was better positioned with Newton and Willard Gibbs. Newton held a high profile, serving as president of the American Association for the Advancement of Science in 1885. His research was on meteors and comets. Gibbs had done groundbreaking work in physical chemistry and possessed a profound understanding of mathematics. However, it would take time for his ideas to be understood and appreciated in the United States. As a reclusive professor of mathematical physics, he saw few students. Perhaps the best American mathematician was George William Hill, who worked on celestial mechanics at the Nautical Almanac Office.

Thus the state of American mathematical scholarship in the mid-1880s was not unlike what it had been prior to the creation of Johns Hopkins. There were a few outstanding applied workers, meager pure research, and no domestic graduate program near the level of European programs.

Frank Nelson Cole



Left to right: James Pierpont, W. F. Osgood, Maxime Bôcher, 1899.

Several mathematically inclined students nevertheless passed through Harvard during this period. They then went to Germany on traveling fellowships and returned to play crucial roles in upgrading mathematical research and graduate study in the United States.

Frank Nelson Cole finished second in the Harvard class of 1882 with highest honors in mathematics. He remained in Cambridge for the first year of his Parker Fellowship to do graduate work in mathematics. Cole then went to Leipzig with the intention of studying the mathematical side of physics [Parker]. During the summer of 1884 he came under the spell of Felix Klein [Parshall and Rowe, 1994, 192, 196–197]. One year later his Htf expired. Cole returned to Harvard with a lectureship and a thesis problem on sixth-degree equations.

Despite his hard work, Cole was unable to make much progress doing research in the mathematical isolation of the United States [Cole, 1886]. Even so, Harvard awarded him a Ph.D. In the classroom Cole lectured on Klein's approach to substitution groups and function theory. More important, Cole opened a channel between Harvard and Klein that would endure after Klein's move to Göttingen in 1886. This connection would prove to be of great significance to American mathematics.

However, with just one or two Parker Fellowships vacated each year, the competition was intense for new scholarships. Some candidates may have queued into the Harvard graduate school, where they could prove their worth for a subsequent offering. In 1885 mathematics graduate student Mellen Haskell, who had received his A.B. two years earlier, was awarded a Parker Fellowship. Haskell went to Leipzig to study with Klein.

In 1886 only one new Parker was available. The Harvard graduating class that year included future distinguished scholars George Santayana, Theodore Richards, and William Fogg Osgood. Santayana received another Htf earmarked for students in his field of philosophy. The Parker was assigned to mathematics instructor Arthur Gordon Webster, who had graduated the previous year with highest honors in mathematics. Webster would earn his Ph.D. in physics at Berlin. Richards and Osgood remained at Harvard for graduate study in chemistry and mathematics, respectively.

Osgood had graduated second in his class with highest honors in mathematics. In 1887 he received his A.M. Inspired by Cole's course on function theory, Osgood applied for an Htf to study with Klein at Göttingen. Having lost his father two years earlier, Osgood was in need of financial support to pursue a Ph.D. abroad. Once again, just a single new Parker Fellowship was available. It was awarded to law school graduate Julian William Mack. Osgood received the Harris Fellowship, which carried a lower stipend than the Parker. He went to Göttingen, and the following year his fellowship was upgraded to a Parker, opening the Harris.

One additional Parker was available in 1888. Mathematics had a strong candidate, Maxime Bôcher, who was graduating with highest honors. Bôcher met the means criteria for a Parker Fellowship. His father, Ferdinand, was a Harvard French professor on a salary of $4,000. If Maxime were unsuccessful in his application for an Htf, he planned to remain at Harvard for an additional year of course work and independent study. The recipient of the Parker was Theodore Richards, from Osgood's class, who had just completed his Ph.D. Richards would do postdoctoral work in Berlin and go on to win the Nobel Prize in chemistry. Bôcher received the Harris. Like Osgood, Bôcher traveled to Göttingen to study with Klein and was promoted to the Parker in his second year.

Haskell, Osgood, and Bôcher each completed his Ph.D. in Germany. Haskell received his degree in 1889 and took a job at Michigan. From there Haskell was hired by Stringham in 1890 to become the second mathematics Ph.D. on the Berkeley faculty [Moore, 2007, 33]. Osgood and Bôcher returned to Harvard in 1890 and 1891, respectively, as instructors of mathematics.

Cole's story was less happy. He had resigned his Harvard position a few years earlier after experiencing a breakdown from overwork [Cole, 1887]. No longer able to concentrate, Cole took a job as an assistant to a railroad engineer. Working in the outdoor air, he slowly regained his health. In 1888 Cole had sufficiently recovered to become the first mathematics Ph.D. on the Michigan faculty. By the second semester he was reprising his course on function theory for faculty and advanced students [Cole, 1889].

Over the 1880s Parker Fellows Stringham, Cole, and Haskell brought Klein's mathematics to Berkeley, Harvard, and Michigan. No new significant

American mathematics Ph.D. program was initiated, however, until 1889 with the opening of Clark University. The president of Clark, G. Stanley Hall, was a psychologist from Johns Hopkins. Hall set out to emulate German scholarship by establishing strong graduate departments in the sciences. When overtures to Klein failed, Hall selected his Hopkins colleague Story to lead mathematics [Cooke and Rickey, 1989].

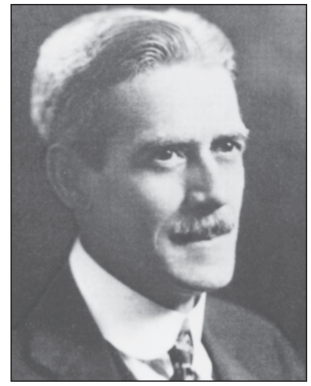
Among those joining Story in the new department were two Klein students. First came the recent German emigré Oskar Bolza. In the second year the mathematics roster added American Henry Seely White, who had been contemporaneous with Osgood at Göttingen. In that same year Arthur Webster received a postdoctoral appointment in physics.

The course offerings at Clark provided the broadest array of European-level mathematics available in the United States. Unfortunately, the university was insufficiently funded to meet its obligations. Over its third year the founder, president, and faculty became embroiled in devastating disputes. In 1892 many faculty members departed for other institutions. The university lost White and Bolza but retained Story and Webster. The second promising American mathematics graduate program left little mark.

The collapse of Clark took place during Bôcher's first year on the Harvard faculty. Meanwhile he and Osgood attempted to continue their research in the face of serious obstacles. Both labored under one-year contracts for $1,250 with four-course teaching loads [Batterson, 2009]. Their (senior) colleagues, Byerly and the Peirces, were not engaged in research, nor were they knowledgeable about recent developments. In this respect the Harvard mathematics department was out of step with the rest of the university [Morison, 1936, 378]. During Eliot's presidency, research had become expected of Harvard faculty. Graduate programs in other subjects were advancing.

The time was ripe for mathematical scholarship at Harvard. Despite their heavy teaching loads, Bôcher and Osgood made it happen. Over his first two years Bôcher obtained significant new results and expanded his thesis into a book on potential theory. The German text, with a foreword by Klein, was published in Leipzig. The book became known among the European mathematical community, providing evidence of substantial American scholarship. Bôcher made a number of other notable contributions, particularly to the theory of ordinary differential equations.

In 1896 Osgood proved a version of the bounded convergence theorem for term-by-term integration of series of continuous functions. Unknown to Osgood was an earlier formulation of Arzelà. Even so, Osgood's profound analysis would be important to Lebesgue in his subsequent development of integration [Hawkins, 1970]. But the greatest American mathematical accomplishment of this time was Osgood's 1900 proof of the Riemann Mapping Theorem for simply connected domains. In removing restrictions on the boundary, Osgood succeeded on a problem that had been pursued by Schwarz and Poincaré [Walsh, 1973].



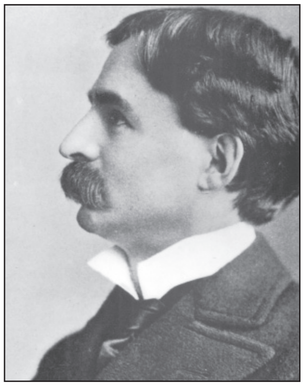
Earle Hedrick

Bôcher and Osgood worked together to elevate the Harvard mathematics curriculum. Both strove to present current ideas in their advanced courses. Although Byerly and the Peirces shared the instructional duties, the Harvard offerings were steadily modernized, course by course. Bôcher, the department's most junior member, took the lead in thesis direction. He was promoted to assistant professor in 1894, and his first Ph.D. student, James Glover, finished the following year.

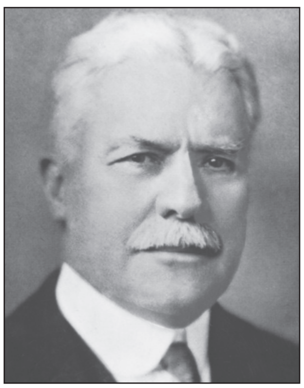
The rate of Harvard mathematics Ph.D. output increased to about one per year, a large number for the time. The school's most promising students, however, continued to go abroad for their doctorates. Charles Bouton and Earle Hedrick came to Harvard for graduate study in the final decade of the nineteenth century. Both earned their master's degrees and then went to Germany on Parker Fellowships. Bouton received his Ph.D. with Sophus Lie at Leipzig and then returned to teach at Harvard. Hedrick took his doctorate under David Hilbert at Göttingen. Following a brief stint at Yale, he became chair at Missouri and then at UCLA. As students of Lie and Hilbert, the traveling fellows had established connections to two of the greatest mathematical thinkers in the world.

Paralleling these developments at Harvard were even more dramatic advances at a new institution [Parshall and Rowe, 1994]. The University of Chicago opened in 1892 with E. H. Moore as acting head of mathematics [Parshall and Rowe, 1994, 283–284]. As already mentioned, Moore had received his Ph.D. under H. A. Newton at Yale seven years earlier. Newton then supported Moore for a year of postdoctoral study in Germany. Returning to the United States, Moore wrote a few papers while holding lower-level teaching positions at Yale and Northwestern. He was one of a handful of Americans committed to mathematical research but had no major results or any experience with thesis direction.

To fill out his team Moore hired Bolza and Heinrich Maschke. Bolza and Maschke had studied together with Klein in their native Germany [Parshall and Rowe, 1994, 289–292]. With this roster Chicago's graduate offerings, from the beginning, were strong and balanced, surpassing the program at Harvard. In 1896 Leonard Dickson completed his thesis at Chicago under Moore. Over the next few years, the flow of promising American students to Europe abated but did not halt. Gilbert Bliss and Oswald Veblen would begin study for their doctorates at Chicago, but Bouton, Hedrick, Edward Huntington, and Max Mason went to Europe.



E. H. Moore



Thomas Fiske

Moore's own research blossomed at Chicago. He, Osgood, and Bôcher established themselves as the leading American mathematicians of the rising generation. They worked together and with others to promote mathematical research nationwide through the American Mathematical Society (AMS).

The history of the AMS is itself instructive. Founded in 1888 at Columbia University with the name the New York Mathematical Society, it began through the initiative of a graduate student, Thomas Fiske. Fiske was inspired by the mathematical activity he witnessed during a residence in England. What Fiske began as a campus interest group quickly grew into a national organization that supported meetings, lectures, and publication.

A dozen years after its birth, a generational and cultural change took place in the AMS leadership. All of the nineteenth-century AMS presidents were born in the 1830s and 1840s. Beginning with E. H. Moore in 1901, the next seven presidents[3] were born in the interval 1858–1865. Each had an earned Ph.D., and all but Fiske had studied in Germany during the decade 1884–1893. Cole, from the same demographic, served throughout as secretary. These younger men sought to transplant the European mathematical ethos to the United States. They succeeded in creating an American identity that was worthy of international respect.

Meanwhile, Harvard students continued to take advantage of the Htf. A gradual change began in the use of the fellowships, with C. N. Haskins and D. R. Curtiss taking Ph.D.s at Harvard and then using the Htf for postdoctoral work in Göttingen and Paris, respectively. Haskins and Curtiss spent their professional careers at Dartmouth and Northwestern. Over the same period J. W. Bradshaw, Elijah Swift, and Leroy Howland went abroad on Htfs for their doctorates. They then carried the benefits of their cosmopolitan experiences to the campuses of Michigan, Vermont, and Wesleyan. Swift and Howland had worked with Hermann Minkowski and Ferdinand Lindemann.

It would take a few more years for the Htf to become a mostly postdoctoral fellowship. In 1908 and 1909 W. A. Hurwitz and Dunham Jackson went to Göttingen on Htfs. There they obtained their doctoral degrees from Hilbert and Edmund Landau. Griffith Evans received his A.B. from Harvard and remained there for his Ph.D. under Bôcher in 1910. Evans then was awarded an Htf to do postdoctoral work with Vito Volterra in Rome. Upon his return to the United States, Evans became the first faculty appointment at Rice [Moore, 2007, 51]. With several strong and many developing domestic graduate programs, it became unusual, after 1910, for Americans to pursue a mathematics Ph.D. abroad.

Conclusion

A number of factors contributed to the sudden rise of American mathematics at the turn of the twentieth century. Most notable were the individuals who overcame huge teaching loads to establish mathematical research as an essential ingredient of campus scholarship. The common thread distinguishing these agents of change from their predecessors was coming of age mathematically in Europe.

Each pioneer required the financial means to study abroad. In many cases the support was provided by an Htf. Not surprisingly, the greatest institutional beneficiary was Harvard, which made striking mathematical advances following the returns from Germany of Cole, Osgood, and Bôcher. However, the reach of the Harvard fellowships was both pervasive and lasting. Through the Htfs the ideas of Klein, Hilbert, Lie, Minkowski, and Volterra were transmitted to campuses across the United States. From 1882 to 1949 Berkeley was led by Htf recipients Stringham, Haskell, and Evans [Moore, 2007]. During these years the foundation was laid, albeit unevenly, for the department's future distinction. Dunham Jackson went to Minnesota in 1919 and Earle Hedrick to UCLA five years later.

Given the range of financial eligibility for the fellowships, it is unclear which Harvard students would have reached Europe without an Htf. Cole was from a farming family and had nine siblings. A department recommendation for Hurwitz's renewal mentioned that his parents needed his support and that he would probably not accept a loan. It seems likely that some recipients would have managed to study in Europe through a loan or on their own resources, whereas more would have remained in the United States to continue at Harvard or to work.

The development of American mathematics owes much to an unlikely patron. John Parker Jr. grew up in Boston just after the American Revolution. He was from a generation that held a fierce

[3] *The presidents were Moore, Fiske, Osgood, White, Bôcher, H. B. Fine, and Edward Van Vleck, each serving a two-year term.*

pride in the young democracy and considerable uncertainty over the long-term stability of the federation of states. Growing up in this hopeful, patriotic environment, Parker became privileged as his father accumulated fabulous wealth. In his will, Parker provided for his own family and friends. Then he carefully devised provisions to assist the less fortunate through education and health care. Although Parker made no mention of mathematics, he no doubt would have been pleased by the success of his beneficiaries and their role in elevating American mathematics to a level that commanded international respect.

Acknowledgments

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Photo credits

Bust of John Parker Jr.: courtesy Harvard Art Museums, University Portrait Collection, Bequest of Mrs. Anna Parker, 1873 (photo by Imaging Department, ©President and Fellows of Harvard College); Thomas Fiske, Earle Hedrick, E.H. Moore: AMS Archives; portrait of George Ticknor by Thomas Sully: Hood Museum of Art, Dartmouth College, Hanover New Hampshire, gift of Constance V. R. White, Nathaniel T. Dexter, Philip Dexter, and Mary Ann Streeter; photo of James Pierpont, W. F. Osgood, and Maxime Bôcher: courtesy of Theodore K. Osgood; H. A. Newton: MADID #11811, Photographs of Yale affiliated individuals maintained by the Office of Public Affairs, Yale University, 1879–1989 (RU 686), Manuscripts and Archives, Yale University Library; Washington Irving Stringham: UCB Mathematics Department; Felix Klein: ©Göttingen State and University Library; Frank Nelson Cole: David Eugene Smith Collection, Rare Book and Manuscript Library, Columbia University; William Story: Clark University Archives.

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A Tribute to Henry Helson

Donald Sarason

Henry Helson, a leading figure in harmonic analysis whose ideas have had an enormous impact, died on January 10, 2010. Born on June 2, 1927, Henry grew up in Bryn Mawr, where his father Harry was a professor, carrying out the research that would make him one of the eminent psychologists of his era. His mother Lida, despite her upbringing as a fundamentalist Lutheran, found her way to the Quaker faith when Henry and his sister Martha were children. All three of them became and remained committed Quakers.

Henry's exceptional mathematical abilities shone through early. Although his father Harry hoped Henry would go into physics, Henry was hooked on mathematics. He entered Harvard as an undergraduate, graduating in 1947. His education had been interrupted for thirteen months by military service, which for him, as a Quaker, was a loathsome experience. On the eve of his graduation Henry was awarded a Harvard traveling fellowship. The fellowship enabled him to fulfill an intense desire he had nurtured to visit Europe. In the academic year 1947–1948 he visited London, Paris, Prague, and Vienna, but he spent most of the year in Poland, first in Warsaw, then in Wroclaw.

Why Poland? The circumstances are explained in Henry's essay [5]. He writes: "… [I] was crazy to see the destruction caused by the war in Europe. … The most desperate place in Europe seemed to be Poland, and there were mathematicians in Poland." Another passage from the same essay reads:

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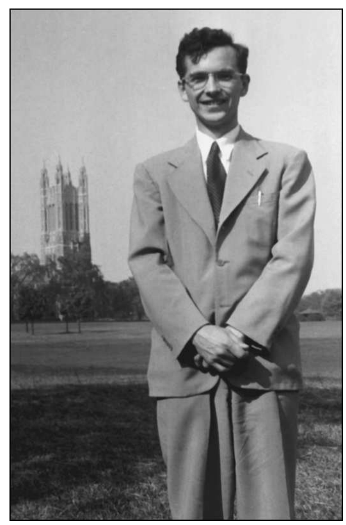


Young Henry, location and date uncertain.

> In college I had spent much time browsing in the mathematics library, and even in the stacks, and I found *Fundamenta Mathematicae* and *Studia Mathematica* of special interest. There were magic names in them: Sierpiński, Steinhaus, Kuratowski, Ulam, Ostrowski, Szegö, Borsuk, Tarski, and of course Banach, and a name that appeared in two forms, sometimes as Szpilrajn and sometimes as Szpilrajn–Marczewski, which I could not explain. Many papers seemed related, many were joint work, all were quite short. I could

not understand most of them, but I tried.

In spring 1948 Henry was in Wroclaw. He writes:

> Szpilrajn had become Marczewski by this time. In Polish law, you had to keep both names for a while if you wanted to change. I suppose that Szpilrajn was a Jewish name. Marczewski took me in hand and gave me good problems, in analysis, to work on. I solved two of them and they made my first publications, and he was pleased. (So was I.)

The publications referred to are [2] and [3], which appeared in *Studia Math.* and *Colloquium Math.*, respectively.

In fall 1948 Henry enrolled in the Harvard mathematics Ph.D. program. His thesis advisor was Lynn Loomis, but by happenstance Arne Beurling turned out to be the source of his mathematical inspiration. In an essay that did not receive wide circulation [6], Henry writes:

> When I came back to Harvard in 1948 Arne Beurling came to visit the department. His course looked hard and unrewarding but my mentor, Lynn Loomis, told me I had to take it. Of course the department had to get together some students to listen to Beurling! So I went, and that set my subsequent mathematical trail.... Beurling was full of wonderful ideas, and I soaked them up. In this time I was the twin of my friend John Wermer, whom I had known for years as an undergraduate. At the conclusion of this year, John went back to Uppsala with Beurling; I stayed another year to write my thesis.

Henry received his Ph.D. in spring 1950 but was late getting on the job market. An offer from UCLA eventually came, but this was during the California loyalty oath controversy; all University of California faculty were required to sign the oath, which Henry would not do. He wrote to Wermer in Uppsala, requesting that John ask Beurling if he could support another student. A positive answer came back, so Henry spent the academic year 1950–1951 in Uppsala, his stay there interspersed with trips elsewhere in Europe. One trip was to Nancy, France, which then housed Laurent Schwartz, Jean Dieudonné, Roger Godement, and Alexander Grothendieck (then a student), among others. The Nancy trip resulted in the offer of a fellowship for the following year, but Henry's parents prevailed on him to stay in the United States. He accepted an instructorship at Yale and was eventually promoted to assistant professor. However, Yale at that time, despite housing an impressive group of young nontenured mathematics faculty, regarded their positions as strictly temporary. In 1955 Henry accepted an assistant professorship offer from Berkeley. (The loyalty oath had by then been declared unconstitutional.) Henry remained at Berkeley for the remainder of his career, becoming associate professor in 1958, professor in 1961, and emeritus professor in 1993.

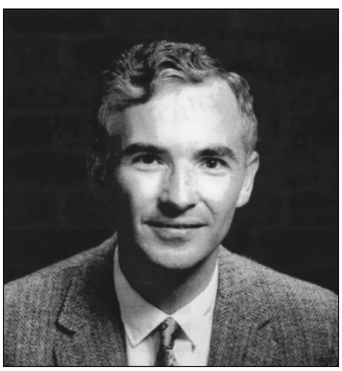
Henry carving turkey, Berkeley, 1950s.

Henry, Ravenna, and children Ravenna, David, and Harold (on Ravenna's lap), Berkeley, 1960.

Henry and his wife Ravenna were married in 1954. Ravenna had received a Ph.D. in psychology from Berkeley and was eager to return there, which enhanced Berkeley's intrinsic attractions for the couple. They raised three children, born between the years 1956 and 1960. Ravenna had a distinguished career as research psychologist at Berkeley's Institute of Personality and Social Research, including serving as director of the Mills Longitudinal Study, which she founded.

Henry's early research, which quickly gained recognition, is discussed below by Izzy Katznelson and John Wermer. I'll dwell here a bit on Henry's two most influential papers, [8] and [9], coauthored with David Lowdenslager. Bill Arveson will describe how those papers were instrumental in the development of the theory of nonself-adjoint operator algebras, a direction likely not envisioned by the paper's authors. The pieces by Kathy Merrill and Jun-ichi Tanaka concern another direction initiated in [8] and [9], involving invariant subspaces.

As the titles of [8] and [9] inform us, the papers arise from certain questions in probability. Attached to a stationary stochastic process with discrete time is its so-called spectral measure, a positive Borel measure μ on the unit circle. Certain questions about the process can be recast as questions about the geometry of the (complex) Hilbert space $L^2(\mu)$. Thus questions in probability are transformed into questions in function theory. Prior to the work of Helson and Lowdenslager, a number of those questions had been answered through the work of, for example, Gabor Szegö, Norbert Wiener, Andreĭ N. Kolmogorov, and Mark G. Kreĭn. Helson and Lowdenslager propelled this theory in two new directions.



Henry, early 1960s.

First, Helson and Lowdenslager replaced the classical one-parameter stationary stochastic processes referred to in the preceding paragraph by multiparameter stationary stochastic processes of a certain kind. In those processes, the unit circle from the classical theory is replaced by a compact abelian group B dual to a dense discrete subgroup of $\mathbb{R}$—a torus of dimension larger than 1, possibly of infinite dimension. The Bohr compactification of $\mathbb{R}$, the dual of the entire discrete real line, is a prime example. Fourier analysis on B leads to the Fourier series in several variables of the papers' titles.

Helson and Lowdenslager established versions in their setting of some basic theorems from the classical setting, for instance, Szegö's theorem from prediction theory and the famous theorem of F. and M. Riesz on analytic measures. This required new ideas, because Helson and Lowdenslager did not have available the classical techniques.

While Helson and Lowdenslager emphasized the group setting in which they worked, it was quickly realized that their ideas extended far beyond that context. In particular, the ideas were employed by people working in the then emerging theory of function algebras, providing decisive impetus. The ideas also suggested new ways to approach the study of Hardy spaces in the unit disk, ideas adopted by Kenneth Hoffman in his well-known book [11].

Helson and Lowdenslager's F. & M. Riesz theorem attracted particular attention. Earlier, Solomon Bochner [1] had established a generalization of the original F. & M. Riesz theorem in which the circle is replaced by the two-torus. A maximal theorem underlies Bochner's method, which can be applied to give a proof of the original F. & M. Riesz theorem, a proof different from the several that had been previously found. Bochner's F. & M. Riesz theorem is a very special case of Helson and Lowdenslager's theorem, which they proved with what were at the time completely new methods, an elegant blend of function theory and operator theory. Those methods can be applied to give yet another and a remarkably simple proof of the original F. & M. Riesz theorem.

In a second direction, Helson and Lowdenslager studied multivariate stochastic processes, replacing the scalar-valued processes from the classical theory by vector-valued processes. This led to questions about factoring matrix-valued functions on the unit circle and corresponding new results. This part of their work hooks up with what eventually became the theory of operator models of Belá Sz.-Nagy and Ciprian Foiaş, a central part of operator theory.

The brief remarks above are but a very rough sketch of the basic papers of Helson and Lowdenslager. The papers are too rich to review adequately in a few lines. They are well worth studying even today; in them one will find the seeds of much of Henry's subsequent research.

David Lowdenslager died tragically in September 1963 at the age of thirty-three. This must have been a terrible blow to Henry. But Henry never spoke of it to me, and of course I never pressed him to do so.

As anyone who has read some of his papers knows, Henry was a masterful expositor. His style is exceptionally clear, economical, vigorous, and fluid, altogether worthy of emulation.

In 1992 Henry started his own publishing firm, Berkeley Books, motivated by his distaste for our present-day crop of calculus textbooks, by the high cost of mathematics books at all levels, and also by what he terms "a passion for entrepreneurial activity". Berkeley Books was a one-man operation except for the actual printing of the prepared text, which Henry contracted to a local printer. He published mostly his own textbooks, but also two by Berkeley colleagues (George Bergman and me). Clearly, Henry's overhead was far below that of mainstream publishers, but also clearly, Henry was at a severe disadvantage when it came to marketing his products. Henry describes his experiences as a publisher in [7]. As he states there, Berkeley Books finished in the black every year (at least until [7] was written).

Mathematics was not Henry's only passion. Another was music. He was an accomplished violinist and violist, and after his retirement, freed from many prior responsibilities, he practiced daily for two hours to enhance his skills. He joined a local music group, Amphion, and performed publicly, especially with a talented and congenial quartet he assembled. Henry's last performance, at Amphion, and the last time he played the violin, was in September 2009, just a few months before his death. He bravely fought off the terrible effects of his disease and chemotherapy and performed Brahms's First Violin Sonata.

Travel was another of Henry's passions. He was keenly interested in other places and other cultures. While on the Berkeley faculty he spent sabbaticals in Sweden, Orsay, Ghana, Montpellier, Florence, Bonn, Marseille, and India. India held a special fascination for Henry; between 1980 and 2000 he made four extensive visits there.

My own indebtedness to Henry is profound, both for the mathematics I learned from him and for his support and kindness during our forty-six years as colleagues. I have illustrated the last remark elsewhere [13], but the story will bear repeating here, in abbreviated form.

The story involves a fundamental theorem of Zeev Nehari about Hankel operators [12]. For present purposes, a Hankel operator is an operator on $\ell^2(\mathbb{Z}_+)$ induced by a Hankel matrix, a complex matrix with constant cross diagonals (the $(j,k)^{th}$ entry depends only on $j+k$). Nehari's theorem answers the question of when such an operator is bounded. The answer, in qualitative form, is that boundedness happens if and only if the matrix entries in the first column are the forward Fourier coefficients of a function in L^∞ of the circle. The quantitative version adds that the L^∞ function can be so chosen that its norm equals the norm of the operator.

As a by-product of his joint paper with Szegö [10], Henry found an elegant and insightful proof of Nehari's theorem using a duality argument facilitated by a factorization result of Frigyes Riesz for functions in the Hardy space H^1. Henry showed me his proof in fall 1965, during one of our frequent mathematical chats. The following day it suddenly dawned on me that Henry's technique was exactly what I needed to complete the proof of a theorem I had been working on for two years. The paper that grew out of this breakthrough [14], I would guess, was a big help in my getting tenure at Berkeley, as was no doubt Henry's support behind the scenes.

Henry's final paper [4] appeared recently in *Studia Mathematica*, sixty-two years after his first paper appeared there. In it, Henry addresses the question of whether Nehari's theorem can be generalized from its original one-dimensional setting to an infinite-dimensional setting. He surmises that the answer is negative, and he maps out a possible way to produce the needed counterexample. At the end he states: "But we cannot go further."

The paper was submitted on October 20, 2009, less than three months before Henry died. He must have feared death was imminent and known that he had better tell other harmonic analysts what he knew before it came to pass.

Henry is survived by his wife Ravenna, sister, Martha Wilson, daughter Ravenna Helson Lipchik, sons David and Harold, and three grandchildren.

(I was guided in composing the preceding remarks, in addition to the sources cited, by a sketch of Henry's life distributed at his memorial service on February 20, 2010, at the Berkeley Friends Meeting Hall, and by advice from Ravenna, Steve Krantz, and several of my coauthors.)

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Y. Katznelson

Some of Henry's Early-Fifties Work

Much of Henry's work in the early 1950s dealt with two problems that, retrospectively, turned out to be very closely related.

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1) The description of homomorphisms of group algebras (in terms of homomorphisms of the underlying groups).

Beurling and Helson [5] prove that automorphisms of $L^1(\mathbb{R})$ are trivial, i.e., given by a linear change of variables on $\hat{\mathbb{R}}$ and, using some earlier results of Beurling and Henry's paper [4], show:

If G and H are locally compact abelian groups at least one of which has a connected dual, and if their group algebras are isomorphic, then G and H are isomorphic.

2) The description of idempotent measures (under convolution) on locally compact abelian groups.

A theorem of Szegö [2], states that if a function $f = \sum a_n z^n$ is holomorphic in the unit disc and admits some analytic continuation across some arc, and if its Taylor coefficients $\{a_n\}$ assume only a finite number of distinct values, then f is rational: $f(z) = P(z)/Q(z)$, with all the zeros of the denominator on the unit circle. The conclusion is equivalent to the statement that the sequence $\{a_n\}$ is ultimately periodic to the right.

Henry [1] extended the theorem, showing that the condition "holomorphic f with some analytic continuation" is too strict and can be replaced by a "harmonic function $u(r,t) = \sum_{-\infty}^{\infty} a_n r^{|n|} e^{int}$ for which there exists an arc (α, β) on the unit circle such that $\int_\alpha^\beta |u(r,t)|dt$ is bounded for $\{r : r < 1\}$". The statement now is that if the coefficients assume only a finite number of distinct values, then the sequence $\{a_n\}$ is ultimately periodic to the right and ultimately periodic to the left.

In the special case, which Henry proved in 1953 [6], the arc (α, β) is the entire circle, the boundedness of $\int_{\mathbb{T}} |u(r,t)|dt$ implies that u is the Poisson integral of a measure μ on the unit circle whose Fourier coefficients $\hat{\mu}(n) = a_n$ assume only a finite set of distinct values,

The stage is now taken by measures whose Fourier coefficients assume only a finite number of distinct values, and in particular idempotent measures (whose Fourier coefficients are necessarily 0 or 1).

The ultimate periodicity of $\hat{\mu}$ is shown to imply that μ is the sum of a discrete measure carried by a finite subgroup of the circle and an absolutely continuous part whose density is a trigonometric polynomial.

This description of idempotent measures (on $\mathbb{T}$) was the basis for Rudin's work on isomorphisms of group algebras and paved the way for Cohen's eventual complete description of idempotent measures (on general groups) and solution of the homomorphism problem.

Helson Sets

Another theorem that Henry proved in 1953 [3] inspired much interest and research activity. It has to do with perfect sets $E \subset \mathbb{T}$ (the circle group) that have the property that every continuous complex-valued function on E is the restriction to E of an absolutely convergent Fourier series. (Such sets are now commonly referred to as *Helson sets*.) Henry proved that these are *sets of uniqueness for measures,* that is, a nontrivial measure μ carried by such sets cannot have its Fourier-Stieltjes coefficients $\hat{\mu}(n)$ tend to zero as $n \to \infty$.

Top, left to right: Hans Lewy, Ravenna, Henry; bottom, left to right: Harold Helson and Michael Lewy. Drakes Bay near Berkeley, early 1960s.

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John Wermer

Henry Helson's work has played a central role in modern harmonic analysis and its applications since the 1950s.

In [1] Henry proved a conjecture of Steinhaus. The conjecture states that if the partial sums of a trigonometric series are all nonnegative, then the

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coefficients of the series tend to 0. Henry in fact proved something stronger: If the trigonometric series $\sum_{-\infty}^{\infty} a_n e^{inx}$ satisfies

$$\sup_{N>0} \int_0^{2\pi} \left| \sum_{n=-N}^{N} a_n e^{inx} \right| dx < \infty,$$

then $a_n \to 0$ as $|n| \to \infty$.

Another early theorem concerns the Banach algebra A of Fourier–Stieltjes transforms $f(t) = \int_{-\infty}^{\infty} \exp(itx)d\sigma(x)$ with $\|f\| = \int_{-\infty}^{\infty} |d\sigma(x)|$, where σ is a finite measure on $-\infty, \infty$. The problem is to find all $f \epsilon A$ such that $\|f^n\|$ is bounded for $-\infty < n < \infty$. The obvious such f are the functions $\exp(a + bt)i$, where a, b are constants. Are these all? Henry worked on this question jointly with Arne Beurling during his stay in Uppsala in 1950–51. Making use of a very ingenious argument Henry found back in the States, they jointly showed that the answer is Yes [2].

In a related paper [3], Henry classified isomorphisms of group algebras of certain abelian topological groups.

During the next decade, Henry studied the following problem: Let μ be a finite positive measure on $[0, 2\pi]$. Form the weighted L^2 space on this, with weight μ. Under what conditions on μ is the conjugation operator on this space a bounded operator? Jointly with Gabor Szegö, Henry found a necessary and sufficient condition on μ [4]. This result has been important in further developments in real analysis.

Moving to a different terrain, Henry and Frank Quigley made a contribution to the study of function algebras. They gave some results on maximal function algebras and found conditions on a smooth closed curve γ in C^n in order that every continuous function on γ can be uniformly approximated by polynomials in the complex coordinates [5].

Among Helson's most influential papers were [6] and [7], written jointly with David Lowdenslager. They deal with a generalization of classical harmonic analysis from the circle to a class of compact abelian topological groups. Observe that the integers Z form a discrete, ordered abelian group with dual group the circle Γ. The characters of Γ are the functions $\exp(in\theta)$ on Γ, labeled by n in Z. Using only half of the characters, we can form functions $\sum_0^{\infty} c_n \exp(in\theta)$, whose Fourier coefficients with negative index are 0. Such functions are boundary functions of holomorphic functions in the unit disk. The closure of these functions in the L^p norm on the circle, using the measure $d\theta$, are the Hardy spaces H^p. A beautiful, powerful theory of these Hardy spaces was developed in the 1920s and 1930s, based on the classical theorems of Hardy, F. and M. Riesz, Szegö, Nevanlinna, Beurling, and others.

In the 1950s people started to extend Fourier series theory by replacing the circle group by a compact abelian group G with suitably ordered dual group and using the characters χ_λ on G, labeled by λ in the dual group: the counterparts of the series $\sum_0^{\infty} c_n \exp(in\theta)$ on Γ are the series $\sum_{\lambda \geq 0} c_\lambda \chi_\lambda$, where $\lambda \geq 0$ is taken in the sense of the ordering on the dual group of G.

Using Haar measure on G to replace the measure $\frac{1}{2\pi} d\theta$ on Γ, one then defines H^p spaces on G to replace the Hardy spaces on Γ. An example is provided by taking G to be the 2-torus, with dual group the group $Z + Z$, and $(p, q) > 0$ in $Z + Z$ if (p, q) lies above the line $n + \alpha m = 0$, where α is a fixed positive irrational number. The characters χ_λ now are the functions $\exp(in\theta)\exp(im\phi)$ on the torus G.

Henry with violin bow, Berkeley, 1980s.

Helson and Lowdenslager succeeded in generalizing classical H^p theory to this new context in a remarkably elegant and complete way. Their work had important consequences both for stochastic processes and for the study of a class of commutative Banach algebras, the "sup norm" algebras. Kenneth Hoffman in [8] was able to use their ideas to make significant progress in this field.

I myself came to interact with their work in my effort to solve a problem in the theory of sup norm algebras posed by the work of Andrew Gleason in [9]. Gleason had introduced the notion of "parts" in the maximal ideal space of a sup norm algebra and conjectured the existence of complex analytic structure in parts that are not a single point. I was able, using methods from the Helson–Lowdenslager theory, to prove this conjecture for the case of "Dirichlet" algebras. These are sup norm algebras on a compact space X such that the real parts of the functions in the algebra are uniformly dense in the real continuous functions on X [10].

My own personal connection with Henry goes back a long way. We were classmates in the Harvard class of 1947, and for some years in the early 1950s, we were fellow instructors at Yale. In that period we shared an apartment in New Haven. Henry was a quite competent cook, and I was a loyal assistant, and we ran some quite successful dinner parties. In the spring of 1952 I got married, and Christine and I needed an apartment. Henry, very generously, was willing to move out and allow me to trade apartment mates. He also said to me at that time, "You are lucky, John. She thinks you are funny." I took it as a compliment.

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William A. Veech

Henry Helson was a wonderful human being. An invitation to contribute to this memorial article immediately brought to mind a very important episode from the early part of one career.

After spending a first academic year, in this case 1966–67, in the legendary permanent temporary building, T4, a newcomer to Berkeley would be relocated to Campbell Hall, as often as not in the office of a permanent faculty member on leave for the year. In 1967–68 Jack Feldman's office was made available thanks to his being on leave for a year in Russia. However, Jack cut short his leave and returned to Berkeley at midyear with the prospect of being cramped in a small office with a visitor. While Jack accepted this imposition with grace, Henry Helson saved the day with an unexpected invitation to join him in his much larger office. This was a little disconcerting to a young mathematician who barely knew Henry and who, having read Helson, especially Helson-Lowdenslager, in his graduate school days, had viewed Henry as something of a hero.

During the several months of joint officing, my "hero" became a great friend. Henry offered constant encouragement to an officemate who was absorbed, even fixated, with a problem about $\mathbb{Z}_2$-skew products. The relaxed atmosphere created by daily conversations with Henry about all matters, mathematical and nonmathematical, probably contributed as much to the solution as did my own efforts. Thank you and Godspeed, Henry!

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William Arveson

Helson and Subdiagonal Operator Algebras

Commutative Origins

Henry Helson is known for his work in harmonic analysis, function theory, invariant subspaces, and related areas of commutative functional analysis. I don't know the extent to which Henry realized, however, that some of his early work inspired significant developments in noncommutative directions—in the area of nonself-adjoint operator algebras. Some of the most definitive results were obtained quite recently. I think he would have been pleased by that—while vigorously disclaiming any credit. But surely credit is due; and in this note I will discuss how his ideas contributed to the noncommutative world of operator algebras.

It was my good fortune to be a graduate student at UCLA in the early 1960s, when the place was buzzing with exciting new ideas that had grown out of the merger of classical function theory and the more abstract theory of commutative Banach algebras as developed by Gelfand, Naimark, Raikov, Silov, and others. At the same time, the emerging theory of von Neumann algebras and C^*-algebras was undergoing rapid and exciting development of its own. One of the directions of that noncommutative development—though it went unrecognized for many years—was the role of ergodic theory in the structure of von Neumann algebras that was pioneered by Henry Dye [Dye59], [Dye63]. *That* Henry would become my thesis advisor. I won't say more about the remarkable development of noncommutative ergodic theory that is evolving even today, because it is peripheral to what I want to say here. I do want to describe the development of a class of nonself-adjoint operator algebras that relates to analytic function theory, prediction theory, and invariant subspaces: subdiagonal operator algebras.

It is rare to run across a reference to Norbert Wiener's book on prediction theory [Wie57] in the mathematical literature. That may be partly because the book is directed toward an engineering audience and partly because it was buried as a classified document during the war years. Like all of Wiener's books, it is remarkable and fascinating, but not an easy read for students. It was inspirational for me and was the source from which I had learned the rudiments of prediction theory that I brought with me to UCLA as a graduate student. Wiener was my first mathematical hero.

Dirichlet algebras are a broad class of function algebras that originated in efforts to understand

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the *disk algebra* $A \subseteq C(\mathbb{T})$ of continuous complex-valued functions on the unit circle whose negative Fourier coefficients vanish. Several paths through harmonic analysis or complex function theory or prediction theory lead naturally to this function algebra. I remind the reader that a *Dirichlet algebra* is a unital subalgebra $A \subseteq C(X)$ (X being a compact Hausdorff space) with the property that $A + A^* = \{f + \bar{g} : f, g \in A\}$ is sup-norm-dense in $C(X)$; equivalently, the real parts of the functions in A are dense in the space of real valued continuous functions. One cannot overestimate the influence of the two papers of Helson and Lowdenslager ([HL58], [HL61]) in abstract function theory and especially Dirichlet algebras. Their main results are beautifully summarized in Chapter 4 of Ken Hoffman's book [Hof62].

Along with a given Dirichlet algebra $A \subseteq C(X)$, one is frequently presented with a distinguished complex homomorphism

$$\phi : A \to \mathbb{C},$$

and because $A + A^*$ is dense in $C(X)$, one finds that there is a unique probability measure μ on X (of course I really mean unique *regular Borel* probability measure) that represents ϕ in the sense that

$$\phi(f) = \int_X f \, d\mu, \qquad f \in A. \tag{1.1}$$

Here we are more concerned with the closely related notion of weak*-Dirichlet algebra $A \subseteq L^\infty(X, \mu)$, in which uniform density of $A + A^*$ in $C(X)$ is weakened to the requirement that $A + A^*$ be dense in $L^\infty(X, \mu)$ relative to the weak*-topology of L^∞. Of course we continue to require that the linear functional (1) should be multiplicative on A.

Going Noncommutative

Von Neumann algebras and C^*-algebras of operators on a Hilbert space H are self-adjoint—closed under the $*$-operation of $\mathcal{B}(H)$. But most operator algebras do not have that symmetry; and for nonself-adjoint algebras, there was little theory and few general principles in the early 1960s beyond the Kadison-Singer paper [KS60] on triangular operator algebras (Ringrose's work on nest algebras was not to appear until several years later).

While trolling the waters for a thesis topic, I was struck by the fact that so much of prediction theory and analytic function theory had been captured by Helson and Lowdenslager, while at the same time I could see diverse examples of operator algebras that seemed to satisfy noncommutative variations of the axioms for weak*-Dirichlet algebras. There had to be a way to put it all together in an appropriate noncommutative context that would retain the essence of prediction theory and contain important examples of operator algebras. I worked on that idea for a year or two and produced a Ph.D. thesis in 1964—which evolved into a more definitive paper [Arv67]. At the time I wanted to call these algebras *triangular*, but Kadison and Singer had already taken the term for their algebras [KS60]. Instead, these later algebras became known as *subdiagonal* operator algebras.

Here are the axioms for a (concretely acting) subdiagonal algebra of operators in $\mathcal{B}(H)$. It is a pair $(\mathcal{A}, \phi)$ consisting of a subalgebra $\mathcal{A}$ of $\mathcal{B}(H)$ that contains the identity operator and is closed in the weak*-topology of $\mathcal{B}(H)$, all of which satisfy

- SD1: $\mathcal{A} + \mathcal{A}^*$ is weak*-dense in the von Neumann algebra $\mathcal{M}$ it generates.
- SD2: ϕ is a conditional expectation, mapping $\mathcal{M}$ onto the von Neumann subalgebra $\mathcal{A} \cap \mathcal{A}^*$.
- SD3: $\phi(AB) = \phi(A)\phi(B)$, for all $A, B \in \mathcal{A}$.

What SD2 means is that ϕ should be an idempotent linear map from $\mathcal{M}$ onto $\mathcal{A} \cap \mathcal{A}^*$ that carries positive operators to positive operators, is continuous with respect to the weak*-topology, and is faithful in the sense that for every positive operator $X \in \mathcal{M}$, $\phi(X) = 0 \implies X = 0$.

We also point out that these axioms differ slightly from the original axioms of [Arv67] but are equivalent when the algebras are weak*-closed.

Examples of subdiagonal algebras:

(1) The pair $(\mathcal{A}, \phi)$, $\mathcal{A}$ being the algebra of all lower triangular $n \times n$ matrices, $\mathcal{A} \cap \mathcal{A}^*$ is the algebra of diagonal matrices, and $\phi : M_n \to \mathcal{A} \cap \mathcal{A}^*$ is the map that replaces a matrix with its diagonal part.

(2) Let G be a countable discrete group which can be totally ordered by a relation $\le$ satisfying $a \le b \implies xa \le xb$ for all $x \in G$. There are many such groups, including finitely generated free groups (commutative or noncommutative). Fix such an order $\le$ on G and let $x \mapsto \ell_x$ be the natural (left regular) unitary representation of G on its intrinsic Hilbert space $\ell^2(G)$, let $\mathcal{M}$ be the weak*-closed linear span of all operators of the form ℓ_x, $x \in G$, and let $\mathcal{A}$ be the weak*-closed linear span of operators of the form ℓ_x, $x \ge e$, e denoting the identity element of G. Finally, let ϕ be the state of M defined by

$$\phi(X) = \langle X\xi, \xi \rangle, \qquad X \in M, \quad \xi = \chi_e.$$

If we view ϕ as a conditional expectation from $\mathcal{M}$ to the algebra of scalar multiples of the identity operator by way of $X \mapsto \phi(X)\mathbf{1}$, then we obtain a subdiagonal algebra of operators $(\mathcal{A}, \phi)$.

(3) There are natural examples of subdiagonal algebras in II_1 factors $\mathcal{M}$ that are based on ergodic measure-preserving transformations that will be familiar to operator algebraists (see [Arv67]).

In order to formulate the most important connections with function theory and prediction theory, one requires an additional property called *finiteness* in [Arv67]: there should be a distinguished tracial state τ on the von Neumann algebra $\mathcal{M}$ generated by $\mathcal{A}$ that preserves ϕ in the sense that $\tau \circ \phi = \tau$. Perhaps we should indicate the choice of τ by writing $(\mathcal{A}, \phi, \tau)$ rather than $(\mathcal{A}, \phi)$, but we shall economize on notation by not doing so.

Recall that the simplest form of *Jensen's inequality* makes the following assertion about functions $f \neq 0$ in the disk algebra: $\log|f|$ *is integrable around the unit circle, and the geometric mean of* $|f|$ *satisfies*

$$(2.1) \qquad |\frac{1}{2\pi}\int_{\mathbb{T}} f(e^{i\theta})\,d\theta| \leq \exp\frac{1}{2\pi}\int_{\mathbb{T}}\log|f(e^{i\theta})|\,d\theta.$$

In order to formulate this property for subdiagonal operator algebras, we require the determinant function of Fuglede and Kadison [FK52]—defined as follows for invertible operators X in $\mathcal{M}$:

$$\Delta(X) = \exp\tau(\log|X|),$$

$|X|$ denoting the positive square root of X^*X. There is a natural way to extend the definition of Δ to arbitrary (noninvertible) operators in $\mathcal{M}$. For example, when $\mathcal{M}$ is the algebra of $n \times n$ complex matrices and τ is the tracial state, $\Delta(X)$ turns out to be the positive nth root of $|\det X|$.

Corresponding to (2.1), we will say that a finite subdiagonal algebra $(\mathcal{A}, \phi)$ with tracial state τ satisfies *Jensen's inequality* if

$$(2.2) \qquad \Delta(\phi(A)) \leq \Delta(A), \qquad A \in \mathcal{A},$$

and we say that $(\mathcal{A}, \phi)$ satisfies *Jensen's formula* if

$$(2.3) \qquad \Delta(\phi(A)) = \Delta(A), \qquad A \in \mathcal{A} \cap \mathcal{A}^{-1}.$$

It is not hard to show that (2.2) $\implies$ (2.3).

Finally, the connection with prediction theory is made by reformulating a classical theorem of Szegö, one version of which can be stated as follows: For every positive function $w \in L^1(\mathbb{T}, d\theta)$ one has

$$\inf_f \int_{\mathbb{T}} |1 + f|^2 w\,d\theta = \exp\int_{\mathbb{T}} \log w\,d\theta,$$

f ranging over trigonometric polynomials of the form $a_1 e^{i\theta} + \cdots + a_n e^{in\theta}$. In the noncommutative setting, there is a natural way to extend the definition of determinant to weak*-continuous positive linear functionals ρ on $\mathcal{M}$, and the proper replacement for Szegö's theorem turns out to be the following somewhat peculiar statement: For every weak*-continuous state ρ on $\mathcal{M}$,

$$(2.4) \qquad \inf \rho(|D + A|^2) = \Delta(\rho),$$

the infimum taken over $D \in \mathcal{A} \cap \mathcal{A}^*$ and $A \in \mathcal{A}$ with $\phi(A) = 0$ and $\Delta(D) \geq 1$.

In the 1960s there were several important examples for which I could prove properties (2.2), (2.3), and (2.4), but I was unable to establish them in general. The paper [Arv67] contains the results of that effort. Among other things, it was shown that every subdiagonal algebra is contained in a unique *maximal* one and that maximal subdiagonal algebras admit *factorization*: *Every invertible positive operator in* $\mathcal{M}$ *has the form* $X = A^*A$ *for some* $A \in \mathcal{A} \cap \mathcal{A}^{-1}$. Factorization was then used to show the equivalence of these three properties for arbitrary *maximal* subdiagonal algebras.

Resurrection and Resurgence

I don't have to say precisely what *maximality* means because, in an important development twenty years later, Ruy Exel [Exe88] showed that the concept is unnecessary by proving the following theorem: *Every (necessarily weak*-closed) subdiagonal algebra is maximal.* Thus, factorization holds *in general* and the three properties (2.2), (2.3), and (2.4) are *always* equivalent.

Encouraging as Exel's result was, the theory remained unfinished because no proof existed that Jensen's inequality, for example, was true in general. Twenty more years were to pass before the mystery was lifted. In penetrating work of Louis Labuschagne and David Blecher [Lab05], [BL08], [BL07a], [BL07b], it was shown that not only are the three desired properties true in general but virtually all of the classical theory of weak*-Dirichlet function algebras generalizes appropriately to subdiagonal operator algebras.

I hope I have persuaded the reader that there is an evolutionary path from the original ideas of Helson and Lowdenslager, through forty years of sporadic progress, to a finished and elegant theory of noncommutative operator algebras that embodies a remarkable blend of complex function theory, prediction theory, and invariant subspaces.

Henry with Jun-ichi Tanaka, Berkeley campus, May 1984.

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Kathy Merrill

I first met Henry Helson on a bus commuting to a regional AMS meeting in Salt Lake City one spring morning in 1983. Later that morning, Henry told the group attending his talk in our special session that I knew more about cocycles of an irrational rotation than he did. This of course was not true, but I was to learn that it was typical of Henry's kindness and generosity. I was just finishing my Ph.D. at the University of Colorado and was on the job market, hungry for praise but not so deserving of it. Henry, on the other hand, was already well known for using cocycles to mine information about invariant subspaces of functions on compact groups with ordered duals [2]. Many of his results had important applications to ergodic theory and Diophantine approximation as well.

Our conversations in Salt Lake City began a mathematical friendship that added joy as well as practical support to the beginning of my career. Henry helped me get a (non-tenure-track) job offer at Berkeley, which was quite helpful to my confidence even though I had to turn it down. He went on to grace me with a regular stream of correspondence over the next ten years, including conjectures, examples, references, and odd bits of mathematical beauty. One result of this interchange was a joint publication [4], leading to my enviable Helson number of 1. Henry also wrote to me about his philosophy of teaching: "It is simply not true that ordinary students are incapable of abstraction, and must be jollied with a constant stream of pseudo-applications." He sent reassurance when I expressed self-doubt. When I told

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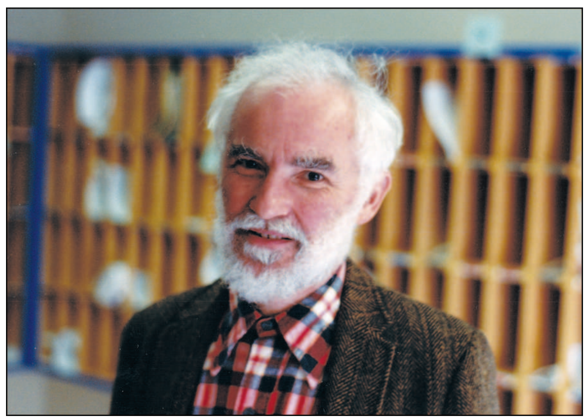
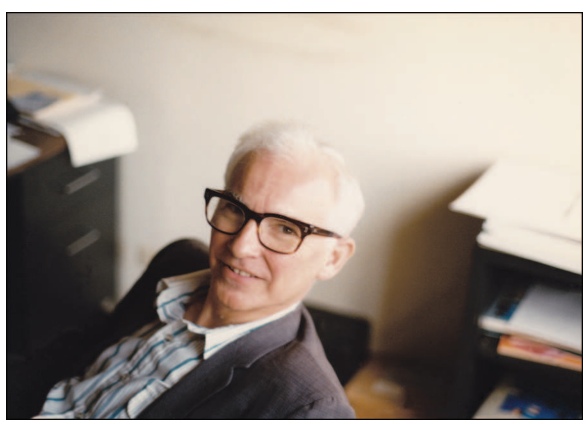


Henry before a shave, Evans Hall, Berkeley, 1984 (from George Bergman's collection, with his permission).



Henry after a shave, Evans Hall, Berkeley, 1984 (from George Bergman's collection, with his permission).

him I was afraid I would never have another new idea, he told me he always feared that each paper he wrote would be his last, but so far it never was. When I apologized for being slow to understand something he had sent me, he assured me that he was even slower. He said that while he could write mathematical papers, he could not read them.

I spent part of my first sabbatical in 1991 visiting Henry at Berkeley. He arranged for me to speak in the functional analysis seminar there and afterward chided me for making cocycles appear too simple. We spoke of difficulties faced by women in mathematics, and particularly of the controversy brewing at Berkeley at the time. Henry's perspective, as fit his character, was thoughtful and kind, avoiding angry rhetoric and supporting a valued colleague. During that visit, he introduced me to his student Herbert Medina, with whom I began a fruitful and pleasurable collaboration. My work with Herbert and fellow coauthor Larry Baggett soon moved from cocycles to wavelets. Henry Helson's work was lurking under the central idea there as well. Our approach to wavelets depended on applying the multiplicity function of a spectral measure to the representation of the integers as translations in the core subspace of a multiresolution analysis. In his book, *The Spectral Theorem* [3], Henry gives a simple and elegant account of the multiplicity function, using range functions instead of direct integrals. Indeed, a referee for our wavelet paper [1] commented that he had to read Henry Helson to understand our approach. Many other wavelet researchers have also found Helson's range functions a useful tool for understanding the shift-invariant subspaces associated with wavelets and multiresolution analyses.

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Jun-ichi Tanaka

Recollections of Henry Helson

In early spring of 1980, Ravenna and Henry stopped at Tokyo on their way to Calcutta. M. Hasumi and I met them at the airport, which was the first time I saw Henry in person. At that time I was studying function algebras and got interested in function theory associated with flows, developed by F. Forelli and P. Muhly. This research direction was pioneered primarily by the series of works by Helson and Lowdenslager.

A few years later, 1984–1985, I had a chance to stay at Berkeley as a postdoc, and Henry was willing to take me in as an advisee. It was a fantastic experience, although my poor English caused a lot of trouble. I joined in the Arveson seminar and occasionally attended colloquia with Henry. I remember vividly D. Sarason's lecture on the Bieberbach conjecture, P. Jones's one on the corona problem, and so on.

I felt rather shy around Henry, but he was always kind and friendly. He would look at what I was reading over my shoulder, then explain to me related topics. I felt very lucky to learn function theory on groups directly from one of its founders. Henry sometimes talked about mathematicians whom he had met in the past. I was surprised a little to learn that he knew S. Kakutani and his family well.

Helson and Lowdenslager developed their theory on a compact abelian group K dual to a dense discrete subgroup Γ of $\mathbf{R}$. (This setting is more general than it may look.) Let σ be the normalized Haar measure on K. The *Hardy space* $H^p(\sigma), 1 \le p \le \infty$, is defined to be the space of all functions in $L^p(\sigma)$ whose Fourier coefficients with negative indices vanish. Since $H^\infty(\sigma)$ is a weak*-Dirichlet algebra in $L^\infty(\sigma)$, we are led to study the structure of invariant subspaces, which can be investigated in detail by virtue of the group structure. An ergodic flow $x \to x + e_t$ is introduced naturally on K, called an *almost periodic flow*. Then ϕ lies in $H^p(\sigma)$ if and only if for almost every x, $t \to \phi(x + e_t)$ lies in $H^p(dt/\pi(1 + t^2))$, the closure of $H^\infty(\mathbf{R})$ in $L^p(dt/\pi(1 + t^2))$. This observation enables us to extend such Hardy spaces to the case of general ergodic flows. Recall that a closed subspace $\mathfrak{M}$ of $L^p(\sigma)$ is *invariant* if $H^\infty(\sigma) \cdot \mathfrak{M}$ is contained in $\mathfrak{M}$. Denote by $H_0^p(\sigma)$ the subspace of all functions in $H^p(\sigma)$ with mean value zero. This invariant subspace $H_0^p(\sigma)$ plays an important role. Dealing with invariant subspaces, we may restrict our attention to the case of $p = 2$. There is a correspondence between invariant subspaces and certain unitary functions on $K \times \mathbf{R}$, called *cocycles*.

Henry raised repeatedly the problem of whether every invariant subspace can be represented as the smallest one containing a definite element, the so-called *single generator problem* (SGP). And he showed me a lot of equivalent and sufficient conditions. In his survey article [5, 1975], he writes "Is every simply invariant subspace of L^2 generated by one of its elements? Has H_0^2 a single generator? These problems are old, and apparently untouched by all that precedes. Answers will be interesting

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and probably difficult." Henry once told me he raised the problem in 1957, but it seems to appear in [6, p. 183] for the first time, in an equivalent form related to evanescent stochastic processes. The difficulty of SGP seemed to center on the case of $H_0^2(\sigma)$.

This case had been the main subject of my research for a long time. (Yes, this sentence is written in the past perfect tense. I believe that I finally succeeded recently in settling it.) The answer is negative in the context of Helson and Lowdenslager: In the case of almost periodic flows, H_0^2 has no single generator, and there are many kinds of simply invariant subspaces of L^2 with no generator. In view of these facts it may sound strange to say that it can be shown that there is an ergodic flow on which H_0^2 has a single generator. Thus, in the setting of weak*-Dirichlet algebras, it is meaningful to characterize the extreme points of the unit ball of H_0^1, which is due to Gamelin. Henry seemed to be very satisfied with this report.

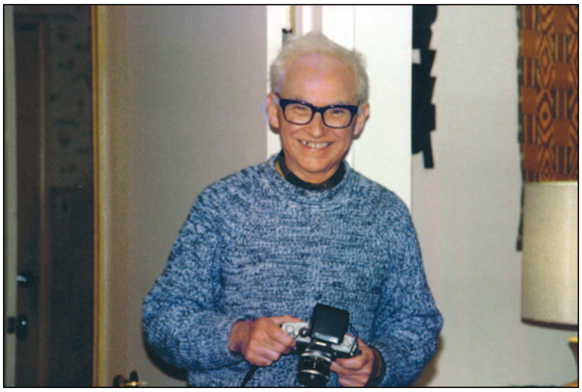
In the process of investigating SGP, I learned a lot of things and had some results on flows. M. Nadkarni once wrote me that he had also gotten interested in SGP soon after he finished his Ph.D. in 1965, and had obtained a number of results that he had been able to publish. We seemed to share the same experience in this regard.

Henry was very encouraging in spite of my rather slow progress and was always pleased when I could obtain new ideas to attack the problem. As an example of by-products of my endeavor, let me explain how one can settle, by using my ideas, both questions of Forelli's stated in his report [2] for the Nice Congress, which sits back-to-back with Henry's [3]. At the end of the 1980s, I was eager to look for an ergodic flow on which H_0^2 is singly generated. The shift on the integers $\mathbf{Z}$ induces a homeomorphism τ on the Stone-Čech compactification $\beta\mathbf{Z}$ of $\mathbf{Z}$. Let X be the quotient space of $\beta\mathbf{Z}\times[0,1]$, by identifying $(\omega,1)$ with $(\tau(\omega),0)$. Then we have a continuous flow on X being an extension of the translation on $\mathbf{R}$. I am still interested in this flow in connection with the corona theorem. Indeed, by a normal families argument, one can show that $H^\infty(\mathbf{R})$ is isometrically isomorphic to $H^\infty(X)$, the algebra of all bounded Borel functions which are analytic on each orbit. Then $X\times(0,\infty)$ represents concretely a certain part of the fiber of the maximal ideal space of $H^\infty(\mathbf{R})$. Notice that the upper half-plane $\mathbf{R}\times(0,\infty)$ is a dense open subset of $X\times(0,\infty)$. There are a lot of representing measures for $H^\infty(X)$ that are invariant, which suggests that there may exist a relation between the corona problem and the individual ergodic theorem, via Wiener's Tauberian theorem. Somehow, I felt as if I understood vaguely why Kakutani had said occasionally, "There should be a simple direct proof of the corona theorem" (I learned Kakutani's remark from O. Hatori). In any event, I had not only a desired flow in a quotient of X but also a negative answer to one of Forelli's questions: There is a minimal flow on which the induced uniform algebra is not a Dirichlet algebra, which holds on each minimal set in X. The other question was also settled in the course of looking into the absolute values of single generators. It runs "What makes a positive measure the total variation measure of an analytic measure?" (This is also a title of one of Forelli's papers [1].) The answer turned out to be: the necessary and sufficient condition for this to be true is that the distant future equals $\{0\}$. Of course, both results heavily relied on the preceding work of Muhly.

What do you imagine from these key words: "analytic almost periodic functions", "the behavior around infinity", and "the distribution of zeros"? One always confronts these phrases when dealing with the structure of invariant subspaces in the almost periodic setting. In the autumn of 1984, the rumor that a Japanese mathematician staying in France at the time had solved the Riemann Hypothesis (RH) spread fast in the Berkeley mathematics community. Henry seemed to get excited about it and wanted to know about this mathematician and the group of number theorists in Japan, although I did not know anything at all about them. As it happened, the rumor turned out to be premature, and the story died down soon thereafter. I felt that just like A. Beurling, Henry might have had RH as one of his problems he should challenge, as Dirichlet series was central to his life's work, and Cassels's book was always his favorite. He once provided another proof of the convergence of $\zeta(1)^{-1}$, which seems roughly to have the depth of the prime number theorem [4]. I learned from him how to extend $\zeta(s)$ and $\zeta(s)^{-1}$ to outer functions in $H^2(\mathbf{T}^\infty)$, and the similarity between Carleson's convergence theorem and the convergence of $\zeta(s)^{-1}$ on the critical strip, which is equivalent to RH.

Although I had no chance to visit Berkeley again after that, we became close friends, keeping in touch until his death. Henry was a good correspondent and a masterly writer. After discussing mathematical ideas, he usually wrote about the recent situation of himself and his family. I remember how he was delighted when his tender for a fine old Italian instrument was accepted (he was a well-known violinist) and how he was happy when his elder son David got married to an English lady in London. He also reported his regrets when it became too hard to continue making wine by himself.

Last autumn, in connection with Rudin's autobiography [7], we corresponded about polydisc algebras. The closed subalgebra of $H^\infty(\mathbf{R})$ obtained by restricting $A(\mathbf{T}^2)$ to one orbit is the simplest one for which the corona theorem fails. A



Henry with camera, Berkeley, early 1990s.

cocycle argument provides easily the fact that an entire inner function in $A(\mathbf{T}^n)$ is a monomial (the theorem of Bojanic and Stoll), as it gives rise to a cocycle of the form $\exp(i\lambda t)$, a weight at infinity.

At the end of last year, I sent season's greetings as usual, together with my new idea to attack the corona problem on polydiscs, depending on tensor products. However, I had no answer from him, and his poor health lay heavily on my mind.

Then, early in the new year, I received the sad news from Hasumi and Nadkarni one after another.

I feel so lucky to have met such a first-rate mind at an early stage of my career and to have received his affectionate interest in my research. He will be greatly missed by everyone who has known him.

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John E. McCarthy

Henry Helson was a transforming influence on me. In my last year of graduate school, his work on the Smirnov class [5] had led him to ask the following question: What functions in the Hardy space H^2 are in the range of every coanalytic Toeplitz operator?

In other words, what functions f in H^2 have the property that for every nonzero function m in H^∞, there is a function g in H^2 such that

$$T_{\bar{m}} g \;:=\; P_{L^2 \to H^2}\, \bar{m} g \;=\; f. \tag{1}$$

D. Sarason observed that such functions must have a certain regularity. Indeed, let $m(z) = 1 - z$ and $g(z) = \sum_{n=0}^\infty a_n z^n$ in (1). Then

$$f(z) \;=\; \sum_{n=0}^{\infty} (a_n - a_{n+1}) z^n.$$

So, in particular, f must have a convergent Fourier series at 1.

Replacing m by other functions, one can get more and more regularity conditions that f needs to satisfy. Conversely, one can show that if the coefficients of f decay rapidly enough, then it will be in the range of every coanalytic Toeplitz operator. The trick was to find the right rate of decay. Several people worked on this problem around the time 1988–89.

Why was this problem transformative? Because of the way Henry discussed it with me. Even though I was just a graduate student, he treated me as an equal, with courtesy and respect. He was genuinely interested in what I had to say, or if I made ε progress. This gave me the confidence I needed to think I might be able to solve it, and in the summer of 1989, just after I graduated from Berkeley, I succeeded [7]. My work relied heavily on earlier results of N. Yanagahira [8], but nonetheless I was proud of it. I saw that paper, far more than my dissertation, as my transition from student to mathematician.

After leaving Berkeley, I kept up a frequent correspondence with Henry. He gave me a lot of encouragement and advice both mathematical and personal. We wrote one small paper together [6], but mainly wrote letters about what we were thinking about. (My missives were mainly electronic; his frequently handwritten.) He had strong opinions about most things, including grammar and style. To this day, I cannot use the word "factorization", since, as he pointed out, there is no verb "to factorize"; the correct noun is "factoring".

One piece of advice he gave me was "Every mathematician should look at the ζ-function. It is a mirror." I looked, and having spent much of my early career studying the Hardy space and related objects, I decided to study spaces of Dirichlet series $\sum a_n n^{-s}$ with square-summable coefficients. This was not a new idea—a beautiful paper on this space had been written by Hedenmalm, Lindqvist, and Seip [1]—but it gave me a mental hook to pull myself into this fascinating area. Henry himself had pioneered the introduction of functional analysis into the study of Dirichlet series [2, 3, 4],

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and he returned to the subject in the past few years. The last letter I received from him was about the generalization of Nehari's theorem on boundedness of Hankel operators to infinitely many variables. (As Harald Bohr had observed, Dirichlet series provide a convenient setting for studying functions of infinitely many variables—essentially, for each prime p, the function p^{-s} plays the role of a variable.)

When I think of Helson's work, the adjective that comes to mind is "elegant". His proofs were always clever, insightful, and graceful. As a person, he was honest and clear-seeing. Like all those who were privileged to know him, I shall miss him.

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Herbert Medina

Reminiscences of a Former Student

Henry Helson was a mathematician who wanted his students to learn to appreciate, to work on, and to love "true analysis". He was eager for us to tackle concrete problems that usually required delicate estimates. He did not believe in analysis that used "sledgehammer" theorems not requiring the estimates and inequalities that are so important to an analyst. I remember him telling me that he thought the real interesting problems were not ones with general hypotheses (e.g., "let G be a topological group"), but ones in specific domains, function spaces, or with specific operators. He shared these views with his students, and the way in which I view mathematics has very much been influenced by Helson's views.

As far as my specific dissertation work, he was the mathematician who introduced me to cocycles and operators that arose from irrational rotations on the circle and showed me how to use the spectral theorem to study and classify these. Work on problems related to these topics was the focus of the first few years of my mathematical career.

There are many memories that I have of Professor Helson, but I'll only relate one anecdote. I remember one time going into his office feeling not terribly excited about something that I was reading and communicated this to him. He simply said, "We are interested in what we work on." In other words, I simply should get to work on the mathematics in the paper and that would get me excited about the topic. He was right! I have used that phrase with my own students several times.

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Farhad Zabihi

I was one of Henry Helson's last students. I began the research on my dissertation in the summer of 1991 amid personal and financial difficulties that had troubled me for several years. Helson was aware of my circumstances but nevertheless wanted to start working together once he returned from Toulouse in late spring. What followed were two years full of excitement and mathematical discovery.

Helson proposed that I work on a problem that interpolates between the Riemann mapping theorem and a theorem of Szegö. The Riemann mapping theorem produces an analytic function on the unit disk whose boundary values lie on the boundary curve of a simply connected region. Szegö's theorem, when interpreted geometrically, says that an analytic function in H^2 can take its values on different circles centered at the origin. Helson thought these theorems could be married together, and perhaps one could find analytic functions that take their values on prescribed curves of a general type.

In our initial approach, Helson suggested that I find analytic functions whose values lie on curves given by formulas, and he thought we should try fixed point methods. Indeed, using Banach's fixed point theorem, I was able to solve the problem for families of lines, hyperbolas, and ellipses under strict conditions. Helson was particularly interested in the ellipse case because it generalized Szegö's theorem (in a narrow sense) and indicated that a solution may exist for a more general class of curves.

With much greater effort, I was able to find solutions for a class of curves that bound convex regions. The proof involved a complicated construction of analytic functions, the limit of which

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Henry and Ravenna, Berkeley, circa 2007.

yielded a solution. I was convinced the construction worked, but Helson was not satisfied. He thought that the proof was unwieldy and computational. "The hardest problems," he told me, "are solved with the simplest ideas." So I continued to work on my proof and found a new approach using ideas from the proof of the Riemann mapping theorem. It cracked the problem wide open and led to the main theorem in my dissertation.

Helson's insistence on brevity and simplicity also carried over to writing the dissertation. His writing style is clean, clear, and engaging, and he expected no less from his students. It had taken me a year to complete my research, and it took almost as long to simplify the proofs and distill them to their final form. We published these results, along with some extensions that I discovered later, in a joint paper [1]. In the course of writing my dissertation and the paper, Helson taught me as much about the art of presentation as he had about rigorous research.

By the time I graduated in 1993, the difficulties I faced early on were mostly behind me. I am sure Helson was in part responsible, for he had made a profound impression on me. I kept in touch with him through the years after my graduation. We often had lunch at his favorite Indian restaurant in Berkeley, and I later introduced him to Persian cuisine, to his delight. On reflection, he once said that he could have been more supportive of his students. I disagreed, as my own debt of gratitude to Henry Helson is immeasurable.

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Acknowledgments

Unless otherwise noted, all photos are from the Helson family collection, courtesy of Ravenna Helson.

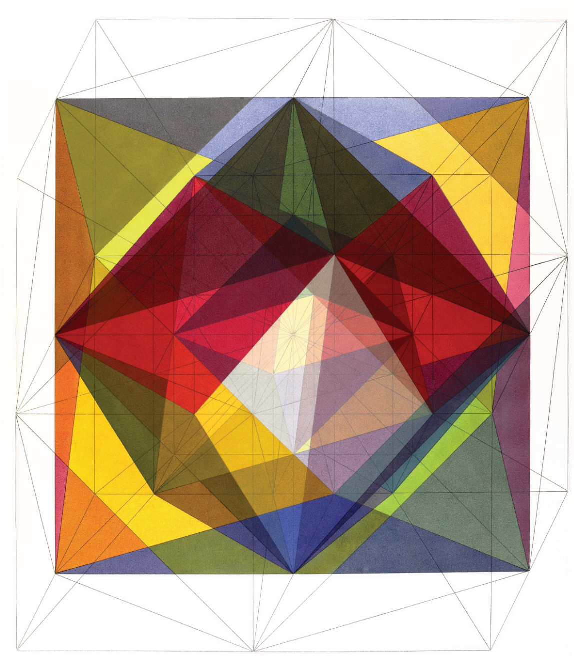
We thank Mary Jennings for her work on the photos for the article.

Ph.D. Students of Henry Helson (all are from UC Berkeley)

Frank Forelli Jr.	1961
Michael Cambern	1963
Ernest Fickas	1964
Malcolm Sherman	1964
Stephen Gerig	1966
Iri Yale	1966
Shirley Jackson Jr.	1967
Malayattil Rabindranathan	1968
Udai Tewari	1969
Samuel Ebenstein	1970
Chester Jacewitz	1970
Bonnie Saunders	1978
Geon Choe	1987
Carlos Carbonera	1990
Trent Eggleston	1990
Zachary Franko	1990
Herbert Medina	1992
Adrian Zidaritz	1992
Farhad Zabihi	1993
Nicholas James	1996

Bridges Pécs 2010

Paul Gailiunas



***Chrome 163*, John Hiigli, 2005.**

The Bridges Conferences, running annually since 1998, bring together mathematicians, scientists, artists, educators, musicians, writers, computer scientists, sculptors, dancers, weavers, model builders, and more in a lively atmosphere of exchange and mutual encouragement. Each participant will experience the conference differently, and I can only record my personal recollections of the 2010 Bridges in Pécs, Hungary, the widest ranging conference so far.

Bridges is not just talks. An innovation in 2010 was a film fest, held on the first evening. One item illustrates the informality and dynamism of Bridges. The night before the opening, people were sitting around talking, and a newcomer showed an animation. The moderators of the film fest happened to be present and were so impressed that with less than twenty-four hours' notice the schedule was changed to include this work.

The art exhibitions provide another example of the flexibility of Bridges. Work in the official exhibition has been formally submitted and appears in the catalogue: Mingjang Chen's *Chaotic Landscape Painting* uses elements that arise naturally from iterated function systems to produce a convincing landscape in the traditional Chinese style. John Hiigli's *Chrome 163* depicts overlaid images of polyhedra, and for me the challenge is to understand how they are related. Another piece that challenges the viewer to apprehend multiple images is *Coquillage* by Jacques Beck. A highlight of the informal exhibition was pieces created by

***Coquillage*, Jacques Beck, 2009.**

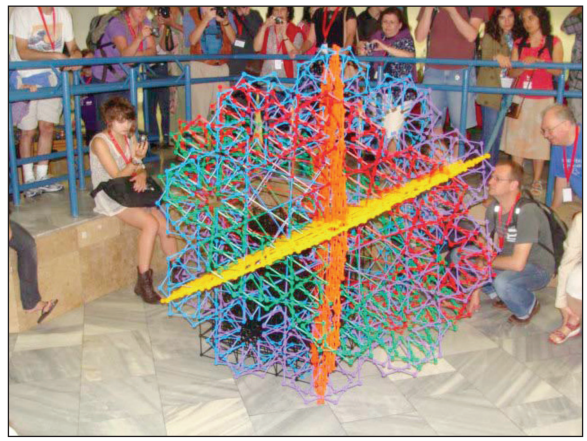
Paul Gailiunas is retired. His email address is `paulgailiunas@yahoo.co.uk`. *This article is a shortened version of a piece that is available on the AMS website at* `http://www.ams.org/meetings/Bridges2010-Gailiunas.pdf`.

scribing circles onto stainless-steel plates. Shine a light and a three-dimensional image appears. The stainless-steel versions of some of Lajos Szilassi's regular toroids have also stayed in my memory.

Another innovation was the ScienTile Competition. There were seventy-two designs that included several imaginative ideas, as well as straightforward decorative tiles of high quality. Of course there were the regular Bridges extra events: the music evening, the informal music evening, the theatre night, and the excursion.

I had visited the region of Pécs before, but I was very happy to visit some of the attractions again. The sculpture park at Nagyharsány was certainly worth a second visit. The Apáczai Education Centre, a school complex apparently attended by almost all the children in Pécs, was new to me. The main reason for visiting it was to see the Bridges Pécs 2010 Zome sculpture in its permanent place. Since it had to be transported, it was quite small, but, as usual, it was an impressive creation.

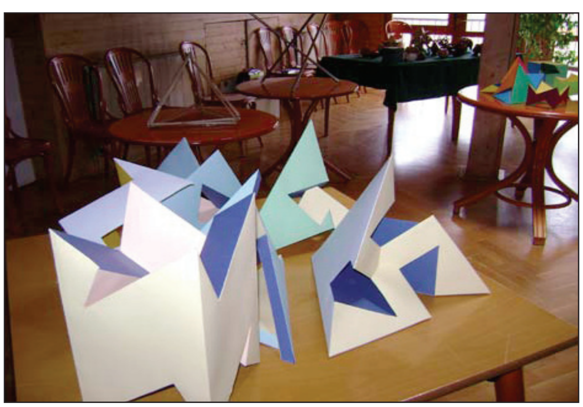
Building the Zome Sculpture



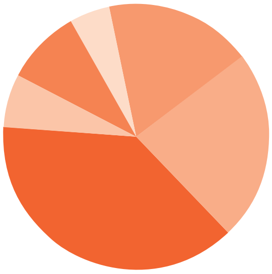
The Zome sculpture installed.

Theatre Night was a performance by the Schaffer and Stern Dance Ensemble, who use the medium of dance to explore and communicate mathematical concepts. The evening was entertaining, as well as thought-provoking, and thoroughly enjoyed by the audience. The informal music event was as varied as ever, with a selection of Bridges participants displaying an eclectic mix of styles to a high musical standard. The formal event consisted mainly of modern jazz and improvisation with the Ávéd-Fenyvesi Quartet and The *Well-Tempered Universe* by Sc.Art. We also had the ANK Pécs Children's Handbell Choir and the staggering virtuosity of Katalin Gál Poór, who performed his own composition based on the opening digits of the decimal expansion of π.

Even if you did not come to Pécs, if you have ever been to Bridges you will recognize the special character of the conference I have described. If you have never been to a Bridges conference, all I can say is, you do not know what you are missing.



The informal art exhibition.



Preliminary Report on the 2009-2010 New Doctoral Recipients

Richard Cleary, James W. Maxwell, and Colleen Rose

This report presents a statistical profile of recipients of doctoral degrees awarded by departments in the mathematical sciences at universities in the United States during the period July 1, 2009, through June 30, 2010. All information in the report was provided over the summer and early fall of 2010 by the departments that awarded the degrees. The report includes a preliminary analysis of the fall 2010 employment plans of 2009–2010 doctoral recipients and a demographic profile summarizing characteristics of citizenship status, sex, and racial/ethnic group. This preliminary report will be updated by the Report on the 2009–2010 New Doctorates to reflect subsequent reports of additional 2009-2010 doctoral recipients from the departments that did not respond in time for this report, along with additional information provided by the doctoral recipients themselves. A list of the nonresponding departments is on page 299.

Detailed information, including tables which traditionally appeared in this report, is available on the AMS website at `www.ams.org/annual-survey/survey-reports`.

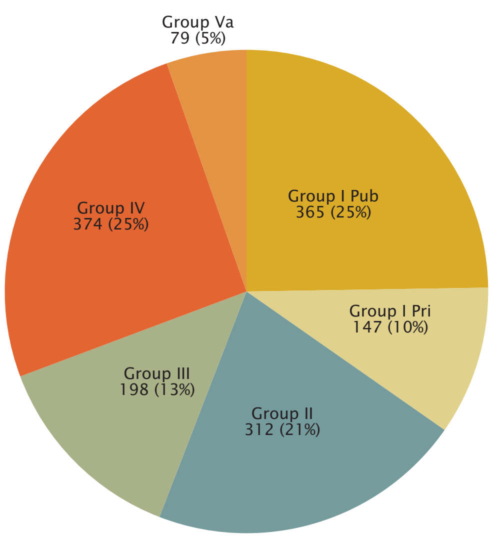
Doctoral Degrees Awarded

Based on the data collected it appears the number of Ph.D.s being awarded is still increasing. The 258 departments responding this year and last year reported a total of 1,474 new doctoral recipients, an increase of 58 over the 1,416 new doctoral recipients they reported last year. Based on this and an analysis of the departments that have yet to respond, it is likely that the final count for 2010 will exceed last year's final count of 1,605.

Again considering only the 258 departments responding both years, all groups report increases in the number of new Ph.D.s except for Group I Private. The nineteen departments in Group 1 Private responding to both surveys reported 45 fewer new doctoral recipients for 2010, 147 for 2009-2010 compared to 192 for 2008-2009. (See page 298 for a description of the department groupings.)

30% (440) of the new Ph.D.s had a dissertation in statistics/biostatistics, followed by applied mathematics (213) and algebra/number theory (206) both with 14%.

Figure A.1: Number and Percentage of Degrees Awarded by Department Groupings



Total Degrees Awarded: 1,475

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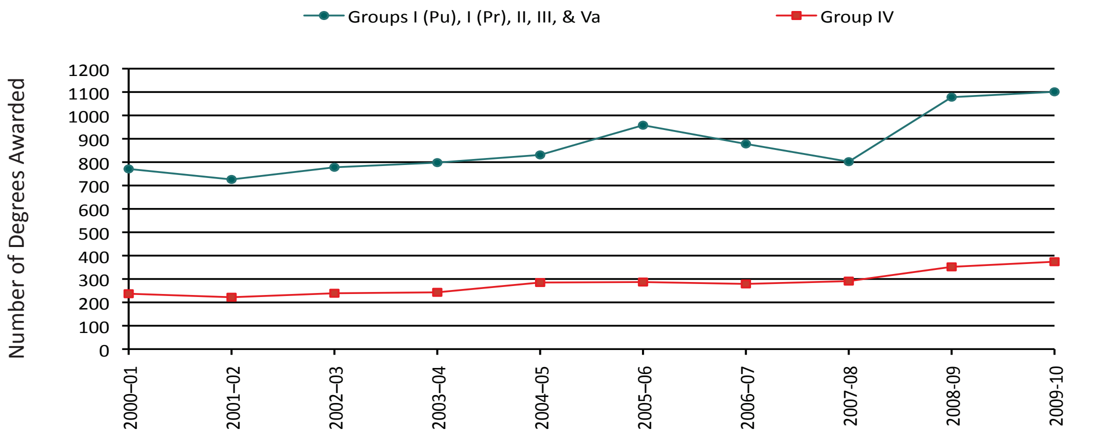
Doctoral Degrees Awarded

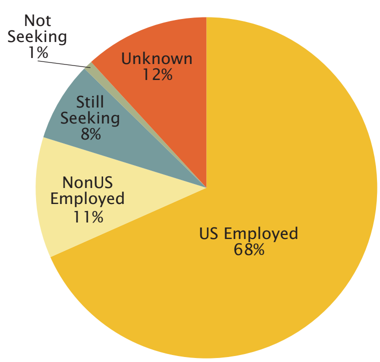
Figure A.2: New Doctoral Degrees Awarded by Combined Groups, Preliminary Counts



Employment

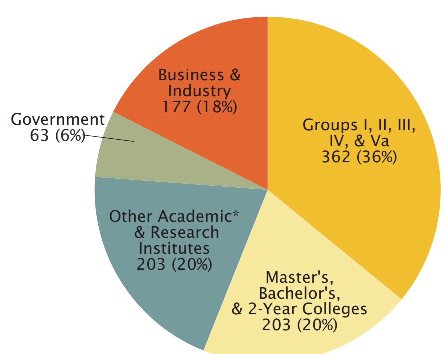
The overall preliminary unemployment rate is 9.9%, up from 7.9% last year. The employment plans are known for 1,301 of the 1,475 new doctoral recipients. The number of new doctoral recipients employed in the U.S. is 1,008, up 21 from last year's preliminary number. Employment in the U.S. increased in all categories except "Business and Industry" which decreased 4%. The number of new Ph.D.s taking positions in government has remained relatively stable at 63 this year. Academic hiring of new doctoral recipients increased to 768, compared to 741 last year.

Figure E.1: Employment Status



- 9% of new Ph.D.s are working at the institution which granted their degree, up from 6% last year and higher than recent years when it was 7% and 8%.
- 53% (531) of those employed in the U.S. are U.S. citizens, up from 51% last year.
- 80% (477) of non-U.S. citizens known to have employment are employed in the U.S., the remaining 175 non-U.S. citizens are either employed outside of the U.S. or unemployed.

Figure E.2: U.S. Employed by Type of Employer

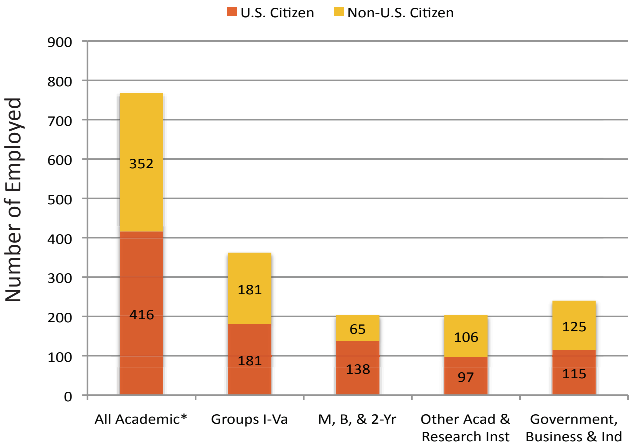


*Other Academic consists of departments outside the mathematical sciences including numerous medical related units.

- Positions in Business & Industry increased slightly in Group IV (from 85 to 90); remained the same for Group II at 23 and decreased slightly in Groups I (Pu), I (Pr) and III.

Employment

Figure E.3: Employment in the U.S. by Type of Employer and Citizenship



*Includes Groups I-Va, M, B, 2-Yr, other academic and research institutes/nonprofit.

Looking at U.S. citizens whose employment status is known:

- 82% (531) are employed in the U.S., of these:
  - 34% are employed in Ph.D.-granting departments
  - 44% are employed in all other academic positions
  - 22% are employed in government, business and industry positions

Figure E.4: New Ph.D.s in Postdocs by Degree-Granting Department

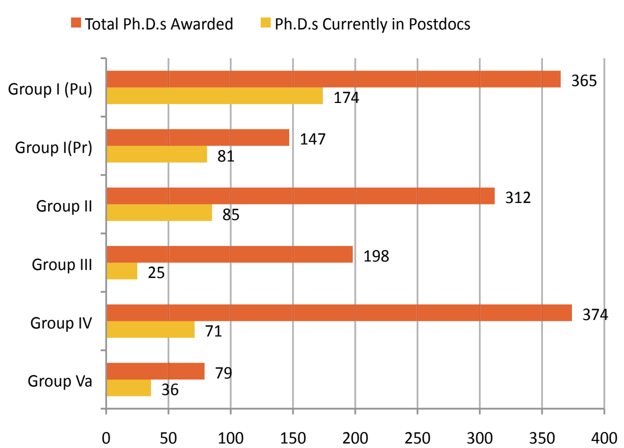
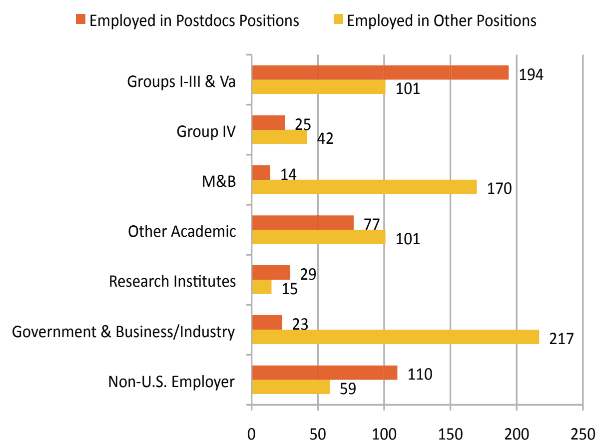


Figure E.5: Employment in Postdocs by Type of Employer

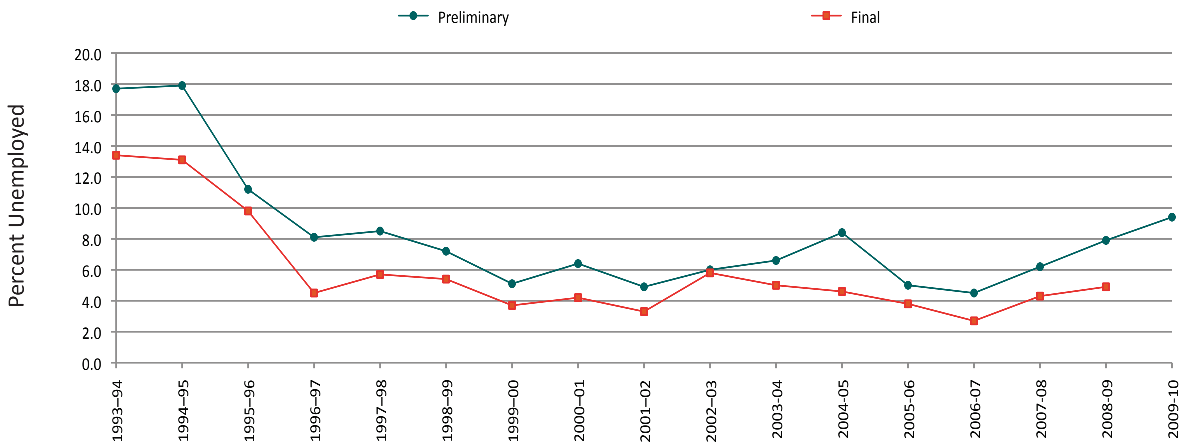


- 31% (472) of the new Ph.D.s are reported to be in postdoc positions.
- 23% of the new Ph.D.s in postdoc positions are employed outside the U.S.
- 47% of the new Ph.D.s having U.S. academic employment are in postdocs; last year this percentage was 42%.
- 60% of the new Ph.D.s employed in Groups I-Va are in postdoc positions, 42% of these postdocs received their Ph.D.s from Group I (Pu) institutions.
- 55% of the new Ph.D.s in Group I (Pr) are employed in postdocs, while only 13% of new Ph.D.s awarded by Group III are in postdocs.

- Unemployment among those whose employment status is known is 9.9%, up from 8% for fall 2009.
- Group III reported highest unemployment at 14.6%.
- Group IV reported the lowest unemployment at 4.2%.
- 10.3% of U.S. citizens are unemployed, compared to 8.6% in fall 2009.
- 9.5% of non-U.S. citizens are unemployed; the rates by visa status are 12.7% for those holding a permanent visa and 9.1% for those with a temporary visa.

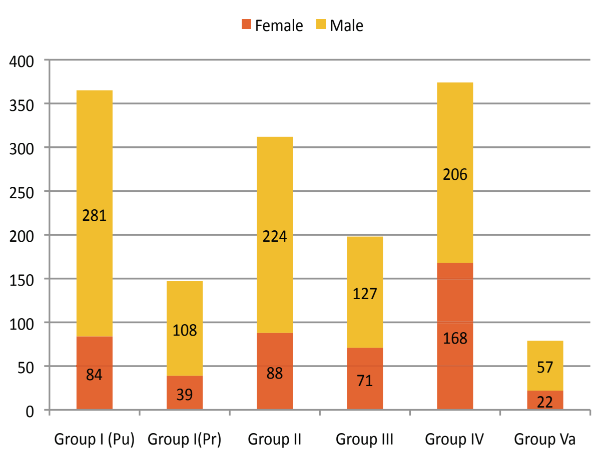
Employment

Figure E.6: Percentage of New Doctoral Recipients Unemployed 1993-2010



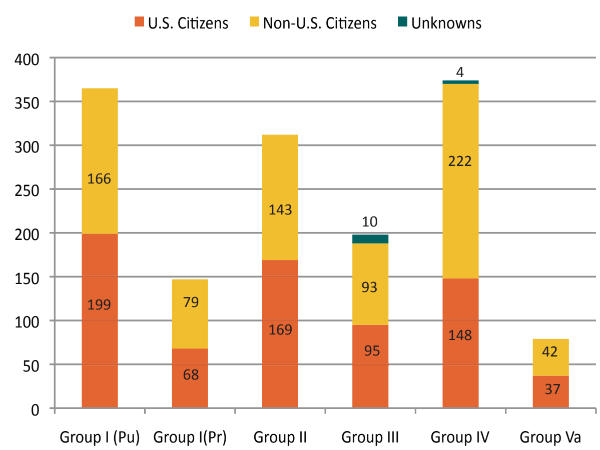
Demographics

Figure D.1: Gender of Doctoral Recipients by Degree-Granting Department



- Females account for 32% (472) of the 1,475 Ph.D.s, the same percentage reported in fall 2009 and 2008.

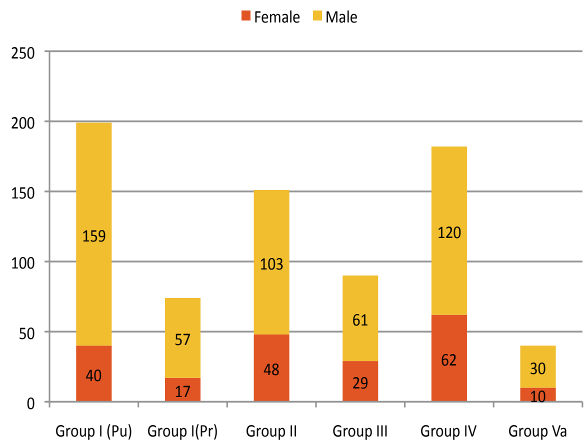
Figure D.2: Citizenship of Doctoral Recipients by Degree-Granting Department



- 49% (716) of Ph.D.s are U.S. citizens; last year's figure was 47%.

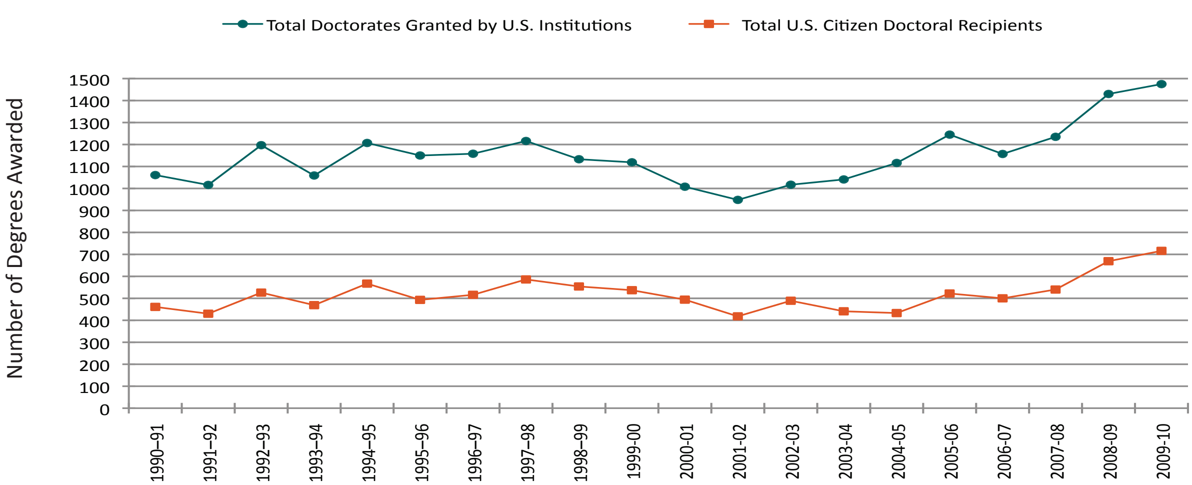
Demographics

Figure D.3: Gender of U.S. Citizen Doctoral Recipients by Degree-Granting Department



- 29% (206) of the U.S. citizens are female; last year's figure was 30%.
- Among the U.S. citizens: 4 are American Indian or Alaska Native, 23 are Asian, 22 are Black or African American, 19 are Hispanic or Latino, 5 are Native Hawaiian or Other Pacific Islander, 461 are White, and 6 are of unknown race/ethnicity.

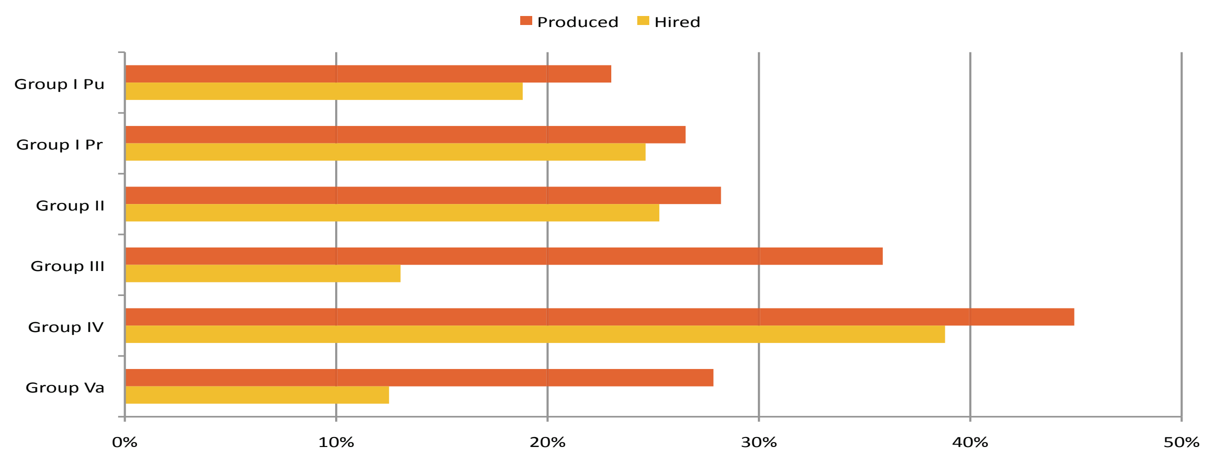
Figure D.4: U.S. Citizen Doctoral Recipients Preliminary Counts



Female New Doctoral Recipients

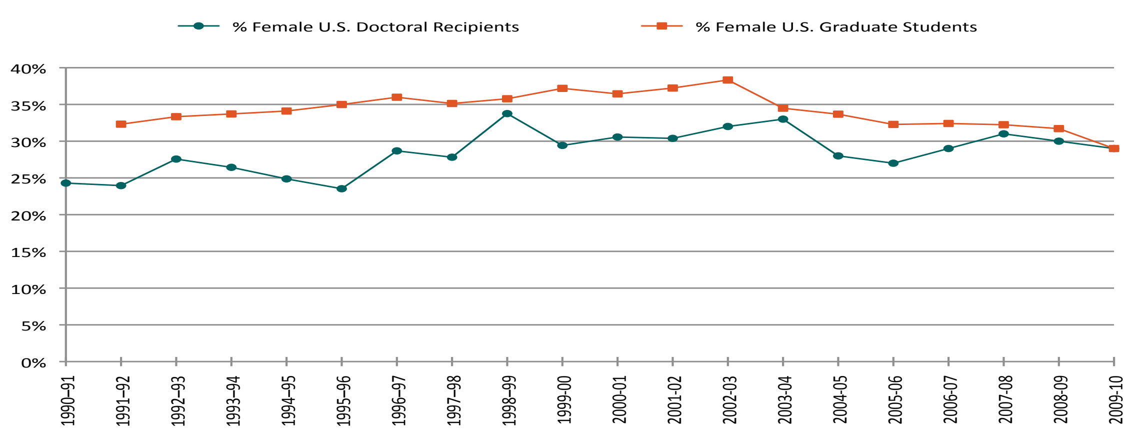
For the third consecutive year the proportion of female new doctoral recipients is 32%, based on preliminary counts. While the percentage has remained flat, the number of females receiving Ph.D.s has increased from 388 in 2008, to 462 in 2009, to 472 this year. The unemployment rate for females is 8.9%, compared to 10.4% for males and 9.9% overall.

Figure F.1: Females as a Percentage of New Doctoral Recipients Produced and Hired by Doctoral-Granting Department



- 44% of those hired by Group B were women (up from 43% last year) and 43% of those hired by Group M were women (up from 40% last year).
- 25% of those reporting having postdoc positions (472) are women.
- 43% of the women employed in Groups I-Va are in postdoc positions.

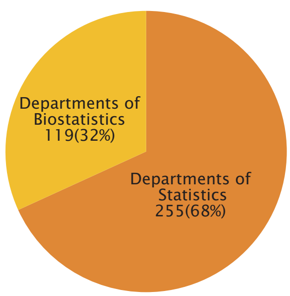
Figure F.2: Females as a Percentage of U.S. Citizen Doctoral Recipients



Ph.D.s Awarded in Group IV (Statistics/Biostatistics)

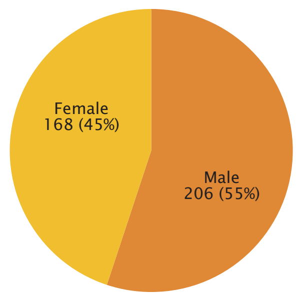
This section contains information about new doctoral recipients in Group IV. Group IV produced 374 new doctorates, of which all but 6 had dissertations in statistics/biostatistics. This is a 6% increase over the number reported for fall 2009 of 352. In addition, Groups I–III and Va combined had 72 Ph.D. recipients with dissertations in statistics. In Group IV, 148 (40%) of the new doctoral recipients are U.S. citizens (while in the other groups 52% are U.S. citizens). The 68 departments responding last year and this year reported a total of 368 new doctoral recipients, an increase of 2% over last year. While the unemployment rate for new Ph.D.s with dissertations in statistics or probability has increased to 6.5%, the unemployment among the Group IV new Ph.D.s is 4.2%.

Figure S.1: Ph.D.s Awarded in Group IV



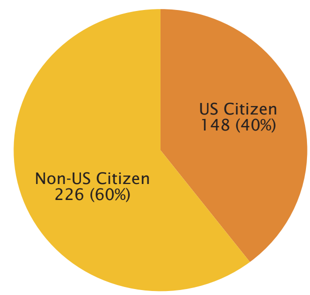
- 25% of all Ph.D.s awarded were in Group IV.
- Females account for 40% of statistics and 55% of biostatistics Ph.D.s awarded.

Figure S.2: Gender of Group IV Ph.D. Recipients



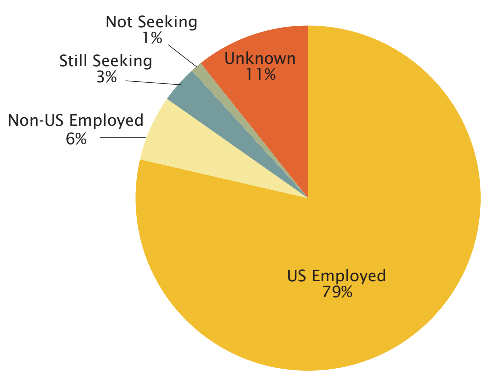
- Females accounted for 45% of the 374 Ph.D.s in statistics/biostatistics, compared to all other groups combined, where 28% (304) are female.

Figure S.3: Citizenship of Group IV Ph.D. Recipients



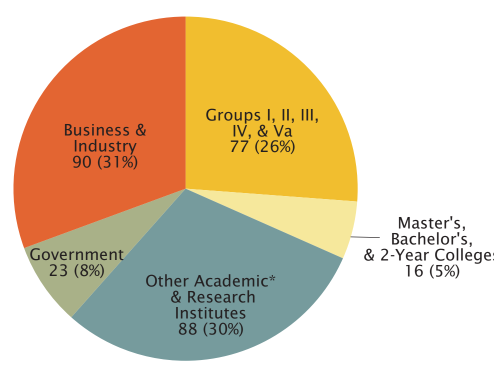
- 42% of of Group IV U.S. citizens are females, while in all other groups 25% are females.

Figure S.4: Employment Status of Group IV Ph.D. Recipients



- 4.2% of Group IV Ph.D.s are unemployed compared to 12.1% among all other groups. This is up from 3.3% last year.
- Unemployment among new Ph.D.s with dissertations in statistics/probability is 6%, up from 5%. Among all other dissertation groupings 9.9% are unemployed.

Figure S.5: U.S. Employed Group IV Ph.D. Recipients by Type of Employer



*Other Academic consists of departments outside the mathematical sciences including numerous medical related units.

- 31% of Group IV Ph.D.s are employed in Business/Industry, compared to 12% in all other groups.
- 39% of those hired by Group IV were females, compared to 21% in all other groups.

Other Information

Survey Response Rates

Doctorates Granted
Departmental Response Rates

Group I (Pu)	25 of 25 including	0 with no degrees
Group I (Pr)	19 of 23 including	0 with no degrees
Group II	51 of 56 including	2 with no degrees
Group III	74 of 81 including	8 with no degrees
Group IV	74 of 92 including	5 with no degrees
Statistics	47 of 57 including	2 with no degrees
Biostatistics	27 of 35 including	3 with no degrees
Group Va	16 of 22 including	2 with no degrees

A list of departments still to respond with their doctoral degrees awarded is available on page 299.

Previous Annual Survey Reports

The 2009 First, Second, and Third Annual Survey Reports were published in the *Notices of the AMS* in the February, August, and November 2009 issues, respectively. These reports and earlier reports, as well as a wealth of other information from these surveys, are available on the AMS website at `www.ams.org/annual-survey/survey-reports`.

Acknowledgements

The Annual Survey attempts to provide an accurate appraisal and analysis of various aspects of the academic mathematical sciences scene for the use and benefit of the community and for filling the information needs of the professional organizations. Every year, college and university departments in the United States are invited to respond. The Annual Survey relies heavily on the conscientious efforts of the dedicated staff members of these departments for the quality of its information. On behalf of the Data Committee and the Annual Survey Staff, we thank the many secretarial and administrative staff members in the mathematical sciences departments for their cooperation and assistance in responding to the survey questionnaires.

Other Sources of Data

Visit the AMS website at `www.ams.org/annual-survey/other-sources` for a listing of additional sources of data on the Mathematical Sciences.

Group Descriptions

Group I is composed of 48 departments with scores in the 3.00–5.00 range. Group I Public and Group I Private are Group I departments at public institutions and private institutions, respectively.

Group II is composed of 56 departments with scores in the 2.00–2.99 range.

Group III contains the remaining U.S. departments reporting a doctoral program, including a number of departments not included in the 1995 ranking of program faculty.

Group IV contains U.S. departments (or programs) of statistics, biostatistics, and biometrics reporting a doctoral program.

Group V contains U.S. departments (or programs) in applied mathematics/applied science, operations research, and management science which report a doctoral program.

Group Va is applied mathematics/applied science; Group Vb, which was no longer surveyed as of 1998–99, was operations research and management science.

Group M contains U.S. departments granting a master's degree as the highest graduate degree.

Group B contains U.S. departments granting a baccalaureate degree only.

Listings of the actual departments which compose these groups are available on the AMS website at `www.ams.org/annual-survey/groups_des`.

About the Annual Survey

The Annual Survey series, begun in 1957 by the American Mathematical Society, is currently under the direction of the Data Committee, a joint committee of the American Mathematical Society, the American Statistical Association, the Mathematical Association of America, and the Society of Industrial and Applied Mathematics. The current members of this committee are Pam Arroway, Richard Cleary (chair), Steven R. Dunbar, Susan Geller, Abbe H. Herzig, Ellen Kirkman, Joanna Mitro, James W. Maxwell (ex officio), Bart S. Ng, Douglas Ravanel, and Marie Vitulli. The committee is assisted by AMS survey analyst Colleen A. Rose. In addition, the Annual Survey is sponsored by the Institute of Mathematical Statistics. Comments or suggestions regarding this Survey Report may be directed to the committee.

Doctoral Degrees Not Yet Reported

The following mathematical sciences, statistics, biostatistics, and applied mathematics departments have not yet responded with their doctoral degrees awarded. Every effort will be made to collect this information for inclusion in the New Doctoral Recipients Report which will be published in the August 2011 issue of *Notices of the AMS.*

Departments yet to respond can obtain copies of the Doctorates Granted survey forms on the AMS website at `www.ams.org/annual-survey/surveyforms`, by sending email to `ams-survey@ams.org`, or by calling 1-800-321-4267, ext. 4124.

Group I (Public)

All departments have responded.

Group I (Private)

Brandeis University
Princeton University
Rice University
Stanford University

Group II

Claremont Graduate University
Clemson University
Dartmouth College
University of California, Davis
University of California, Santa Cruz

Group III

Missouri University of Science and Technology
New Mexico State University, Las Cruces
Old Dominion University
Stevens Institute of Technology
University of Missouri-St. Louis
University of Nevada, Las Vegas
University of New Mexico

Group IV (Statistics)

Harvard University
New York University, Stern School of Business
North Dakota State University, Fargo
Northwestern University
University of Alabama-Tuscaloosa
University of California, Davis
University of Chicago
University of Wisconsin, Madison
Western Michigan University
Yale University

Group IV (Biostatistics)

Cornell University
Louisiana State University, Health Science Center
The University of Albany, SUNY
University of Cincinnati, Medical College
University of Louisville
University of Massachusetts, Amherst
University of South Carolina
Yale University

Group Va (Applied Mathematics)

Illinois Institute of Technology
Louisiana Technology University
Princeton University
State University of New York at Stony Brook
University of California-Merced
University of Washington

WHAT IS... a Free Cumulant?

Jonathan Novak and Piotr Śniady

Dedicated to Roland Speicher on the occasion of his fiftieth birthday.

Cumulants are quantities coupled to combinatorial notions of "connectivity" and probabilistic notions of "independence". There are two principal species of cumulants: classical and free. Our discussion begins with the more widely known classical cumulants.

Suppose one wishes to determine the number c_n of connected graphs on the vertex set $[n] = \{1, \ldots, n\}$. The total number of (possibly disconnected) graphs on $[n]$ is $m_n = 2^{\binom{n}{2}}$, and because any graph is the disjoint union of its connected components, we have

$$m_n = \sum_{\pi \in \mathcal{P}(n)} \prod_{B \in \pi} c_{|B|}, \tag{1}$$

where the sum runs over all partitions $\pi = B_1 \sqcup B_2 \sqcup \ldots$ of $[n]$ into disjoint nonempty subsets. The sequence (m_n) thus recursively determines (c_n) via (1).

More generally, if m_n is the number of "structures" that can be placed on $[n]$, and c_n is the number of "connected structures" on $[n]$ of the same type, then the sequences (m_n) and (c_n) are related as in (1). This fundamental enumerative link, which is often expressed in terms of generating functions, is ubiquitous in mathematics. Prominent examples come from enumerative algebraic geometry, where connected covers of Riemann surfaces are counted in terms of all covers, and quantum field theory, where sums over connected Feynman diagrams are computed in terms of sums over all diagrams.

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Formula (1) is also well known to probabilists. In stochastic applications, $m_n = m_n(X) = \mathbb{E}X^n$ is the moment sequence of a random variable, and the quantities $c_n(X)$ implicitly defined by (1) are called the *cumulants* of X. This term was coined by R. Fisher and J. Wishart in 1932. Cumulants were, however, first investigated by the Danish astronomer T. Thiele as early as 1889. He called them *half-invariants* because

$$X, Y \text{ independent} \implies c_n(X+Y) = c_n(X) + c_n(Y).$$

This linear behavior is what gives cumulants their advantage: in most situations cumulants are the "right" quantities to work with. For example, the universality of the standard Gaussian distribution is reflected in the simplicity of its cumulant sequence $0, 1, 0, 0, 0, \ldots$.

Let us now consider a geometric variation of our initial graph-counting problem. Given a graph G on $[n]$, we may represent the vertices of G as points on a circle and the edges of G as line segments joining these points. The resulting picture may be connected even if G is not (Figure 1). Let κ_n denote the number of "geometrically connected" graphs on $[n]$. As before, we consider a partition π of $[n]$ which tells which vertices of G belong to the same connected component of its geometric realization.

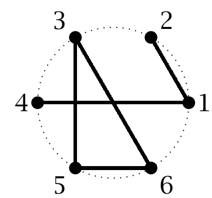

Figure 1. Example of a geometrically connected graph.

It turns out that π is always *noncrossing*: if we represent its blocks as convex polygons, they will be disjoint (Figure 2). It follows that

$$m_n = \sum_{\pi \in \mathcal{NC}(n)} \prod_{B \in \pi} \kappa_{|B|}, \tag{2}$$

where now the summation runs over noncrossing partitions of $[n]$.

Remarkably, just like the c_n's of formula (1), the κ_n's of formula (2) may be realized as half-invariants. Recall that a noncommutative probability space is a complex algebra $\mathcal{A}$ together with a linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ sending $1_{\mathcal{A}}$ to 1. One regards the elements of $\mathcal{A}$ as random variables, with φ playing the role of expectation. The decision to view noncommutative algebras as probability spaces is justified a posteriori by the fact that this framework supports a new notion of independence, called *free independence*, modeled on the free product of algebras. Free independence is the key to a rich noncommutative probability theory—D. Voiculescu's *free probability theory*—which is now widely studied and well known to appear in a wide variety of contexts, for example, in the large dimension limit of random matrix theory.

The realization that formula (2) holds the key to free half-invariants is due to R. Speicher. Given a random variable X in a noncommutative probability space, we associate with its moment sequence $m_n(X) = \varphi(X^n)$ the *free cumulant* sequence $\kappa_n(X)$ via (2). This is completely analogous to the construction of classical cumulants, the only difference being that the lattice of all partitions has been replaced by the lattice of noncrossing partitions. Fundamentally, this means that set-theoretic connectivity has been replaced with geometric connectivity. Speicher showed that

$$X, Y \text{ freely indep.} \implies \kappa_n(X+Y) = \kappa_n(X) + \kappa_n(Y).$$

Indeed, the relationship between free independence and free cumulants mirrors the relationship between classical independence and classical cumulants in every conceivable way. For example, the analogue of the Gaussian distribution in free probability theory is the Wigner semicircle distribution, and its universality is reflected in the simplicity of its free cumulant sequence $0, 1, 0, 0, 0, \ldots$.

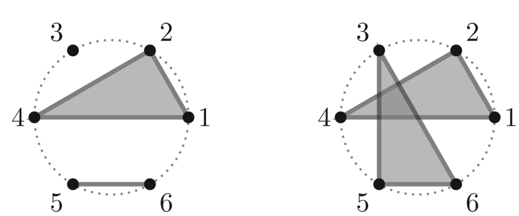

Figure 2. Noncrossing (left) versus a crossing (right) partition.

Free probability theory was initiated by Voiculescu in the 1980s, who used it to solve several previously intractable problems in operator algebras. Thus it is a very new field of mathematics, albeit an exceptionally successful one. It therefore came as a surprise when P. Biane and S. Kerov discovered that free cumulants play an important role in a very classical theory, namely the representation theory of the symmetric groups.

Given a finite group G and a conjugacy class C of G (viewed as an element of the group algebra $\mathbb{C}[G]$), one knows that C acts as a scalar operator in any irreducible representation of $\mathbb{C}[G]$. The value of this scalar defines a function on conjugacy classes, called the *central character* of the irreducible representation. Understanding an irreducible representation amounts to computing its central character which, typically, is very difficult. In the case of the symmetric group $G = S_N$, free cumulants shed substantial light on this question. One begins by considering the *Jucys-Murphy element* $X_N = (1\ N+1) + \cdots + (N\ N+1) \in \mathbb{C}[S_{N+1}]$, which is the sum of all transpositions interchanging $N+1$ with a smaller number. The spectrum of X_N can be completely understood in any irreducible representation; thus X_N and simple functions of X_N may be considered "known". One then defines the nth moment $m_n = m_n(X_N)$ of X_N to be that part of the expansion of X_N^n which belongs to $\mathbb{C}[S_N]$. These moments are not scalars but central elements in $\mathbb{C}[S_N]$, and they give rise to free cumulants $\kappa_n = \kappa_n(X_N)$ via (2). Remarkably, the conjugacy class C_k of k-cycles can be expressed as a polynomial in the free cumulants of X_n, and furthermore one has the first-order approximation $C_k = \kappa_{k+1}$ + lower order terms.

This is not the only surprising appearance of the free cumulant concept. For example, R. Stanley has shown that the free cumulants of M. Haiman's parking function symmetric functions are precisely the complete symmetric functions. M. Lassalle has formulated conjectures linking free cumulants to Jack polynomials. Where will free cumulants appear next?

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Thinking about Technology in the Classroom

Kurt Kreith

Decisions regarding the use of technology in the mathematics classroom are subject to powerful market forces. Amazing products, exorbitant claims, and tools in search of a problem abound. Fortunately, the underlying market is also complex, with thoughtful educators, private and public granting agencies, and (most recently) freeware available on the Internet serving to shape this market and its bottom line. What seems to be needed is the development of principles that can help schools, teachers, and authors of standards navigate these stormy waters. Much as Deborah Ball has worked to define mathematical knowledge for teaching, so is there a need to develop a better understanding of the characteristics and uses of technology that make it effective as a teaching tool.

Here we might do well to begin with the writings of Norbert Wiener. In God and Golem, Inc, Wiener shares some ominous thoughts about a world in which technology plays the role of master rather than servant. He raises the specter of a form of natural selection between man and machine, one whose outcome remains to be determined. In the mathematics classroom, such musings may lead us to reflect on the practice of asking students to use computer technology for operations that they themselves are unable to perform. By way of example, it has become commonplace to ask students as young as middle school to turn to calculators for the correlation coefficients of hypothetical data sets. While the mathematics of correlation may be beyond both teacher and student, such exercises are defended on the grounds that they convey a qualitative understanding of numerical measures that are frequently alluded to. Fair enough—as long as we are aware of the fateful step this form of instruction can represent.

One way of addressing such concerns is to focus on the transparency of the technology under consideration. To what extent is the user aware of what the machine is doing vs. dealing with it as a black box? Here the common spreadsheet provides an attractive alternative to many of its glitzier counterparts. In the case of correlation, Excel has a built-in function that corresponds to the black box approach. However, it also provides a format in which, based only on the machine's ability to do basic arithmetic, the student can perform and visualize the underlying calculations. Were there a classroom version of Excel that gives the teacher control of when the built-in function CORREL is to be made accessible, we would have a teaching tool in which the machine plays the role of servant throughout.

Setting aside such issues of student-machine relationship, it becomes important to consider the questions confronting teachers. "How would a particular form of technology support me in the challenges I face?" is sure to be on the table. And since issues of motivation and student involvement are high on many teachers' lists, it would be surprising if these did not figure into decisions regarding the use of technology. While the importance of such considerations should not be minimized, teachers should also be encouraged to move beyond them with questions such as, "How will my students' interaction with this form

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of technology affect their mathematical development?" and, "Given the role that procedural skills and proof play in our standards for instruction, how can technology help me meet my obligations in these areas?"

Considering these in reverse order, technology is an unlikely tool for teaching proofs; also there exist deep concerns about the role that technology might play in *undermining* the acquisition of procedural skills. As such, it is important to backtrack to the issue of mathematical development which thoughtful teachers see as central to their enterprise. Separating "procedural skills" and "proof" is a vast chasm that might be termed "structural understanding". And here we make the following assertion: Properly used, technology can be a powerful tool in cultivating structural understanding; badly used, it becomes an impediment.

Such assertions about structural understanding are likely to lead to requests for a more precise definition. One approach is to sidestep such requests on the grounds that the concept is highly dependent on the topic and level at which it is being applied. Alternatively, one could assert that structural understanding is what will enable students to make the transition from procedural skill to proof. Or one can be honest and admit that, like "mathematical knowledge for teaching", it is a term in need of better understanding. In adopting the latter approach, it seems appropriate to begin with some examples that might help us arrive at a definition.

One of the services that technology provides is the calculation of remainders arising in whole number division problems. Excel's MOD function can free its users from this form of drudgery, enabling them to reflect on the structure of the problem that requires such calculations. In this context, spreadsheets can be used to reenact the discovery of base-10 positional notation by simulating the making of "stacks of ten" while also converting pennies to dimes, dimes to dollars, etc. In the quest for structural understanding, students can be provided with an Excel template which they are expected to complete as follows.

◇	A	B	C	D	E	F	G	H	I
1	This spreadsheet allows you to enter a wholle number M in cell B3. It then simulates the makeing of "stacks fo ten" to convert M into "pennies", "dimes", "dollars", ...								
2							dollars	dimes	pennies
3	**M=**	473				**0**	**4**	**7**	**3**
4									
5			*scratchwork*						
6						0	4	47	473
7						0	4	7	3

With such a spreadsheet in hand, it becomes natural to consider "stacks of 8", peasant multiplication, the Euclidean algorithm, etc.

Creative attempts to cultivate structural understanding abound in the book *The Enjoyment of Mathematics* by Rademacher and Toeplitz. Here Chapter 23 deals with "Periodic Decimal Fractions" and, as survivors of the math wars will recall, this topic figured prominently into debates about the place of long division in the curriculum. But in their remarkable exposition of Euler's theorem, these authors confront the problem of finding the decimal expansion of 3/41 in the following format:

$$\frac{3}{41} = \underset{3}{0}\,.\,\overline{\underset{30}{0}\,\underset{13}{7}\,\underset{7}{3}\,\underset{29}{1}\,\underset{3}{7}}\ldots$$

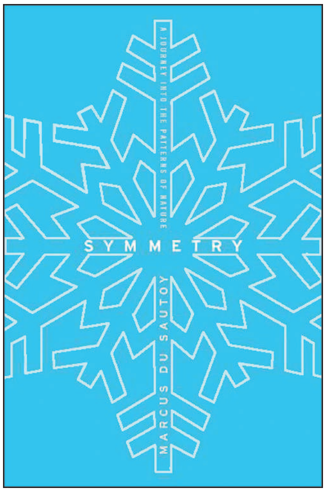
This technique (closely related to what was once called "short division") focuses attention on the central structural issue at hand, namely that of cycles in the sequence of remainders. Had spreadsheets existed in 1929, might these authors have employed the following?

◇	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	This spreadsheet allows you to enter a proper fraction in A3:A4. It then uses the MOD function to perform "short division" by determining the remainders to be generated in Row 4 and the fraction's decimal expansion in Row 3.														
2															
3	3	=	.	0	7	3	1	7	0	7	3	1	7	0	7
4	41		3	30	13	7	29	3	30	13	7	29	3	30	13

Absent until now has been any mention of geometry, where straightedge and compass constructions are a likely starting point for efforts to achieve structural understanding. Here there exist remarkable forms of dynamic geometry software that harness technology in the implementation of the underlying procedures and in the continuous deformation of the resulting figures. While such software has much to offer, there exist questions analogous to ones discussed above. In discovering that "the medians of a triangle seem to intersect at a single point," should the surprising role of their trisection points be arrived at synthetically or numerically? Should it be possible for the teacher to specify that it is synthetic geometry that is under study? Do the marketers of such software have a tendency to identify computational power with pedagogical effectiveness?

Such questions I am pleased to leave to teachers with greater experience in the classroom use of these products. In my own work with high school students enrolled in California's State Summer School for Mathematics and Science (`http://www.ucop.edu/cosmos/`) and in prerequisite-free freshman seminars at UC Davis, I have tended to emphasize spreadsheet templates such as those described above. For anyone interested in trying their hand at some templates whose completion is intended to cultivate "structural understanding", I would be pleased to respond with an email attachment. On such a basis, contributions towards a definition of structural understanding would be welcome as well.

Book Review



Symmetry: A Journey into the Patterns of Nature

Reviewed by Brian E. Blank

Symmetry: A Journey into the Patterns of Nature
Marcus du Sautoy
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In January 1975, Jacques Tits gave a lecture during which he wrote down the order, $2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$, or about $8 \cdot 10^{53}$, of a sporadic simple group M that Bernd Fischer and, independently, Robert Griess had predicted in 1973. Andrew Ogg, an expert in modular functions who happened to be in the audience, was uniquely qualified to recognize something peculiar about this order: he had recently determined the primes that give rise to a certain family of genus 0 modular curves, and the primes he had discovered were precisely the fifteen prime divisors on the blackboard. It was an astonishing coincidence. At the time, to suggest anything more than that would have been moonshine.

Three years later, in 1978, Fischer, Donald Livingstone, and Michael Thorne calculated the character table of the Monster, as John Horton Conway had christened M, by assuming an additional prediction of Griess, the existence of an irreducible character of degree $\chi_1 = 196883$. Shortly thereafter, John McKay noticed that the coefficient c_1 of the elliptic modular function,

$$j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 + 20245856256q^4 + \cdots$$

$$= \frac{1}{q} + 744 + \sum_{k=1}^{\infty} c_k q^k \qquad (q = \exp(2\pi i \tau)),$$

is given by $c_1 = 1 + \chi_1$. The evidence of a link between M and the j-function became too overwhelming to dismiss when McKay and John Thompson discovered additional equations involving the characters of M and the Fourier coefficients of j, of which the first few are $c_2 = c_1 + \chi_2$, $c_3 = c_1 + c_2 + \chi_3$, and $c_4 = c_1 + c_3 + \chi_3 + \chi_4$. At this stage, Thompson cautiously referred to these curious equalities as "numerology", but, prompted by some of his suggestions, Conway and Simon Norton discovered further evidence of a deep relationship between the Monster and modular functions, a relationship they dubbed *monstrous moonshine*.

The whimsical term "moonshine" had a certain suitability. There was something humorous about so unexpected a connection between seemingly unrelated branches of mathematics. Writing in 1979, six months before Griess constructed M, Ogg remarked, "It is particularly amusing that new light should be shed on the function j, one of the most intensely studied in all of mathematics, by the most exotic group there is (or is not, as the case may be)." At the time, the light to which Ogg referred was not the bright, direct light that a theoretical connection would have shone. It was a dimmer, reflected light. As Conway put it, "The stuff we were getting… had the feeling of mysterious moonbeams lighting up dancing Irish leprechauns."

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Few writing tasks could seem more foolhardy than authoring a popular book about monstrous moonshine, but the publication of Marcus du Sautoy's *Finding Moonshine* in Great Britain gave the appearance that one had emerged. Concerned, perhaps, that *Finding Moonshine* might be mistaken for a revenuer's memoir, du Sautoy's American publisher issued it as *Symmetry: A Journey into the Patterns of Nature*.

It is difficult to categorize du Sautoy's book in a few sentences, but the mathematical content is centered around finite simple groups and their classification (CFSG). After setting the scene with some autobiography, du Sautoy uses a discussion of symmetry to introduce the finite group concept. The groups of rotations of regular prime-sided polygons become the first simple groups the reader encounters. After learning about the regular solids of the Pythagoreans and Theaetetus, du Sautoy's reader visits the Alhambra with the author, as he hunts for the seventeen symmetries of its decorations. After finding the last, du Sautoy asks, rhetorically, "How can I be sure there isn't an 18th one out there to be discovered?" It is a question to which he will return.

Finding a seamless transition, du Sautoy directs his attention to the theory of equations. Sometime late in the 1400s or early in the 1500s, 125 years or so after the completion of the Alhambra, the Bolognese mathematician Scipione del Ferro discovered formulas that express the roots of cubic polynomials in terms of radicals. Another Bolognese mathematician, Lodovico Ferrari, solved the quartic in 1540. In du Sautoy's formula-free discussion of these advances, algebra takes a back seat to the antics of the cast of eccentric characters. The work of Niels Abel and Évariste Galois on higher degree equations follows, with biographies of these interesting but short-lived mathematicians. The breakthrough of Galois in 1832 brings us not only to the notion of the simple group but also to noncyclic examples ($\mathrm{PSL}(2, p)$ and A_n).

Camille Jordan, Felix Klein, Sophus Lie, Émile Mathieu, Arthur Cayley, and William Burnside are the remaining nineteenth-century group theorists to whom du Sautoy accords more than a few lines of attention. The contributions of Jordan and Lie were fundamental to the emerging theory of groups, but, more germane to the subject at hand, Lie introduced an important family of continuous groups, and Jordan enriched the supply of simple groups with several infinite families of finite analogues of Lie groups. In publications of 1861 and 1873, Mathieu complicated the developing pattern by discovering five finite simple groups that did not belong to the known families. Burnside inadvertently gave a name to such groups many years later when he remarked, "These apparently sporadic simple groups would probably repay a closer examination than they have yet received" [3, p. 504].

The inclusion of Cayley in *Symmetry* is welcome, if surprising. Although he did not play a direct role in the classification of simple groups, his indirect influence was instrumental, for it was Cayley who, in a series of papers published in 1854, introduced the abstract group concept. Cayley's dictum, "A group is defined by means of the laws of combination of its symbols," translated by du Sautoy as, "Forget the equation and its solutions, just look at the interaction of the permutations," provides an effective segue from the theory of equations to modern group theory.

The abstract approach did more than unify group-theoretic results that were arising in geometry, number theory, the theory of equations, and the theory of invariants. It also suggested questions that would otherwise have seemed pointless. Thus it was Cayley who first asked, What are all the possible groups of a given order? Among the mathematicians whom Cayley influenced, Otto Hölder is particularly noteworthy (but is overlooked by du Sautoy). In 1889, Hölder completed the basic theorem on composition series that Jordan began in 1870, and, in 1892, it was Hölder who kicked off the CFSG program when he stated, "It would be of the greatest interest to gain an overview of the entire collection of simple groups." In the same article, he proved that there is no simple group with order p_1p_2 or $p_1p_2p_3$, where the p_i's are primes, not necessarily distinct. He also determined all simple groups up to order 200. Later that year, Frank Nelson Cole, remarking that it was "desirable to extend this census as far as possible", stretched it to 500 (and to 660 the next year). The progress of subsequent censuses may be found in [4].

The decade of the 1890s saw several other developments related to the topics of *Symmetry*. In a coda to his chapter, *The Palace of Symmetry*, du Sautoy states that "The language of group theory gives us the means to prove that 17—and no more—different symmetry groups are possible on a two-dimensional wall." This imperceptive assertion, which refers to the independent determinations of the plane crystallographic groups by William Barlow, Yevgraf Stepanovich Fyodorov, and Arthur Schönflies between 1891 and 1894, comes as a letdown. In the Alhambra chapter, du Sautoy had promised, "As we shall see later in our story, [the proof that there cannot be an 18$^{\text{th}}$ pattern] depends on mastering group theory." *Language* is not *mastery*, and we do not see. (Although we know how many symmetries *could* be at the Alhambra, there is some controversy about how many there actually are [10]. I would rather gaze at a Mark Rothko canvas than say anything more about these busy ornamentations.)

Some other important steps taken in the 1890s are not mentioned. Examples of such omissions are the theory of group characters, introduced by Gerhard Frobenius in 1896, and the development of Lie theory and its finite analogues. The last of Wilhelm Killing's papers on the classification of the finite dimensional simple Lie algebras over $\mathbb{C}$ appeared in 1890. Later in the decade, beginning with his thesis in 1894, Élie Cartan completed Killing's project by constructing the exceptional simple Lie algebras. In 1896 Leonard Eugene Dickson received his doctorate, the first in mathematics awarded by the University of Chicago. His dissertation, supplemented by the more than thirty papers on group theory he wrote over the next few years, was the basis of his 1901 book, *Linear Groups with an Exposition of the Galois Field Theory*. In it, Dickson included all the classical projective groups over finite fields and listed fifty-three of the fifty-six noncyclic simple groups of order less than 1 000 000. Shortly thereafter, in 1905, Dickson constructed finite simple analogues of the exceptional Lie group G_2. Additionally, he introduced finite analogues of E_6. Dickson is cited but once in *Symmetry*, and that reference concerns a remark Dickson made decades after this work.

One additional event of the 1890s that was auspicious for CFSG was Burnside's switch in 1893, at the age of forty-one, from applied mathematics to group theory. His first result was to show that the alternating group A_5 is the only simple group with order $p_1p_2p_3p_4$, where the p_i's are primes, not necessarily distinct. (At more or less the same time, Frobenius proved that a group of order $p_1p_2 \cdots p_n$ cannot be simple if $n \geq 2$ and the p_i's *are* distinct primes.) In 1904, Burnside proved the p^aq^b theorem, which states that a group whose order is divisible by fewer than three distinct primes is solvable. As a result, such a group is not simple unless it is cyclic of prime order.

In his history of CFSG, Solomon describes Burnside's p^aq^b theorem as the final triumph of the first era of investigation [13]. In the second edition of his book, Burnside made room for more promising techniques, such as group representations, by jettisoning antiquated material. The verification of the list of simple groups to order 660, which is found in the last chapter of the first edition, was among the discards. Surveying Dickson's greatly extended list, Burnside noted, "An examination of the orders of known non-cyclical simple groups brings out the remarkable fact that all of them are divisible by 12" [3, p. 330]. He did not elevate that observation to a conjecture, but, having pondered the divisor 2 for fifteen years, he was ready to assert, "The contrast. . .between groups of odd and even order suggests inevitably that [non-cyclic] simple groups of odd order do not exist" [3, p. 503]. Burnside's restraint about the divisor 3 proved to be well judged—in 1960, Michio Suzuki discovered a simple group of order 29120, the first addition to Dickson's sixty-year-old list.

With its first ten chapters, which comprise about 300 pages, *Symmetry* outlines the development of group theory from its origins through the first twelve years of CFSG. The next half-century did not see continuous progress, but it did span important advances, such as the beautiful characterization of finite solvable groups by Philip Hall, du Sautoy's mathematical great-great-grandfather. Because these results are too technical for a book aimed at the lay person, *Symmetry* skips over them. Its final two chapters, which total some 50 pages, are concerned with the concluding forty-five years of CFSG, as well as monstrous moonshine. When du Sautoy picks up the trail, he describes the Odd Order Theorem of Walter Feit and John Thompson, which states that all odd order groups are solvable, a result that affirms Burnside's conjecture. Du Sautoy does his best to convey the difficulty of the theorem, but there is no way for his reader to gauge the sea change in methodology that has occurred since the Burnside epoch.

Ironically, the Feit-Thompson Theorem, "the single result that, more than any other, opened up the field" [7, p. 1], is a dead end in *Symmetry*. We are told that it "inspired a whole generation of young mathematicians", but, for du Sautoy's reader, it is a largely anonymous generation pursuing undisclosed paths. The true completion of CFSG came, we are told, when Michael Aschbacher and Stephen Smith plugged a "gap" in a "missing step" of a "16-point plan". That is the extent of what we learn about the classification machinery. After Feit-Thompson, even the *statements* of the theorems are too technical.

As good luck would have it, a long-dormant byway of CFSG was about to awaken. In 1965 Zvonimir Janko announced the discovery of a new sporadic simple group. This sixth sporadic group, the first to be found in more than ninety years (and, with order 175 560, the second addition to Dickson's list), was the start of an enormously successful eleven-year treasure hunt, which ended in 1975 with Janko's discovery of his fourth new sporadic group, the twenty-sixth and, as it turned out, last of these exceptions. To fully appreciate the feverish activity involved, read du Sautoy's account of Conway's twelve-hour-and-twenty-minute determination of the 4 157 776 806 543 360 000 element sporadic group Co_1. Or how Donald Higman and Charles Sims took a stroll around a quadrangle while dinner plates were being removed, and, by the time they sat down to dessert, had the 44 352 000 element sporadic group HS in their possession (modulo some paper and pencil calculations that went on into the early hours of the next morning). Further details, as told by Sims himself, may be found in a survey of the mathematics of Higman [2].

In his last chapter, du Sautoy describes the constructions of M and the final sporadic group, J_4. His account highlights quite a contrast: the team assembled by Norton relied on Richard Parker's computer program to answer *Yea* or *Nay*, whereas Griess accomplished his construction of M entirely by bare-handed calculations. Du Sautoy also devotes eight pages of his last chapter to moonshine, a topic he introduced near the beginning of his book. In that earlier discussion, he describes how he learned of moonshine directly from Conway and Norton during a visit to Cambridge in 1985 as a prospective graduate student. At the end of Chapter 1, moonshine is left a tantalizing mystery. It remains a mystery, still tantalizing, one hopes, at the end of Chapter 12. At least readers will know that the experts have figured out quite a bit in the years since 1985. For those who want to learn more, Chapter 0 of [5] makes an excellent continuation.

I have come to the end of du Sautoy's book but have touched on less than half its content. That is because the author has managed to effectively embed selected episodes from the history of group theory into a narrative that sometimes resembles a personal journal and sometimes a travelogue. In the mix, we find many extended, nonmathematical discussions of symmetry as it is manifested in a broad panorama of guises. Facts about icosahedral Chinese incense burners from the first millennium CE and dodecahedral Roman dice from the fifth century BCE stay within our comfort zone. When du Sautoy strays from such topics, the results are dicier. He tells us that "Studies indicate that the more symmetrical among us are more likely to start having sex at an earlier age." Ovulating women, du Sautoy continues, can sniff out symmetry from the sweaty tee-shirts men have worn, but men, apparently, do not pick up the scent of a symmetrical woman. Enquiring minds that want to know more are out of luck: *Symmetry* does not come with any notes. One discussion that has no need for footnotes is an experiment devised by psychologists to pit symmetry against logic (page 278). You can try it out on your own, as I did when teaching truth tables in a transitions course. The results amused the students and enlightened the instructor.

We have now met du Sautoy the biographer, the historian, and the spokesman for symmetry. He is also an anthropologist who reports on the rituals of the strange tribe in the midst of which he lives. He does not have to go native—as a prominent group theorist, he *is* a native. Readers of the *Notices* will find this aspect of his book highly entertaining. You may not have encountered protesters outside your department bearing "No more group theory" placards, but I guarantee you will flash on scenes from your lives. When the author admits, "Like most mathematicians, I am naturally quite shy. I'm not someone...who likes to introduce myself to people. I hate parties, and I'm terrified of the telephone," I recognize many colleagues around the world, and I recognize myself. From among *Symmetry*'s readers, perhaps a parent or partner of a mathematician will empathize with the mother who complained, "There is something wrong with my son. He sits in his room all day studying mathematics" (page 337). Perhaps the general reader will come away understanding Lie's firsthand observation that, "A mathematician is comparatively well suited to be in prison" (page 228).

Du Sautoy examines the entire mathematical process—the inspiration, the serendipity, the hard labor, the setbacks, the frustration, the elation, the disillusionment, and the competition. "Big theorems are like jigsaw puzzles," he advises. "Who wouldn't enjoy being the person to put in the last piece?" He stresses our compulsion to classify, our obsession with pattern hunting, our reluctance to admit defeat. His book is structured as a year-long diary, and he updates us on the progress he is making, or the ground he is losing, with respect to a research problem concerning the enumeration of finite groups. As he realizes that the number theory of elliptic curves is shedding light on his work, we realize that *Finding Moonshine* was a sly choice of title.

On several occasions du Sautoy emphasizes that mathematics provides "a haven for the weird", a niche in which social oddness is tolerated. However, he does not sound all that tolerant when he describes a colleague as looking "like a tramp", or "a slightly mad clown", or "Neanderthal", or "frothing slightly at the mouth". One mathematician is said to have "distinctly inferior social skills". Another "will avoid eye contact with you at all costs". There is one mathematician who appears to be an outgoing communicator, but, "With him," du Sautoy divulges, "it's one-way traffic.... He doesn't seem remotely interested in what anyone else has to say. It's almost as if he is compelled to...forestall any possibility of normal two-way interaction." And then there is the group theorist who suffers a manic episode brought on by contemplating the ramifications of a new sporadic group. These are not anonymous members of the tribe. In every case, du Sautoy names names.

I have already mentioned the only major problem with *Symmetry*, an absence of footnotes. I do have a few minor quibbles. The term "shuffle" should probably not be used as a synonym for permutation: it suggests the special permutations employed, for example, in the definition of the wedge product. On seven occasions, du Sautoy writes "Lie group" when he means "finite group of Lie type". Three of the nineteen digits of $|Co_1|$ on page 317 are incorrect.

In the chapter on the Alhambra, du Sautoy expresses annoyance at his guide's claim that the Arabs invented zero (page 71). He asserts that "the Indians discovered zero. The Arabs were just good messengers bringing the idea from the East to the West." Perhaps this statement refers to the Hindu-Arabic zero in use today, but, if one goes back further in time, a case can be made that zero originated in the Middle East. A tablet that was unearthed at Kish, about eighty kilometers southeast of present-day Baghdad, indicates that the Babylonians used zero as a placeholder in their system of numeration around 700 BCE. Indeed, Hans Freudenthal advanced the theory that the Babylonian zero migrated to India. Bartel van der Waerden deemed this proposal "quite possible", but it remains a conjecture.

On page 240, du Sautoy declares that Burnside's $p^a q^b$ theorem "proved to be just what was needed to identify the simple groups with a small number of symmetries". The example he gives, the list of noncyclic simple groups of orders up to 200, is an unsatisfactory illustration of his assertion, given that Hölder determined the same list in 1892, twelve years before Burnside proved the $p^a q^b$ theorem. In his next paragraph, du Sautoy asserts that Burnside used the $p^a q^b$ theorem in [3] to determine all simple groups up to order 1092. As remarked earlier, Burnside did not pursue this fruitless path in his second edition; he provided only a reference to this determination, which he published in 1895, nine years before the $p^a q^b$ theorem. Special cases of the theorem, such as $p^a q$ with $p < q$, were known in 1895, and Burnside used them alongside his interesting observation that, Z_2 excepted, no simple group of even order n can exist unless $12 \mid n$, $16 \mid n$, or $56 \mid n$.

We are informed on page 305 that Graham Higman (no relation to Donald Higman) and McKay were in Oxford when they constructed J_3. The next page reminds us of "Graham Higman and John McKay in Oxford". British place names are recorded at every opportunity, with enough detail to separate Oxford from neighboring Chilton, but, with few exceptions, American locations are omitted. As a result, du Sautoy's readers will gain no impression of the disproportionate amount of group theory that has been done at Cal Tech, Rutgers, and the Universities of Chicago, Illinois, and Michigan. By contrast, Cambridge University merits three lines in the index.

Du Sautoy's reportage is spotty in places. Michael O'Nan, Richard Lyons, the sporadic groups they discovered (ON and Ly), and the constructions of these groups by Sims are not mentioned. The construction of the Baby Monster is not ascribed to Jeffrey Leon and Sims. The contribution of David Wales to the construction of the sporadic group Ru is acknowledged, but, for the 604 800 element group J_2, the third and final addition to Dickson's list, only Wales's coauthor, Marshall Hall Jr., is credited. In this context du Sautoy alludes to a "rather tense stand-off that got the whole group theory community talking." Should a group be named after its predictor, or its constructor? "Hall would get rather upset if the group he'd constructed was called simply the second Janko group." There may be some legitimacy to the controversy, but du Sautoy chose an unsuitable example. As Griess has pointed out [8, p. 237], Hall discovered J_2 independently, so, in this case, joint credit is due regardless of the larger controversy. (Chapter 17 of [6] supplemented by [7, p. 110] is a good place to sort out matters of divination and fabrication.)

On pages 310 and 315, du Sautoy spins an entertaining tale of how McKay alerted Conway to the possibility of a simple group associated with the Leech lattice. It was the summer of 1966, and both mathematicians were in Moscow for the International Congress of Mathematicians. Conway, who was manning the *pirozhki* stand McKay approached, handed over a roll, and McKay handed back the Leech. This is almost too good a story to scrutinize, but the problem is that du Sautoy's unsourced version does not agree with the firsthand accounts that McKay and John Leech gave in telephone interviews a quarter century closer to the event [14, pp. 118–119]. Ultimately, it is of no consequence whether the exchange took place in Moscow in 1966 or, as McKay and Leech both say, in Cambridge in 1967. This discrepancy, however, does illustrate the confusion that can arise from the absence of documentation.

On page 330, du Sautoy states that, "By the mid 1970s, a total of 25 different sporadic groups had been discovered or conjectured to exist.... The feeling was that 25 might be the limit of what was possible." Is this portrayal of a consensus accurate? Referring to the late 1970s, by which time Janko had predicted a twenty-sixth sporadic group, du Sautoy remarks, "It seemed that only two of the 26 [sporadic groups] were still unclaimed [i.e., not yet constructed]" (page 335). Was it really so apparent then that there would be no additional sporadic groups to construct? According to Griess [9],

> By the late 1970s the classification program had been making a lot of progress, and there were increasing expectations of eventual closure. However, there was not a firm conjecture that the list of simple groups known or suspected to exist at that time must be the complete list. There seemed to be no certainty that the number of sporadic groups must be a *particular number* (such as 26).

> During the early 1970s there was a gap of about two and a half years in which no new sporadic group was discovered (between the discoveries of the Lyons group and the Rudvalis group). Within that time period, no one felt confidence about predicting numbers of sporadic groups to come. The fourth Janko group was conjectured to exist in 1975. During the remainder of the 1970s it was too soon after Janko's latest discovery to feel strongly that 26 must be the right number.

Du Sautoy reports that Daniel Gorenstein declared CFSG to be "all over" in February 1981, when, *to quote du Sautoy*, "a paper...proved that there couldn't be two different groups that looked like the Monster." This assertion may refer to a footnote in [7, p. 1]. It is true that the uniqueness theorem for M was an essential ingredient of CFSG—in 1900, Ida Schottenfels proved that A_8 and $PSL(3,4)$ are nonisomorphic simple groups of order 20160 (and Dickson proved that there are infinitely many orders to which there correspond pairs of nonisomorphic simple groups). However, du Sautoy's use of the term *paper* is misleading. Gorenstein, at the end of his footnote, added the cautionary remark that the work to which he referred was a manuscript in preparation. When a paper was finally published in 1985, it contained an outline of a procedure for determining uniqueness but no claim that the method had actually been implemented. Griess, Ulrich Meierfrankenfeld, and Yoav Segev established uniqueness of the Monster in 1989 [13, p. 341]. (Additionally, this result finally yielded the irreducible character of degree 196883 whose assumed existence had been the basis of the calculation of the character table of M a decade earlier.)

In his concluding pages, du Sautoy's tone becomes noticeably downcast. Although the aftermath of CFSG was a period of "intense activity" [13, pp. 345–347], du Sautoy portrays the opposite when he asserts, "A sense of anticlimax descended on group theory." With pronouncements such as, "The mathematicians who truly understood all the intricacies of [CFSG] were getting old," "Very special techniques could die out with the passing of this generation of practitioners," and "Few young and aspiring mathematicians were attracted to the field," du Sautoy is neither accurate nor attentive to the lessons that can be gleaned from the histories of CFSG and moonshine. After all, the passing of Hermite, Weierstrass, Dedekind, and Klein did not bequeath a dire future for the modular function. A more fitting outlook is that of Solomon [13, §10], who concludes, "We await the visionaries of new generations...who will shed unexpected new light on this ever-fascinating subject." Visionaries and, perhaps, moonbeams.

Based on subject matter and level, Mark Ronan's *Symmetry and the Monster: One of the Greatest Quests of Mathematics* [11] may be regarded as a competitor of du Sautoy's book. It was released slightly earlier and reviewed (favorably) by Griess in the *Notices* [8]. As the titles suggest, *Journey* and *Quest* treat similar mathematical topics. Sometimes the resemblance is uncanny. Walter Feit's description of his wardrobe, "I now possess five new pairs of trousers, two new jackets plus new shoes," written shortly after he arrived in New York, is perfectly unremarkable, but both authors saw fit to quote it. To Ronan, a simple group is a "symmetry atom". To du Sautoy, it is a "symmetry building block". Ronan nicknames Jacques Tits "The Man from Uccle", a pun on the 1960s television series and a reference to the Belgian town where Tits was born. Also playing on words, du Sautoy labels Griess "the mathematical Doctor Frankenstein". Whereas Ronan is a disciplined chronicler who likes to share interesting tales, du Sautoy is a raconteur who aims to educate his readers by means of stories that unfold vividly.

For anyone who is interested in the CFSG program, both books are recommended as informative supplements to articles such as [1], [2], [4], [12], and [13]. If I were forced to choose only one of these books, and if CFSG were the only criterion, then I would give the nod to Ronan's more focused approach. But du Sautoy's work is also up to the task, and his broad sweep and imaginative writing will have great appeal. He has gathered a wide range of loosely related topics and has very cleverly assembled them into a coherent, absorbing narrative. Readers of the *Notices* will find *Symmetry: A Journey into the Patterns of Nature* enjoyable and worthwhile.

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Book Review

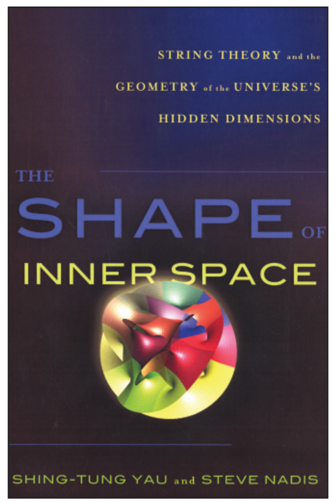
The Shape of Inner Space

Reviewed by Nigel Hitchin

The Shape of Inner Space
Shing-Tung Yau and Steve Nadis
Basic Books, 2010
US$30.00, 40 pages
ISBN-13: 978-0-465-02023-2

"The End of Geometry": a curious title for the final chapter of a book whose first author is perhaps the best-known differential geometer in the world. And this from someone who confesses that geometry is "the field closest to nature and therefore closest to answering the kinds of questions I care most about.'' Yau's concluding theme concerns the challenge to find the mathematical language to describe both general relativity (which differential geometry has done very successfully) and quantum physics (whose language in recent years has penetrated algebraic geometry). It is string theory and its practitioners that have brought these new ideas into geometry, it is they who use and christened *Calabi-Yau* manifolds, and it is they who are demanding from us a new geometry to suit their needs. The book aims to tell us how this came about via two themes: one is a personal story about Yau himself and his Fields Medal-winning proof of the Calabi conjecture, the other a description for the lay reader of string theory and the role of Calabi-Yau manifolds in its attempt to describe the universe.

First the Calabi conjecture. The version most relevant here concerns objects that had been of interest to algebraic geometers for a long time—varieties with trivial canonical bundle. These are higher-dimensional analogues of elliptic curves. In two dimensions they are the so-called K3 surfaces. Yau's theorem asserts the existence of a special kind of Riemannian metric on these—a Kähler metric with zero Ricci tensor, equivalently a Riemannian solution to Einstein's vacuum equations. Until the theorem in 1977, there were no



nontrivial compact examples, but the proof, together with algebraic geometric constructions, generated many. Not only that, but the metric, in particular for K3 surfaces, provided a tool for understanding the algebraic geometry of moduli problems. It gave an analytical object that could somehow interpolate between algebraic ones—just as it helps to see rational numbers inside the reals, one can use the Calabi-Yau metric to pass from one algebraic surface to another.

The path to proving the theorem was not easy, and Yau's description of his successes and setbacks on the way makes for interesting reading and is instructive for young mathematicians to see. His first attempt at a counterexample, presented to an eminent audience, failed. By examining how it failed, he convinced himself the conjecture must be true and then worked hard on it, talking to people, learning new techniques, until finally in a Christmas Day meeting in 1976 with Calabi and Nirenberg, his proof seemed to hold up. "I may have my shortcomings," he confesses, "but no one has ever accused me of being lazy!" The second author, a seasoned science writer, gives a smooth, literate account here, but on occasion one can detect the authentic Yau voice breaking through.

Then comes string theory, historically described in a fashion that is recognizable to readers of, for example, Brian Greene's books. Around 1984 the physicists Michael Green and John Schwarz found that a consistent supersymmetric string theory required six extra real dimensions that had a Kähler metric with

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zero Ricci tensor. It seems that they were erroneously told that no higher-dimensional analogues of K3 surfaces existed, but soon they found out about Yau's theorem and before long the objects acquired the name of Calabi-Yau manifolds. The theorem is an existence theorem and doesn't provide any detailed information about the metric that the physicists might need, so what they actually had from it was a number of examples of the underlying spaces. In fact, too many, because quite soon, in cooperation with algebraic geometers, they were able to generate thousands and thousands of examples. Somewhat difficult to choose from if you want a model of the universe.

But a little later Calabi-Yau manifolds took on a different role, by means of a link with conformal field theories. The symmetries of those theories provided a new viewpoint on Calabi-Yau geometry, and out of it came the phenomenon of mirror symmetry—it seemed as if these manifolds all came in pairs. The thousands of examples provided a good testing ground for this, and in the end, as the authors remark, this symmetry would "revitalize Calabi-Yau manifolds and rejuvenate a somnolent branch of geometry". Those mathematicians working in enumerative algebraic geometry might resent the use of this adjective, but it is true that mirror symmetry awakened them to a new dawn in this area, presenting new problems and new methods. Enumerative geometry concerns itself with counting objects within a bigger one—curves, sheaves, etc.—and the physicist's partition function turns out to be a generating function for some of these. The symmetries of the conformal field theory suggested properties of the manifolds that were to a large extent invisible in the more traditional approach. One steps back and looks at things in a much wider context and in particular finds a systematic way of counting degenerate objects. Indeed, the extra structure that is observed emanates from these degenerations—in fact, it is still not clear whether mirror symmetry applies to a single Calabi-Yau or to a degenerating family. In any case, this symmetry and its ramifications is one of the most profound ways in which physical ideas have been absorbed into pure mathematics.

These, and related issues, are described in the book quite effectively, and it becomes a rather different story from Yau's personal achievement, in his dogged pursuit of the big theorem. Throughout the book attempts are made to describe in layperson's terms the mathematical and physical concepts, but I suspect that many of these will only make sense to a practicing mathematician despite the multiple diagrams and two-dimensional analogies. On the other hand, the lay reader may well enjoy being presented with some views on what geometry is, how it links up with other areas of mathematics, and how it provides a language for expressing some of the key concepts in physics. The reader may also find intriguing the authors' view on the motivation of mathematicians when they attack a problem and how they feel on solving it. Drawing on the analogy of mountain climbing, we are told, for example, that "at the end of a long proof, the scholar does not plant a flag." (I wonder if the reader would be so naive as to believe that.) What is slightly irritating, and seems to be a common method in science writing, is the multiplicity of sound bites from eminent scholars: "… as X of Harvard explains …" In an area in which either the mathematics is very abstract or the physics untested, falling back on the comments of the great and good for justification is a poor substitute for real evidence.

So what really was the impact of Yau's theorem in string theory? It established the existence of Ricci-flat Kähler metrics, but, as we read in the book, this may not actually be what the physicists wanted. Making the beta function vanish involves an infinite number of adjustments to the metric, moving away from the Ricci-flat condition; and non-Kähler Calabi-Yaus seem to be needed to complete the moduli space picture. It seems that the main concrete outcome of the theorem was to open the door for the physicists to invade algebraic geometry, to provide new questions, new tools, and new ways of organizing information.

In his earlier years Einstein commented that "since the mathematicians have invaded the theory of relativity, I do not understand it myself." It may be that some of us geometers have the same feeling about the incursion of ideas from physics today, but it is a fact that is not going to change. So how do we now take that final chapter—is the death of geometry real or simply greatly exaggerated? "Geometry as we know it will undoubtedly come to an end," say the authors, but it seems more likely a statement about string theory. For, despite Yau's energetic pursuit of string-related mathematical problems, despite his important facilitation of interactions between physicists and mathematicians, and despite his encouragement of students and postdocs on these problems, his *oeuvre* contains highly influential results in many other branches of geometry. In the book he steps aside ("one of the luxuries of being a mathematician") from the controversial discussions of the landscape or of multi-universes. Another such luxury is to accept the mathematical challenges of problems from whatever source, so long as they intrigue us, and I suspect the first author will continue to do this.

Weekend Getaway Guide: A Mathematics Research Conference

Katharine A. Ott

Like many mathematicians, I occasionally miss class in order to attend a research conference. When I explain the reason for my upcoming absence, the incredulous look on my students' faces says it all: *You're traveling there? To do math? And listen to math talks all day? Over the weekend?* As a graduate student, you might feel the same way. After all, who wants to do work over an entire weekend? The truth is that mathematics research conferences are an integral part of being a mathematician; and yes, as a graduate student you are a mathematician! Although the idea might be unappealing or intimidating at first, graduate school is the time to begin attending research conferences. Ultimately, attending and participating in research conferences will help you write your dissertation, get a postdoc or a job, and move through the tenure process of academia.

Mathematics conferences are unique in many ways, and even within mathematics they will differ between research areas. This article is designed to be a "travel guide" to help you attend your first several math meetings. If you start attending math conferences early in your mathematical career, and attend them on a regular basis, you will begin to build a network of colleagues in your research area. This network can have a profound influence on your entire mathematical career. Attending conferences will also help you broadcast your research to other mathematicians in your area of specialty and learn new mathematics. In turn, you will be establishing yourself as an active member of your research community.

Before the guide begins, you might be wondering what mathematics research conferences are or why they exist. Like professional conferences in other disciplines and professions, math conferences serve to bring together mathematicians who are all interested in a common topic. Contrary to popular belief, mathematicians are not solitary workers. In fact, we are very social and rely on collaborations, both formal and informal, to make progress in our respective fields. For this reason, mathematicians gather on a regular basis across the world to present new mathematical ideas, to discuss research themes and open questions, and to interact with colleagues who work at different institutions.

Where Should I Go? Choosing a Conference to Attend

You do not necessarily need to have a thesis advisor or a dissertation topic in order to attend a research conference. In fact, there are no prerequisites for attending most math conferences. Once you have decided on an area of interest, you can, and should, start searching for meetings that suit you. A good place to find upcoming conferences is on the AMS website, under the heading Meetings. Here you will find a list of upcoming national, international, and sectional meetings of the AMS (more on these later) and also a Mathematics Calendar, which compiles other upcoming conferences with links to their websites. There are dozens of conferences listed on the Mathematics Calendar, and most will not be interesting to you. Use the search command to find the conferences related to your interests, such as "Analysis" or "Geometry". If you check this calendar regularly, you can also search the term "New" to see what has been added in the past week. Your specialty may be a very specific branch of mathematics, so broaden the subject area when looking for conferences.

Your department may also post announcements for conferences, and you can ask your advisor or another faculty member in your research area to alert you to upcoming meetings. Other good ways to learn about upcoming conferences are to add your address to an email list for your research area and to regularly check the schedule of the NSF-sponsored math institutes: AIM, IAS, ICERM, IMA, IPAM, MBI, MSRI, and SAMSI. In order to find good conferences to attend and to sign up before

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deadlines pass, you should be checking for upcoming meetings on a routine basis.

Math meetings have many different formats. They vary in number of participants, length, and types of talks. On one end of the conference spectrum is a workshop. The goal of a workshop is generally to get mathematicians together in one place to work on a specific problem or group of problems. Some workshops are targeted toward senior researchers (though young mathematicians are still included), while others are specifically aimed at graduate students and recent Ph.D.s (this term usually refers to mathematicians who received their Ph.D.s in the past three or five years). These conferences generally last a week or more and often occur in the summer. A typical workshop for graduate students and recent Ph.D.s is run by several senior researchers and includes talks targeted toward beginners in the field. There will usually be a series of talks on one subject, and in addition there may be some new results or open questions presented. Attending workshops aimed at junior researchers as a graduate student is beneficial on many levels. First and foremost, since the conference is aimed at beginners in the field, you should understand more of the talks. Workshops can supplement what you have learned in your graduate courses and can also introduce you to new research areas. Another benefit of attending workshops is that you will meet other graduate students in your area of specialty. These students will be your colleagues for many years to come, and some will also become coauthors or friends. For this reason, the environment of a workshop is often very conducive to interacting socially with other participants.

Another type of research conference is one at which researchers present new mathematical findings. These conferences often span a weekend (two–three days), but can also run a week or longer. They generally consist of a series of longer talks (roughly one hour each) by invited speakers, who tend to be well-established or up-and-coming researchers in the field. Sometimes these meetings include sessions of shorter talks by participants, usually called contributed talks. You do not have to be invited to give a contributed talk, and once you have research results, you should strongly consider applying to give one. You will be able to share your research with an audience of mathematicians in your research area and gain visibility by giving a contributed talk. Moreover, if you give a contributed talk, the organizers are more likely to provide financial support to attend the conference. The application to give a contributed talk is usually not competitive; applying means submitting a title and abstract, and graduate students are always encouraged to apply. Keep in mind that, although it may seem intimidating to give a talk at a conference attended by experts in your field, presenting your work is a vital part of being a mathematician, and it is easiest to begin practicing as a graduate student.

A third type of conference is an AMS national or sectional meeting. These conferences are run by the AMS and happen at regular intervals. Each AMS section (Eastern, Southeastern, Central, and Western) has a meeting in the fall and in the spring every year at different colleges and universities. AMS sectional meetings occur over a weekend and consist of special sessions and invited talks. Special sessions are organized by volunteers who invite speakers to attend and give talks. The length of talks in a special session is usually set at twenty minutes. The four one-hour invited talks are chosen by the AMS and the local organizing committee. The national conference, also called the Joint Mathematics Meetings, occurs every year in January. This conference has a similar format to the sectional meetings but on a much larger scale. AMS meetings do not provide financial support for any speakers or participants. Other mathematical organizations, such as SIAM and the MAA, also host meetings throughout the year.

Aside from AMS meetings, most conferences have some funds to provide financial support to participants. These funds are usually provided by a national organization, such as the NSF or the NSA, or by the hosting university, and monies are often set aside for graduate students and recent Ph.D.s. In other words, as a graduate student, you can often attend conferences for free! Before you get too excited, keep in mind that this “free trip” is not a vacation. In other words, you most likely won’t be staying at a deluxe resort or visiting popular tourist sites. Instead, you will probably be spending six or eight hours in a classroom or auditorium listening to lectures. Nevertheless, attending conferences does provide opportunities for visiting new places and experiencing other college and university campuses.

In order to receive financial support to attend a conference, it might be necessary to give a contributed talk, submit a CV or short statement about your research, or provide a letter of support from your advisor. After you apply for support, wait to hear back from the organizers before buying plane tickets or making hotel reservations. The organizers will provide you with instructions on how to get reimbursed and the maximum amount that they can reimburse you. This is all important information. A final word on the subject of financial support: always keep your receipts!

What Should I Pack? Things to Do before You Attend a Conference

Once you have decided to attend a conference, there will be several logistical details that require your attention. Visit the conference website to learn the details of where to stay, how to get there, and how to get reimbursed. Be sure to take notice of any deadlines, such as for abstract submission or reserving hotel rooms. Do not assume that these deadlines are soft! Missing deadlines inevitably causes more work for the organizers, and as a graduate student or recent Ph.D., this is not how you want to be remembered.

The most important information available on the conference website is the list of speakers or participants. Look over this list in advance of the conference and do some research on the speakers. You may recognize some names on the list, but probably there will be others that are unfamiliar. Find the speakers' home pages. Look around and try to get a sense of who they are (e.g., institution, rank, Ph.D. advisor, etc.) and what they do mathematically. To answer the latter question, you might also want to use MathSciNet or the Math arXiv to find recent publications and coauthors. There is no need to read every paper to learn the technical details of what they have each proved. Instead, aim to scan some abstracts to get a general sense of the research expertise of each speaker.

Another great way to prepare for conferences is to regularly attend a departmental seminar in your area of interest. By attending seminars, you will begin to learn what types of research questions are interesting to mathematicians in your field. Even if you do not understand the seminar talks, try to understand some general ideas of what the talk is about. If you can, make an exercise of writing down one or two sentences about each talk you attend, and also a question. Save these pieces of paper: they might make more sense to you in the future. Work up the courage to ask your question to the speaker. Also, if there is a group going to lunch or dinner, ask to be included. You will gain valuable experience interacting with mathematicians and get the opportunity to meet researchers from other institutions. The ritual of listening to research talks, discussing mathematics, and sharing a meal will be repeated over and over at conferences, and it is more comfortable to begin doing all of these things at your home institution.

Pack comfortable, casual clothes. By now you have enough experiences with mathematicians to know that they are not formal dressers. And, after all, it is the weekend.

What Should I Do while I'm There? Maximizing Your Time at a Conference

Now that you've arrived at the conference, take a look at the schedule if you have not already seen it posted online. If there are parallel sessions (meaning that two or more talks take place at the same time in different locations), mark the talks that you want to see based on the research you did before the conference. Even if there is only one session of talks occurring, put a star next to the talks that you don't want to miss. It is no secret that mathematicians skip talks, and no one will be taking attendance. Nevertheless, as a graduate student your goal is to network with the participants, so you want to be present as much as possible. As a young researcher you will probably spend most of your time at conferences attending talks. As a more senior researcher, however, you may find it more beneficial to spend less time at talks and more time interacting with other mathematicians. Nonetheless, conference talks are always a touchstone for generating conversation among participants.

Keep your expectations of what you will learn from the talks at a conference reasonable. You will be overwhelmed with information, most of it above your head, in the next few days. If you have been attending seminar talks at home, then this will not be a surprise. Do not feel intimidated or frustrated, and do not give up and leave a talk. Continue your exercise of writing down at least one or two ideas from each talk and a question. If a talk is of interest to you, then you may want to take extensive notes. Another suggestion is to keep track of the names that the speaker references in his or her talk. After keeping track of these individuals for a while, you can begin to see how your research area developed over time. You will also see patterns emerge of certain groups who work on similar problems and thus learn about different subgroups within your specialty area.

If you are giving a contributed talk, make sure that you have prepared well for your presentation (for more on how to prepare a research talk, consult one of many guides available, such as John McCarthy's pamphlet "How to give a good colloquium", distributed by the AMS, or Steven Krantz's book *A Mathematician's Survival Guide*). Make sure to have the proper technology (e.g., laptop, flash drive, or transparencies) and a backup. It is always a good idea to email the organizers before you travel to confirm what will be available in the lecture room. Stay within your allotted time and make sure to thank the organizers for the opportunity to speak. Pay attention to who is in your audience, and introduce yourself to these participants either right after your talk or during another break in the conference. Try to engage these people in a discussion about your talk. You might want to ask them if they had any questions, or you can continue where your presentation left off by discussing what research questions you will be exploring next. Even though giving a talk can be nerve-wracking, having a large audience or several questions at the end of your talk is a good thing. When participants ask questions, it means that they find your research interesting (or, at the very least, they paid attention to you for twenty minutes, which is no small feat). If you get a question that you cannot answer, write down the question and who asked it, and try to find an answer when you get home.

Research talks are the focal point of most mathematical conferences. However, as a participant you should consider the time in between and after talks as equally important components of the conference. Mathematicians have their own unique culture, and many of their cultural rituals happen outside of talks. All conferences have scheduled tea breaks during the morning and afternoon. This break of twenty–thirty minutes is a time set aside for informal discussions and caffeine consumption. Mathematicians will often talk about the results presented in the preceding sessions. As much as possible, integrate yourself into these discussions. At tea time, join a group and introduce yourself. Be assertive and eager to talk about yourself and your research. When others are talking, listen carefully to what is being discussed. Try to get a sense of what they deem important. For instance, did they find one talk particularly interesting? Or did they find one result very surprising? As a graduate student it is often impossible to see the big picture of your research area. Partaking in these informal discussions may help you better understand the history and development of your research area, as well as point you in the direction of what problems are interesting to active mathematicians.

Another opportunity for informal interaction with conference participants occurs in the evenings after the talks have finished. There may be an organized banquet or party, and as a graduate student you may worry that you are "not really invited" or are otherwise excluded from these activities. This is absolutely the wrong approach, and you should participate in every activity the conference has to offer. If there is no organized dinner, most mathematicians will form groups to go out to dinner, and you should invite yourself along. Dinner is usually a relaxed occasion, with many people having a drink and discussing a wide variety of topics in addition to mathematics, such as the weather, sports, or university administrations.

As you get to know the members of your research group, you will learn that they tend to gravitate toward certain conversational topics. You may also learn that they have strong political feelings or other strong opinions. As a newcomer, this may be surprising, and you should carefully observe how the group interacts. You do not have to change yourself in order to fit in, but you want to gauge how to act in a manner that is friendly and professional. In rare cases, you might find yourself in an awkward situation over dinner. Your companions might order a very expensive bottle of wine and expect you to share in the cost even if you did not drink a glass, or you might find yourself seated between two people having a conversation in a foreign language. Find a way to be polite but also assert yourself.

In summary, as a graduate student your goals at a research conference are to meet as many mathematicians in your area as you can and to promote your own research. Remain upbeat and enthusiastic about your research and take every opportunity available to discuss your mathematical results and activities. The mathematicians you meet at conferences will have a huge impact on your mathematical career. These men and women may, in the near future, write you a recommendation letter for a job or be a postdoctoral mentor. In the longer term, these colleagues could potentially write a paper with you, support your job application to their institution, rate your grant application, peer review one of your research papers for a journal, or write a letter for promotion and tenure. In summary, your entire career will be shaped by the opinion of mathematicians in your research area. There is no benefit to "sucking up" to these individuals, but you should make every effort to meet mathematicians in your specialty and share your research with them. These encounters may seem inconsequential, but they can have a huge impact on the entire span of your mathematical career.

What Souvenirs Should I Bring Back from the Trip? Things to Do When You Return from a Conference

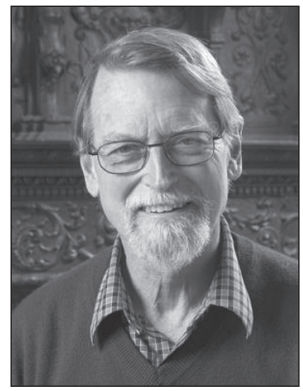
After you return from a conference and catch up on some sleep, take the time to organize everything that you collected from the meeting. If you gave a contributed talk, add the talk to your CV. Save a copy of the program in which your name is printed, and keep this in a folder dedicated to materials of this type. If there were unanswered questions from your talk, you should try to find an answer and, if possible, send an email to whomever posed the question. If there was a preprint that interested you, download the paper from the arXiv or email the author for a copy. Read or file these documents where you can refer to them later. Finally, you should send a short email thanking the organizers and take care of any reimbursement paperwork.

After attending one or two conferences, the overall experience should get easier and more enjoyable. You will begin to see familiar faces at conferences and develop collegial relationships with some participants. In other words, you will no longer be the random person at the dinner table. More important, you will become more comfortable talking about math to new people and more at ease giving research presentations. The purpose of attending conferences as a graduate student is for all of these activities to eventually become second nature.

In time, you will transition from a new researcher to an established researcher in your area, and your role at conferences will change. Remember what it was like at your first few conferences, and help graduate students and junior researchers by including them in discussions and inviting them to dinner.

Mumford Receives National Medal of Science

Elaine Kehoe



David Mumford

On October 15, 2010, President Barack Obama announced the recipients of the National Medal of Science. Among the ten recipients is David Mumford, professor emeritus of applied mathematics at Brown University. Mumford was honored for "extraordinary contributions to the mathematical, engineering and neurobiological sciences."

David Mumford is one of the most influential algebraic geometers of the second half of the twentieth century. A student of Oscar Zariski, he was from the beginning of his career well acquainted with the classical roots of the subject and with its important problems. Mumford was an early adopter and champion of Alexander Grothendieck's approach to algebraic geometry and brought this approach to bear on many long-standing problems. He made seminal contributions to many different aspects of the subject, such as the theory of surfaces and more general questions of classification, the theory of singularities of algebraic varieties, and moduli problems. Mumford is the founder of geometric invariant theory, which provides a framework for treating moduli spaces in many different contexts. Some years ago he left algebraic geometry to pursue other interests and has contributed to such varied fields as brain science, computer vision, neurobiology, cognitive science, and the biology and psychology of perception. Nevertheless, his influence still pervades algebraic geometry, not just through his theorems but also through his books and through the activities of the algebraic geometers he trained as students, many of whom are now senior figures in the subject.

Mumford was born in 1937 in Sussex, England. He studied mathematics at Harvard University, receiving his bachelor's degree in 1957 and his Ph.D. in 1961 under Oscar Zariski. He was a Putnam Fellow (1955–1956) and a Junior Fellow (1958–1961) while at Harvard. From 1961 to 1996 he held various positions at Harvard. He became University Professor at Brown in 1996. He has been a member of the Institute for Advanced Study at Princeton and a visiting professor at the University of Tokyo, both in 1962–1963. He has held numerous other visiting positions, including at the Tata Institute for Fundamental Research, the Institut des Hautes Études Scientifiques, the Institut Henri Poincaré, and the Mathematical Sciences Research Institute in Berkeley. He has also been Nuffield Professor at the University of Warwick and Rothschild Professor at the Isaac Newton Institute of Cambridge University. He served as president of the International Mathematical Union from 1995 to 1998. Mumford was awarded the Fields Medal in 1974. He was co-winner of the Shaw Prize in Mathematical Sciences in 2006 and of the IEEE Longuet-Higgins Prize for fundamental contributions in computer vision in both 2005 and 2009. He received the AMS Steele Prize in 2007 and the Wolf Foundation Prize in Mathematics in 2008. He was a MacArthur Foundation Fellow from 1987 to 1992. He was elected to the U.S. National Academy of Sciences in 1975 and to the American Philosophical Society in 2000, and he is a foreign member of the Accademia Nazionale dei Lincei, Rome, and the Norwegian Academy of Science and Letters.

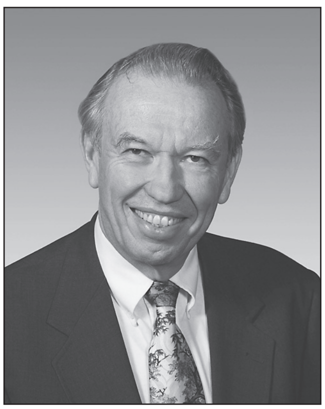
The National Medal of Science is the country's highest distinction for contributions to scientific research. According to a news release from the Office of Science and Technology Policy, "the National Medal of Science honors individuals for pioneering scientific research in a range of fields, including physical, biological, mathematical, social, behavioral, and engineering sciences, that enhances our understanding of the world and leads to innovations and technologies that give the United States its global economic edge." The National Science Foundation administers the award, which was established by Congress in 1959.

Elaine Kehoe is contributing writer for the Notices. *Her email address is* `elainelkehoe@cox.net`.

Presidential Views: Interview with George Andrews

Allyn Jackson

Every other year, when a new AMS president takes office, the *Notices* publishes interviews with the outgoing and incoming presidents. What follows is an edited version of an interview with George E. Andrews, whose two-year term as president ends on January 31, 2011. Andrews is the Evan Pugh Professor in the Department of Mathematics at Pennsylvania State University. The interview was conducted in fall 2010 by *Notices* senior writer and deputy editor Allyn Jackson. An interview with president-elect Eric Friedlander will appear in the March 2011 issue of the *Notices.*



George Andrews

Notices: *What will you remember most about being AMS president?*

Andrews: There are a number of projects that have interested me while I have been president, but the overall importance and significance of the Society and the role the president plays in it are the things that will stick with me the most.

Notices: *What are some of the projects you were involved in?*

Andrews: When I was asked to run for the presidency—which came as a huge surprise to me—I decided that in writing a candidate's statement I would try to pick out just a few things that seemed to me to be of major importance, and if elected I would see what I could do with them. To put them succinctly: I am a supporter of a fellows program and hoped to promote that; I believe that there ought to be a program of small grants, especially for mathematicians early in their careers, and I hoped to see something happen along those lines; I am of the "big tent" philosophy, so I hoped to pursue programs that might build new bridges between the three major societies, the AMS, the MAA [Mathematical Association of America], and SIAM [Society for Industrial and Applied Mathematics]; and I feel that questions of mathematics education, especially in the early years, are of supreme importance, so I interested myself in projects connected with that, particularly professional development for teachers.

Allyn Jackson is senior writer and deputy editor of the Notices. *Her email address is* `axj@ams.org`.

Notices: *Let's start with the fellows program. Can you first describe the idea of this program, for those unfamiliar with it, and then say what the current status is?*

Andrews: The fellows program is designed to create a category of AMS membership called a "fellow", which is an honorary title. Once it is in full operation, the program would name somewhere around fifteen hundred members as AMS Fellows. My personal view is that the program would be a useful thing for the entire mathematics community because it draws attention to people who have contributed substantially to mathematics research. This often has positive effects on awareness and understanding of mathematics by the general public, as well as by members of university administrations.

What has happened over the past two or three years is that the fellows program has been put up for a vote twice in the AMS, and each time it barely lost—it had huge majorities, but not the two-thirds majority that was set as the level at which the program would become a part of the AMS. I have been on a committee trying to design a program that could be put before the membership perhaps in 2011 or the year after. We hope that this new version will address some of the concerns of people opposed to previous versions of the program and that in a future vote it will be accepted. The proposal will be presented to the AMS Council in January 2011.

Notices: *The idea of a fellows program has been controversial. Why?*

Andrews: There are people who love the idea of the egalitarian nature of mathematics and feel the program would create two classes of citizens. My view is that it is important for mathematics to have awards and honorific titles that allow us to draw attention to important achievements and good things that we do. So rather than being something that would harm those who do not become fellows, the program in my view would be good for everybody. Obviously it would be good for the fellows, but it would also make clear that mathematics is doing significant and important things, and that is valuable to all of us.

Notices: *Can you tell me about your work on a program of small grants?*

Andrews: There has been no progress at all with regard to my original hope that the National Science Foundation [NSF] might undertake at least a pilot program of small grants. However, the Simons Foundation has recently established a substantial program of small grants [see "New Program at the Simons Foundation", *Notices*, November 2010, page 1324]. The program is headed by David Eisenbud [director for Mathematics and the Physical Sciences at the Simons Foundation]. I am quite excited about this because it is a big program, something on the order of hundreds of grants. It achieves the sort of thing that I had hoped to see the NSF do, and in some sense it is better than having it done through the NSF, because there are always difficult bureaucratic aspects of dealing with a government agency that are not troublesome with a private foundation.

Notices: *This is something the math community has wanted for a long time.*

Andrews: Everyone I have talked to who has been involved in this makes it clear that this sort of suggestion has been in the air for decades, and as long as we were dealing with the NSF, it has for various reasons gone nowhere. So it is very encouraging to see that this is going to take off from a private foundation. The program is not explicitly devoted to young mathematicians, but it is clear they will be the chief beneficiaries.

Notices: *What about your efforts to build bridges to the other societies?*

Andrews: There are two efforts that I made, and the thing that I am most pleased about is that the AMS will now have a presence in the MAA MathFest. The AMS and MAA sponsored a joint summer meeting until the 1990s, when the AMS pulled out. We have had no presence at the summer MathFest since then. Now there will be a jointly sponsored AMS-MAA hour talk at the MathFest and probably a special session to go along with it. The MAA was quite receptive to this proposal, and the first such talk will be, I believe, at the next MathFest in summer 2011. I am hoping that I can facilitate a similar program between the AMS and SIAM, to have an AMS presence at the SIAM national meeting, but that is not in place yet.

I have also been working with the presidents of the MAA and SIAM on another effort. The presidents of the three organizations for these last two years were all on the faculty at Penn State in the early 1990s, so we are all old friends. We have been exploring the idea of a joint reciprocity relationship among the societies. There are a variety of financial problems connected with this, so it is still under negotiation. It's not quite clear whether it would be a classic reciprocity relationship or something else, but we hope that something can be done to encourage more people to be members of all three societies.

Notices: *Mathematics education is another thing you paid attention to during your presidency. Can you talk about what you did in that area?*

Andrews: In the time I have been president, two national organizations—the National Governor's Association and the Council of Chief State School Officers—decided to put together a committee that would produce a set of national K–12 standards. I was one of the early readers of at least part of the proposal. It has turned out reasonably well in terms of substance. There are things that bother me about it, but not so much that I am seriously critical of it.

My main concern—and one of the reasons I am not as thrilled about it as some people are—is that it is a top-down effort. All that will come out of it is more or less a curriculum and curriculum recommendations. I believe that professional development programs that address the substance of what teachers know—especially elementary school teachers—are much more important. If you have teachers in the classroom who are not up to speed in terms of substance, it doesn't matter what the curriculum is; it is going to be a disaster. On the other hand, if you have teachers in the classroom who really are mathematically capable and comfortable with the mathematics they are teaching, they can rise above a curriculum no matter what it is.

The professional development program that I have concentrated on most heavily is Ken Gross's Vermont Mathematics Initiative. I am doing what I can to encourage that program to be more nationally oriented and to support efforts to obtain funding for a national roll-out.

Notices: *I have heard of Gross's initiative but I don't know much about it.*

Andrews: It is a three-year master's degree program for in-service teachers. The teachers spend time in the program each summer and then one weekend each month throughout the academic year. I visited one of these weekends about a year ago, and I was struck by the care with which the program had been put together, by the dedication of the instructors, but most of all by the motivation and enthusiasm of the in-service teachers. I have taught the mathematics course for elementary school teachers at Penn State, and those future teachers are not thrilled with mathematics and not particularly motivated. The contrast between them and the in-service teachers in Gross's program was the contrast of night and day. The program has been in place for ten years and has measured the increase in achievement by students taught by the teachers who have graduated from the program. There is a significant difference and an especially significant one for the children coming from disadvantaged homes. This program is something that actually works.

Notices: *We have covered the four major points from your candidate's statement. Let me ask about something else. The job market for young mathematicians is very difficult right now. What can the AMS do?*

Andrews: When I was contemplating running for president, the economy was fine, and the job market had not collapsed, so that was not on my list of things I thought I would be concentrating on. Certainly the AMS has at least tried to do a few things. For example, Jim Glimm put together a workshop that we ran just prior to the last national meeting to provide at least hints about and directions for possible retraining so that people could enlarge their job opportunities outside academia. It is very difficult for the AMS to have much impact here because we don't have a lot of money. We can't enter into the job market ourselves and sponsor a lot of postdocs or something like that. We have the AMS Centennial Fellowship, and some people have proposed that we ought to start another fellowship. But that's a microscopic drop in a huge bucket. So the main thing we have been able to do is to offer things like this workshop and to continue running the Employment Register. Just by the nature of the problem, the AMS will not have a major impact.

Notices: *The ICM [International Congress of Mathematicians], which you attended as AMS president, took place in August 2010 in India, the native land of Ramanujan, whose work has been of great interest to you. Did that meeting hold special significance for you?*

Andrews: This was my fifth visit to India. Every other time I have been there, I have been heavily involved as a Ramanujan scholar, but this time my role was quite different, namely, I was there as president of the AMS. There were various aspects that were connected to Ramanujan. There was a film crew making a documentary related to Ramanujan, and I spent one full morning on camera being interviewed. There was an impromptu session of people interested in Ramanujan, which of course I attended, and I also gave an impromptu talk. I suppose the most striking thing that went on at the conference related to Ramanujan was the presentation of the play *A Disappearing Number*, which has been playing in Europe and was simulcast this fall in the United States. Through that play, all of the ICM attendees got some education as to the life of Ramanujan.

Before I went to the ICM, I contacted James Tooley, who wrote the book *The Beautiful Tree*, which is about private schools in the slums of various poor countries throughout the world. I wanted to see these schools in operation, so while I was in Hyderabad his assistant took me to four different schools. It was a very heartening experience, because the schools seem to be making the best of a financially difficult situation. It was interesting to see their success and to see the contrast between how mathematics is taught at the elementary and secondary levels in these private schools and how things are done in the United States. I would say the most striking difference, put simply, is there is much more of what one would call rote learning in India than one typically sees in the U.S. today, at least in elementary education. In a way, that visit had a very strong Ramanujan significance for me because he was someone born into poverty in India and rose to great heights. It was interesting to see the ways in which the poor in India are in some sense managing to help themselves in emerging from poverty. India is an up and coming country. It's a very exciting place. While there is lots of poverty, there is also lots of hope.

Mathematics People

Agrawal and Moreira Awarded TWAS Prize in Mathematics

Manindra Agrawal of the Indian Institute of Technology and Carlos Gustavo Tamm de Araujo Moreira of the Instituto de Matemática Pura e Aplicada have been awarded the 2010 TWAS Prize in Mathematics, given by the Academy of Sciences for the Developing World (TWAS). Agrawal was honored "for his discovery of a novel characterization of prime numbers leading to a deterministic and efficient way of testing primality of a number." Moreira was recognized "for his fundamental contribution to the study of the interplay between fractal geometry and dynamical bifurcations." Each prize carries a cash award of US$15,000. The winners will each present a lecture about his research at the academy's general meeting in 2011.

—From a TWAS announcement

Manin Awarded Bolyai Prize

Yuri Manin of the Max-Planck-Institut für Mathematik and Northwestern University has been awarded the János Bolyai International Mathematical Prize of the Hungarian Academy of Sciences for his book *Frobenius Manifolds, Quantum Cohomology, and Moduli Spaces*, published by the AMS in 1999. The prize, which consists of a cash award of US$25,000 and a medal, was presented at the Hungarian Academy of Sciences on December 1, 2010. The award was established in 1903 by the Hungarian Academy of Sciences in honor of János Bolyai, codiscoverer of non-Euclidean geometry. The prize is awarded every five years to the author of the best mathematical monograph containing original research that was published in the previous ten years.

—From a Hungarian Academy of Sciences announcement

PECASE Awards Announced

Two mathematicians who were nominated by the National Science Foundation received Presidential Early Career Awards for Scientists and Engineers (PECASE) from President Obama. Jose H. Blanchet Mancilla of Columbia University was honored for furthering research using simulation for estimating the likelihood of rare but potentially catastrophic events and for educating students using state-of-the-art Monte Carlo methods. Katrin Wehrheim of the Massachusetts Institute of Technology was honored for contributing to geometry and analysis, for expositions of fundamental and emerging techniques in symplectic geometry, and for encouraging women and girls in mathematics at levels from middle school to junior university faculty.

—From an NSF announcement

Kamnitzer Awarded Aisenstadt Prize

Joel Kamnitzer of the University of Toronto has been awarded the 2011 André-Aisenstadt Prize of the Centre de Recherches Mathematiques (CRM) for his "substantial and deep contributions to the field of geometric representation theory and related topics." The prize recognizes outstanding research achievement by a young Canadian mathematician in pure or applied mathematics.

—From a CRM announcement

CAREER Awards Presented

The Division of Mathematical Sciences (DMS) of the National Science Foundation (NSF) has honored twenty-nine mathematicians in fiscal year 2010 with Faculty Early Career Development (CAREER) awards. The NSF established the awards to support promising scientists, mathematicians, and engineers who are committed to the integration of research and education. The grants provide funding of at least US$400,000 over a five-year period. The 2010 CAREER grant awardees and the titles of their grant projects follow.

Federico Ardila, San Francisco State University, Matroids, Polytopes, and Their Valuations in Algebra and Geometry; Paul Atzberger, University of California, Santa Barbara, Emergent Biological Mechanics of Cellular Microstructures; Erhan Bayraktar, University of Michigan, Ann Arbor, Topics in Optimal Stopping and Control; Janet Best, Ohio State University, Mathematical Questions Arising from Neural Systems: Research and Education; Lewis Bowen, Texas A&M Research Foundation, Ergodic Theory of Nonamenable Group Actions; Yitwah Cheung, San Francisco State University, Diophantine Analysis of Dynamical Systems; Izzet Coskun, University of Illinois, Chicago, The Cohomology and Birational Geometry of Moduli Spaces; Samit Dasgupta, University of California, Santa Cruz, Explicit Class Field Theory, Stark's Conjectures, and Families of Modular Forms; Sarah Day, College of William and Mary, Computational Dynamics and Topology; David Dumas, University of Illinois, Chicago, Complex Projective Structures, Teichmüller Theory, and Character Varieties; Daniel Groves, University of Illinois, Chicago, Surface Bundles and Logic in Geometric Group Theory; Abhinav Kumar, Massachusetts Institute of Technology, Lattices and Sphere Packings, Arithmetic Geometry, and Computational Number Theory; Melvin Leok, University of California, San Diego, Computational Geometric Mechanics: Foundations, Computation, and Applications; Gilad Lerman, University of Minnesota, Twin Cities, New Paradigms in Geometric Analysis of Data Sets and Their Applications; Jinchi Lv, University of Southern California, High-Dimensional Variable Selection and Risk Properties; Dan Margalit, Georgia Institute of Technology, Group-Theoretical, Dynamical, and Combinatorial Aspects of Mapping Class Groups; Yajun Mei, Georgia Institute of Technology, Streaming Data Analysis in Sensor Networks; Irina Mitrea, University of Minnesota, Twin Cities, Spectral Theory for Singular Integrals, Validated Numerics and Elliptic Problems in Non-Lipschitz Polyhedra: Research and Outreach; Julia Pevtsova, University of Washington, From Modular Representation Theory to Geometry: Connections and Interactions; Nicholas Proudfoot, University of Oregon, Eugene, Geometric Category O and Symplectic Duality; Alexander Rakhlin, University of Pennsylvania, Statistical and Computational Complexities of Modern Learning Problems; Dan Romik, University of California, Davis, Combinatorial Probability, Limit Shapes and Enumeration; Juan Souto, University of Michigan, Ann Arbor, Kleinian, Arithmetic, and Mapping Class Groups; Daniel Spirn, University of Minnesota, Twin Cities, Mathematics of Vorticity in Ginzburg-Landau Theory and Fluids; Seth Sullivant, North Carolina State University, Algebraic Problems in Statistics and Biology; Maria Westdickenberg, Georgia Institute of Technology, Combining Research on Dynamic Metastability and Hydrodynamic Limits with a Multifaceted Outreach Plan; Brett Wick, Georgia Institute of Technology, An Integrated Proposal Based on the Corona Problem; Jon Wilkening, University of California, Berkeley, Optimization and Continuation Methods in Fluid Mechanics; Gordan Zitkovic, University of Texas, Austin, Equilibria and Stability in Financial Markets.

—Elaine Kehoe

Chang Chosen Professor of the Year

Ping-Tung Chang, professor of mathematics at Matanuska-Susitna College, Palmer, Alaska, has been selected one of four Professors of the Year by the Council for Advancement and Support of Education (CASE) and the Carnegie Foundation for the Advancement of Teaching. His teaching has been inspired by George Polya's problem-solving method, through which he teaches students the tools they need to solve a problem, what method of solving the problem will be most effective, and how to carry it out. This approach helps students become more confident and engaged in solving and reasoning. He uses tests for formative assessment only and encourages students to retake tests until they master the content. This method has increased students' confidence and promoted a relaxed learning environment. He works with high school students to help them with math and has worked to develop more effective mathematics teaching methods in China.

—Elaine Kehoe

Rhodes Scholarships Awarded

Four students in the mathematical sciences are among thirty-two American men and women who have been chosen as Rhodes Scholars by the Rhodes Scholarship Trust. The Rhodes Scholars were chosen from among 837 students at 309 colleges and universities.

Zachary M. Frankel of Brooklyn, New York, is a senior at Harvard College with concentrations in physics and mathematics. He is fluent in Japanese, is an accomplished debater, and has done advanced graduate work in quantum field theory. He has also been a research assistant at the Kennedy School of Government. He has worked extensively with the Global Viral Forecasting Initiative, taking a semester off to develop models that help global health organizations and governments prevent future pandemics. Frankel plans to do a D.Phil. at Oxford in infectious diseases (zoology).

William J. Zeng of Great Falls, Virginia, is a senior at Yale majoring in physics. His course work ranges from quantum physics and mathematics to comparative literature, philosophy, and Hindi. He has done research at the Massachusetts Institute of Technology and at the Quantum Device Lab in Zurich, and he was an intern in New Delhi with the Indian Youth Climate Network. He has competed internationally on Yale's lightweight crew and has volunteered with the Special Olympics. He plans to do the M.Sc. in mathematics and the foundations of computer science at Oxford.

Esther O. Uduehi, Evansville, Indiana, is a senior at Indiana University, Bloomington, where she majors in biochemistry and mathematics. She was also a visiting student at Oxford. A Wells Scholar, Presidential Intern and Senator Richard Lugar Scholar, and a member of Phi Beta Kappa, she is president of the Indiana University Minority Association of Premedical Students. She has won several awards for her research in organic chemistry and has participated in a United States-Russia global health care study program and has done research at the Broad Institute of the Massachusetts Institute of Technology and at Harvard University. She plans to do the D.Phil. in chemistry at Oxford.

Prerna Nadathur of Roseville, Minnesota, is a senior at the University of Chicago, where she majors in mathematics and minors in linguistics and philosophy. She writes poetry and fiction, plays violin in the university chamber orchestra, performs classical Indian dance, and has won prizes for her piano performances. Prerna has also been a leader in student government and in social justice activities and founded a chapter of Students for a Democratic Society. She has done independent research on social choice theorems, set theory, and homology. At Oxford she will do the M.Phil. in general linguistics and comparative philology.

—From a Rhodes Scholarship Trust announcement

Daniel Rudolf Awarded 2010 IBC Prize

Daniel Rudolf of the University of Jena, Germany, has been awarded the 2010 Information-Based Complexity (IBC) Young Researcher Award. The award is given for significant contributions to information-based complexity by a young researcher who has not reached his or her thirty-fifth birthday by September 30 of the year of the award. The prize consists of US$1,000 and a plaque. The award will be presented at the Foundations of Computational Mathematics (FoCM) conference in Budapest in July 2011.

—Joseph Traub
Columbia University

About the Cover

Random Young diagrams

The cover was suggested (albeit loosely) by the article in this issue on free cumulants, written by Jonathan Novak and Pyotr Śniady. Free cumulants are a tool in the theory of non-abelian probability, and the distribution of Young diagrams was one of the earliest applications of that theory.

The images on the cover were produced by a kind of random walk that generates Young diagrams by adding one box randomly in each step, being sure that the result is again a Young diagram. Such a path of diagrams is in effect a numbering of the boxes in the diagram or, in other words, a Young tableau. Thus, in this process a Young diagram of size n is assigned a probability proportional to the number of Young tableaux giving rise to that diagram. It was proven by S. V. Kerov and A. M. Vershik on the one hand, and B. F. Logan and L. A. Shepp on the other, that as n goes to infinity the random diagrams, scaled by $1\sqrt{n}$, tend to cluster around a fixed shape (indicated in the cover images by a grey curve). This is a non-abelian analogue of tossing coins and getting heads close to half the time. Later work analyzed in more detail the non-abelian Gaussian distribution involved.

This early work has been elaborated extensively, notably in continuing and remarkable work of Kerov and Philippe Biane. It is in Biane's work that free cumulants have proven to be especially valuable. Each Young diagram of size n corresponds to an irreducible representation of $\mathfrak{S}_n$, and the distribution of Young diagrams defined above is associated to the asymptotic behavior of the regular representation of $\mathfrak{S}_n$. Biane has shown how other weightings may be associated to the asymptotic limit of other families of representations of the symmetric group. Kerov and others have explored the asymptotic distribution associated to other infinite series of groups, such as $GL_n(\mathbb{F}_q)$ as $n \to \infty$, but much remains to be discovered.

I have found the written version of Biane's ICM talk in Beijing a good guide to the literature. Along with all the proceedings of the International Congresses, it may be found at

`http://www.mathunion.org/ICM/`

Kerov died at a relatively young age, in 2000. A short memorial article by Vershik can be found in volume 121 of the *Journal of Mathematical Sciences*. The preface of Kerov's book *Asymptotic Representation Theory of the Symmetric Group and Its Applications in Analysis* presents an introduction to his work. Many of his papers can be found at

`http://www.pdmi.ras.ru/~kerov/texfiles/index.html`

I was a little surprised to see that there is no entry on Kerov in Wikipedia, but then, according to Vershik's memorial, this seems to continue a trajectory along which Kerov deserved more recognition than he got.

—Bill Casselman
Graphics Editor
(`notices-covers@ams.org`)

Mathematics Opportunities

Mathematics Research Communities 2011

The American Mathematical Society (AMS) invites mathematicians just beginning their research careers to become part of Mathematics Research Communities, a program to develop and sustain long-lasting cohorts for collaborative research projects in many areas of mathematics. Women and underrepresented minorities are especially encouraged to participate. The AMS will provide a structured program to engage and guide all participants as they start their careers. The program will include a one-week summer conference for each topic, special sessions at the national meeting, discussion networks by research topic, and a longitudinal study of early career mathematicians.

The summer conferences of the Mathematics Research Communities will be held in Snowbird Resort, Utah, where participants can enjoy the natural beauty and a collegial atmosphere. The application deadline for summer 2011 is **March 2, 2011**. This program is supported by a grant from the National Science Foundation. Advanced graduate students and postdoctoral researchers are welcome to apply to be participants.

The topics, dates, and organizers of the 2011 conferences follow.

Week 1: The Geometry of Real Projective Structures, Virginie Charette (Université de Sherbrooke), Daryl Cooper (University of California, Santa Barbara), William Goldman (University of Maryland), and Anna Wienhard (Princeton University).

Week 2: Computational and Applied Topology, Gunnar Carlsson (Stanford University), Robert Ghrist (University of Pennsylvania), and Benjamin Mann (Ayasdi, Inc.).

Week 3: The Pretentious View of Analytic Number Theory, Andrew Granville (Université de Montréal), Dimitris Koukoulopoulos (Université de Montréal), Youness Lamzouri (University of Illinois at Urbana-Champaign), Kannan Soundararajan (Stanford University), and Frank Thorne (Stanford University).

Situated in a breathtakingly beautiful mountain setting, Snowbird Resort provides an extraordinary environment for the MRC. The atmosphere is comparable to the collegial gatherings at Oberwolfach and other conferences that combine peaceful natural ambience with stimulating meetings.

MRC participants have access to a range of activities, such as a tram ride to the top of the mountain, guided hikes, swimming, mountain bike tours, rock climbing, plus heated outdoor pools. More than a dozen walking and hiking trails head deep into the surrounding mountains. Participants can also enjoy the simpler pleasures of convening on the patios at the resort to read, work, and socialize. In the evenings colleagues enjoy informal gatherings to network and continue discussion of the day's sessions over refreshments.

Within a half hour of the University of Utah, Snowbird is easily accessible from the Salt Lake City International Airport. For more information about Snowbird Resort, see `http://www.snowbird.com`.

A report about the 2008 MRC conferences appears in the February 2008 issue of the *Notices* (`www.ams.org/notices/200802/tx080200247p.pdf`). For information on applying for the 2011 program, please visit the website `http://www.ams.org/programs/research-communities/mrc-11`. For further information about the MRC program, please contact AMS associate executive director Ellen Maycock at `ejm@ams.org`.

—AMS announcement

Summer Program for Women Undergraduates

The 2011 Summer Program for Women in Mathematics (SPWM2011) will take place at George Washington University in Washington, DC, from June 25 to July 30, 2011. This is a five-week intensive program for mathematically talented undergraduate women who are completing their junior years and may be contemplating graduate study in mathematical sciences. The goals of this program are to communicate an enthusiasm for mathematics, develop research skills, cultivate mathematical self-confidence and independence, and promote success in graduate school.

Applicants must be U.S. citizens or permanent residents studying at a U.S. university or college who are completing their junior years or the equivalent and have mathematical experience beyond the typical first courses in calculus and linear algebra. Sixteen women will be selected. Each will receive a travel allowance, campus room and board, and a stipend of US$1,750. The deadline for applications is **March 1, 2011.** Early applications are encouraged. Applications are accepted only by mail. For further information, please contact the director, Murli M. Gupta, email: `mmg@gwu.edu`; telephone: 202-994-4857; or visit the program's website at `http://www.gwu.edu/~spwm/`. Application material is available on the website.

—From an SPWM announcement

NSF Support for Undergraduate Training in Biological and Mathematical Sciences

The National Science Foundation (NSF) offers opportunities for support through its Undergraduate Biology and

Mathematics (UBM) program. The goal of the program is to enhance undergraduate education and training at the intersection of the biological and mathematical sciences and to better prepare undergraduate biology or mathematics students to pursue graduate study and careers in fields that integrate the mathematical and biological sciences.

The program will provide support for jointly conducted long-term research experiences for interdisciplinary balanced teams of at least two undergraduates from departments in the biological and the mathematical sciences. Projects should focus on research at the intersection of the mathematical and biological sciences and should provide students exposure to contemporary mathematics and biology addressed with modern research tools and methods. Projects must involve students from both areas in collaborative research experiences and include joint mentorship by faculty in both fields.

Between six and nine awards are expected to be made in 2011. The deadline for full proposals is **February 10, 2011.** For more information, see http://www.nsf.gov/pubs/2008/nsf08510/nsf08510.htm. The UBM program is a joint effort of the Education and Human Resources (EHR), Biological Sciences (BIO), and Mathematical and Physical Sciences (MPS) directorates of the NSF.

—From an NSF announcement

News from the Fields Institute

The Fields Institute will hold a thematic program on Dynamics and Transport in Disordered Systems during the spring of 2011. The following workshops will be held:

February 14-19, 2011: Disordered Polymer Models

April 4–8, 2011: The Fourier Law and Related Topics

June 13–17, 2011: Instabilities in Hamiltonian Systems

Yakov Sinai of Princeton University will deliver the Distinguished Lecture Series, February 22–24, 2011; see www.fields.utoronto.ca/programs/scientific/10-11/disorderedsys/DLS/. Srinivasa Varadhan of the Courant Institute of Mathematical Sciences will give the Coxeter Lecture Series, April 13–15, 2011. See www.fields.utoronto.ca/programs/scientific/10-11/disorderedsys/DLS/. More and up-to-date information on the program can be found at www.fields.utoronto.ca/programs/scientific/10-11/disorderedsys/.

The thematic program during the summer 2011 term will be Mathematical and Algorithmic Aspects of Constraint Satisfaction. It will include a Summer School whose primary goal is to provide participants with a thorough and intense introduction to the main themes of the program. The four threads of the school are: an introduction to the CSP, graph theory and combinatorics, universal algebra, and approximability of CSPs. The following three workshops are planned:

July 11-15, 2011: Graph Homomorphisms

August 2–6, 2011: Algebra and CSPs

August 15–19, 2011: Approximability of CSPs

The Logic in Computer Science (LICS) annual meeting will be held during the week of June 20, 2011, at the Institute and at the University of Toronto. More information is available at www.fields.utoronto.ca/programs/scientific/11-12/constraint/.

Special Lectures

Shing-Tung Yau of the Chinese University of Hong Kong and Harvard University will deliver several lectures as part of the Fields' Distinguished Lecture Series, January 19–21, 2011, at the University of Toronto. See www.fields.utoronto.ca/programs/scientific/10-11/DLS_Yau/.

Piergiorgio Odifreddi of the University of Turin will give a public lecture on February 17, 2011. See www.fields.utoronto.ca/programs/scientific/10-11/public_lectures/.

The annual Nathan and Beatrice Keyfitz Lectures in Mathematics and the Social Sciences will take place on March 14, 2011, delivered by George Lakoff of the Linguistics Department, University of California, Berkeley. The title of the lecture is "The Cognitive and Neural Basis of Mathematics". See www.fields.utoronto.ca/programs/scientific/keyfitz_lectures/.

Future thematic programs include Discrete Geometry and Applications (Fall 2011), Galois Representations (Winter/Spring 2010), and Forcing and Its Applications (Fall 2012).

See www.fields.utoronto.ca/programs/scientific/ for more information on these programs and on all other activities at the institute.

—From a Fields Institute announcement

CRM-PISA Junior Visiting Program

The "Centro di Ricerca Matematica Ennio De Giorgi" (CRM PISA) invites applications for four one-year Junior Visiting Positions for the academic year 2011-2012.

Successful candidates will be new or recent Ph.D.s in mathematics with an exceptional research potential. The annual Junior Visitors' salary is 25,000 Euros tax inclusive (personal income tax) plus a research allowance of approximately 2,500 Euros which Junior Visitors can spend to invite other researchers to CRM PISA. Junior Visitors are expected to start their research activity at CRM no later than October 2011.

Since 2001 CRM PISA has aimed at promoting excellence in a vast spectrum of research fields, from pure mathematics to mathematics applied to the natural and social sciences, and at providing a thriving international and interdisciplinary research environment, particularly for junior visitors who can take part in a great variety of scientific activities and have a unique opportunity to interact with top-class scientists who visit the Centre.

The deadline for application is January 31, 2011. Full particulars about the positions offered and the application procedure are available at http://www.crm.sns.it/grant/18/.

Complete information about the CRM PISA can be found at http://www.crm.sns.it/.

—From a CRM-PISA announcement

Inside the AMS

From the AMS Public Awareness Office

AMS on Social Networks

As part of the Society's commitment to the open flow of communication and community engagement, the AMS uses several social networking tools to supplement the channels currently in place for members, press, and the general public.

We invite you to follow and share content of interest.

* "Like" us—and find others who "like" the AMS—at our AMS Facebook page.

* Follow us on Twitter.

* Read and contribute to the Graduate Student Blog at `mathgradblog.williams.edu/`.

* Subscribe to our videos, share them, comment on them, and embed them in your own sites from the AMS YouTube channel.

* Set up an RSS feed to receive content electronically whenever it is updated on the AMS website.

We invite you to use these networks to find news, updates on AMS programs, commentary, and contests; to connect with others who have similar interests; and to communicate with each other and the AMS. Read more at `www.ams.org/about-us/social`.

2010 Arnold Ross Lecture

"*Thank you for the excellent event. My students and I thoroughly enjoyed the lecture and the competition.*" That was typical of the reaction of about two hundred students and teachers who attended the 2010 Arnold Ross Lecture at the Carnegie Science Center in Pittsburgh, PA. Thomas Hales, Andrew Mellon Professor at the University of Pittsburgh, talked about packings and proof in his lecture "Can Computers Do Math?" Afterward, eight Pittsburgh-area students played *Who Wants to Be a Mathematician*. Read more about the lecture and contest at `www.ams.org/programs/students/wwtbam/arl2010`.

Thomas Hales giving the Arnold Ross Lecture.

—Annette Emerson and Mike Breen
AMS Public Awareness Officers
`paoffice@ams.org`

AMS Email Support for Frequently Asked Questions

A number of email addresses have been established for contacting the AMS staff regarding frequently asked questions. The following is a list of those addresses together with a description of the types of inquiries that should be made through each address.

abs-coord@ams.org for questions regarding a particular abstract or abstracts questions in general.

acquisitions@ams.org to contact the AMS Acquisitions Department.

ams@ams.org to contact the Society's headquarters in Providence, Rhode Island.

amsdc@ams.org to contact the Society's office in Washington, D.C.

amsmem@ams.org to request information about membership in the AMS and about dues payments or to ask any general membership questions; may also be used to submit address changes.

ams-survey@ams.org for information or questions about the Annual Survey of the Mathematical Sciences or to request reprints of survey reports.

bookstore@ams.org for inquiries related to the online AMS Bookstore.

classads@ams.org to submit classified advertising for the *Notices*.

cust-serv@ams.org for general information about AMS products (including electronic products), to send address changes, place credit card orders for AMS products, to correspond regarding a balance due shown on a monthly statement, or conduct any general correspondence with the Society's Customer Services Department.

development@ams.org for information about giving to the AMS, including the Epsilon Fund.

eims-info@ams.org to request information about Employment Information in the Mathematical Sciences (EIMS). For ad rates and to submit ads go to `http://www.eims.ams.org`.

emp-info@ams.org for information regarding AMS employment and career services.

eprod-support@ams.org for technical questions regarding AMS electronic products and services.

mathcal@ams.org to send information to be included in the "Mathematics Calendar" section of the *Notices*.

mathjobs@ams.org for questions about the online job application service for `Mathjobs.org`.

mathrev@ams.org to submit reviews to *Mathematical Reviews* and to send correspondence related to reviews or other editorial questions.

meet@ams.org to request general information about Society meetings and conferences.

mmsb@ams.org for information or questions about registration and housing for the Joint Mathematics Meetings (Mathematics Meetings Service Bureau).

msn-support@ams.org for technical questions regarding MathSciNet.

notices@ams.org to send correspondence to the managing editor of the *Notices*, including items for the news columns. The editor (`notices@math.wustl.edu`) is the person to whom to send articles and letters. Requests for permission to reprint from the *Notices* should be sent to `reprint-permission@ams.org` (see below).

notices-ads@ams.org to submit electronically paid display ads for the *Notices*.

notices-booklist@ams.org to submit suggestions for books to be included in the "Book List" in the *Notices*.

notices-letters@ams.org to submit letters and opinion pieces to the *Notices*.

notices-whatis@ams.org to comment on or send suggestions for topics for the "WHAT IS...?" column in the *Notices*.

paoffice@ams.org to contact the AMS Public Awareness Office.

president@ams.org to contact the president of the American Mathematical Society.

prof-serv@ams.org to send correspondence about AMS professional programs and services.

pub@ams.org to send correspondence to the AMS Publication Division.

pub-submit@ams.org to submit accepted electronic manuscripts to AMS publications (other than Abstracts). See `http://www.ams.org/submit-book-journal` to electronically submit accepted manuscripts to the AMS book and journal programs.

reprint-permission@ams.org to request permission to reprint material from Society publications.

sales@ams.org to inquire about reselling or distributing AMS publications or to send correspondence to the AMS Sales Department.

secretary@ams.org to contact the secretary of the Society.

student-serv@ams.org for questions about AMS programs and services for students.

tech-support@ams.org to contact the Society's typesetting Technical Support Group.

textbooks@ams.org to request examination copies or inquire about using AMS publications as course texts.

webmaster@ams.org for general information or for assistance in accessing and using the AMS website.

Deaths of AMS Members

Leo Esakia, of the Republic of Georgia, died on November 17, 2010. Born on November 14, 1934, he was a member of the Society for 29 years.

Paul S. Herwitz, of Worcester, Massachusetts, died on December 16, 2000. Born on June 10, 1923, he was a member of the Society for 59 years.

Peter J. Hilton, of Binghamton, New York, died on November 6, 2010. Born on April 7, 1923, he was a member of the Society for 51 years.

Lionel W. McKenzie, of Rochester, New York, died on October 12, 2010. Born on January 26, 1919, he was a member of the Society for 48 years.

Joan S. Morrison, of Baltimore, Maryland, died on August 18, 2010. Born on February 16, 1947, she was a member of the Society for 25 years.

Cora Sadosky, of Long Beach, California, died on December 3, 2010. Born on May 23, 1940, she was a member of the Society for 45 years.

Caroline Sweezy, of Las Cruces, New Mexico, died on November 6, 2010. Born on July 13, 1942, she was a member of the Society for 45 years.

William B. Woolf, of Pt. Townsend, Washington, died on December 6, 2010. Born on September 18, 1932, he was a member of the Society for 54 years.

Marie A. Wurster, of Philadelphia, Pennsylvania, died on July 26, 2010. Born on June 27, 1918, she was a member of the Society for 66 years.

Reference and Book List

The ***Reference*** *section of the* Notices *is intended to provide the reader with frequently sought information in an easily accessible manner. New information is printed as it becomes available and is referenced after the first printing. As soon as information is updated or otherwise changed, it will be noted in this section.*

Contacting the *Notices*

The preferred method for contacting the *Notices* is electronic mail. The editor is the person to whom to send articles and letters for consideration. Articles include feature articles, memorial articles, communications, opinion pieces, and book reviews. The editor is also the person to whom to send news of unusual interest about other people's mathematics research.

The managing editor is the person to whom to send items for "Mathematics People", "Mathematics Opportunities", "For Your Information", "Reference and Book List", and "Mathematics Calendar". Requests for permissions, as well as all other inquiries, go to the managing editor.

The electronic-mail addresses are `notices@math.wustl.edu` in the case of the editor and `notices@ams.org` in the case of the managing editor. The fax numbers are 314-935-6839 for the editor and 401-331-3842 for the managing editor. Postal addresses may be found in the masthead.

Upcoming Deadlines

January 21, 2011: Applications for Math for America Foundation (MfA) Fellowship Program. See `http://www.mathforamerica.org/`.

January 27, 2011: Proposals for NSF Computing Equipment and Instrumentation Programs (SCREMS). See `http://www.nsf.gov/pubs/2007/nsf07502/nsf07502.htm`.

January 31, 2011: Nominations for CAIMS/PIMS Early Career Award. See `http://www.pims.math.ca/pims-glance/prizes-awards`.

February 1, 2011: Applications for February review for National Academies Research Associateship Programs. See the National Academies website at `http://sites.nationalacademies.org/PGA/RAP/PGA_050491` or contact Research Associateship Programs, National Research Council, Keck 568, 500 Fifth Street, NW, Washington, DC 20001; telephone 202-334-2760; fax 202-334-2759; email `rap@nas.edu`.

February 1, 2011: Applications for AWM Mentoring Travel Grants. See `http://www.awm-math.org/travelgrants.html#standard`.

February 1, May 1, October 1, 2011: Applications for AWM Travel Grants. See `http://www.awm-math.org/travelgrants.html#standard`.

February 1, 2011: Applications for AMS von Neumann Symposium. See `www.ams.org/meetings/amsconf/symposia/symposia-2011`.

February 4, 2011: Full proposals for NSF Mathematical Sciences Research Institutes. See `http://www.nsf.gov/pubs/2010/nsf10592/nsf10592.htm?WT.mc_id=USNSF_25&WT.mc_ev=click`.

February 10, 2011: Full proposals for NSF Undergraduate Biology and Mathematics (UBM) Program. See "Mathematics Opportunities" in this issue.

February 15, 2011: Applications for AMS Congressional Fellowship. See `http://www.ams.org/programs/ams-fellowships/ams-aaas/ams-aaas-congressional-fellowship` or contact the AMS Washington Office at 202-588-1100, email: `amsdc@ams.org`.

February 21, 2011: Applications for EDGE for Women Summer Program. See `http://www.edgeforwomen.org/?page_id=5`.

February 27, 2011: Entries for Association for Women in Mathematics

Where to Find It

A brief index to information that appears in this and previous issues of the *Notices*.

(AWM) Essay Contest. See `http://www.awm-math.org/biographies/contest.html`.

March 1, 2011: Applications for Summer Program for Women in Mathematics (SPWM2011). See "Mathematics Opportunities" in this issue.

May 1, August 1, November 1, 2011: Applications for May, August, and November reviews for National Academies Research Associateship Programs. See the National Academies website at `http://sites.nationalacademies.org/PGA/RAP/PGA_050491` or contact Research Associateship Programs, National Research Council, Keck 568, 500 Fifth Street, NW, Washington, DC 20001; telephone 202-334-2760; fax 202-334-2759; email `rap@nas.edu`.

May 1, 2011: Applications for National Academies Christine Mirzayan Graduate Fellowship Program for fall 2011. See `http://sites.nationalacademies.org/PGA/policyfellows/index.htm`.

October 1, 2011: Nominations for the 2012 Emanuel and Carol Parzen Prize. Contact Thomas Wehrly, Department of Statistics, 3143 TAMU, Texas A&M University, College Station, Texas 77843-3143.

MPS Advisory Committee

Following are the names and affiliations of the members of the Advisory Committee for Mathematical and Physical Sciences (MPS) of the National Science Foundation. The date of the expiration of each member's term is given after his or her name. The website for the MPS directorate may be found at `www.nsf.gov/home/mps/`. The postal address is Directorate for the Mathematical and Physical Sciences, National Science Foundation, 4201 Wilson Boulevard, Arlington, VA 22230.

Taft Armandroff (10/12)
W. M. Keck Observatory
Kamuela, Hawaii

James Berger (10/11)
Department of Statistical Science
Duke University

Daniela Bortoletto (10/11)
Department of Physics
Purdue University

Kevin Corlette (10/12)
Department of Mathematics
University of Chicago

Juan J. de Pablo (10/12)
Department of Chemical and Biological Engineering
University of Wisconsin-Madison

Joseph M. DeSimone (10/12)
Department of Chemistry
University of North Carolina at Chapel Hill

Barbara J. Finlayson-Pitts (10/11)
Department of Chemistry
University of California, Irvine

Irene Fonseca (10/11)
Department of Mathematical Sciences
Carnegie Mellon University

Sharon C. Glotzer (10/12)
Department of Chemical Engineering
University of Michigan

Suzanne Hawley (10/11)
Astronomy Department
University of Washington

Jerzy Leszczynski (10/12)
Department of Chemistry and Biochemistry
Jackson State University

James W. Mitchell (10/12)
Department of Chemical Engineering
Howard University

Ramesh Narayan (10/11)
Harvard University and Harvard-Smithsonian Center for Astrophysics

Sharon L. Neal (10/11)
Department of Chemistry and Biochemistry
University of Delaware

Luis Orozco (10/12)
Department of Physics
University of Maryland, College Park

John Peoples Jr. (10/11)
Fermilab, Batavia, IL

Elsa Reichmanis (10/11)
School of Chemical and Biomolecular Engineering
Georgia Institute of Technology

Fred S. Roberts (10/12)
DIMACS
Rutgers University

Geoffrey West (10/11)
Santa Fe Institute
Santa Fe, NM

Book List

The Book List highlights books that have mathematical themes and are aimed at a broad audience potentially including mathematicians, students, and the general public. When a book has been reviewed in the Notices, *a reference is given to the review. Generally the list will contain only books published within the last two years, though exceptions may be made in cases where current events (e.g., the death of a prominent mathematician, coverage of a certain piece of mathematics in the news) warrant drawing readers' attention to older books. Suggestions for books to include on the list may be sent to* `notices-booklist@ams.org`.

*Added to "Book List" since the list's last appearance.

Apocalypse When?: Calculating How Long the Human Race Will Survive, by Willard Wells. Springer Praxis, June 2009. ISBN-13:978-03870-983-64.

The Best Writing on Mathematics: 2010, edited by Mircea Pitici. Princeton University Press, December 2010. ISBN-13: 978-06911-484-10.

Bright Boys: The Making of Information Technology, by Tom Green. A K Peters, April 2010. ISBN-13: 978-1-56881-476-6.

The Calculus of Friendship: What a Teacher and Student Learned about Life While Corresponding about Math, by Steven Strogatz. Princeton University Press, August 2009. ISBN-13: 978-0-691-13493-2. (Reviewed June/July 2010.)

The Cult of Statistical Significance: How the Standard Error Costs Us Jobs, Justice, and Lives, by Stephen T. Ziliak and Deirdre N. McCloskey, University of Michigan Press, February 2008. ISBN-13: 978-04720-500-79. (Reviewed October 2010.)

Duel at Dawn: Heroes, Martyrs, and the Rise of Modern Mathematics, by Amir Alexander. Harvard University Press, April 2010. ISBN-13: 978-

06740-466-10. (Reviewed November 2010.)

Euler's Gem: The Polyhedron Formula and the Birth of Topology, by David S. Richeson. Princeton University Press, September 2008. ISBN-13: 97-80691-1267-77. (Reviewed December 2010.)

Here's Looking at Euclid: A Surprising Excursion through the Astonishing World of Math, by Alex Bellos. Free Press, June 2010. ISBN-13: 978-14165-882-52.

The Housekeeper and the Professor, by Yoko Ogawa. Picador, February 2009. ISBN-13: 978-03124-278-01. (Reviewed May 2010.)

How to Read Historical Mathematics, by Benjamin Wardhaugh. Princeton University Press, March 2010. ISBN-13: 978-06911-401-48.

Isaac Newton on Mathematical Certainty and Method, by Niccolò Guicciardini. MIT Press, October 2009. ISBN-13: 978-02620-131-78.

Logicomix: An Epic Search for Truth, by Apostolos Doxiadis and Christos Papadimitriou. Bloomsbury USA, September 2009. ISBN-13: 978-15969-145-20. (Reviewed December 2010.)

The Math Book: From Pythagoras to the 57th Dimension, 250 Milestones in the History of Mathematics, by Clifford A. Pickover. Sterling, September 2009. ISBN-13: 978-14027-579-69.

A Mathematician's Lament: How School Cheats Us Out of Our Most Fascinating and Imaginative Art Form, by Paul Lockhart. Bellevue Literary Press, April, 2009. ISBN-978-1-934137-17-8.

Mathematicians: An Outer View of the Inner World, by Mariana Cook. Princeton University Press, June 2009. ISBN-13: 978-0-691-13951-7. (Reviewed August 2010.)

Mathematicians Fleeing from Nazi Germany: Individual Fates and Global Impact, by Reinhard Siegmund-Schultze. Princeton University Press, July 2009. ISBN-13: 978-0-691-14041-4. (Reviewed November 2010.)

Mathematics in Ancient Iraq: A Social History, by Eleanor Robson. Princeton University Press, August 2008. ISBN-13: 978-06910-918-22. (Reviewed March 2010.)

Mathematics in India, by Kim Plofker. Princeton University Press, January 2009. ISBN-13: 978-06911-206-76. (Reviewed March 2010.)

A Motif of Mathematics: History and Application of the Mediant and the Farey Sequence, by Scott B. Guthery. Docent Press, September 2010. ISBN-13 978-4538-105-76.

Mrs. Perkins's Electric Quilt: And Other Intriguing Stories of Mathematical Physics, Paul J. Nahin, Princeton University Press, August 2009. ISBN-13: 978-06911-354-03.

Naming Infinity: A True Story of Religious Mysticism and Mathematical Creativity, by Loren Graham and Jean-Michel Kantor. Belknap Press of Harvard University Press, March 2009. ISBN-13: 978-06740-329-34.

Nonsense on Stilts: How to Tell Science from Bunk, by Massimo Pigliucci. University of Chicago Press, May 2010. ISBN-13: 978-02266-678-67.

Numbers Rule: The Vexing Mathematics of Democracy, from Plato to the Present, by George G. Szpiro. Princeton University Press, April 2010. ISBN-13: 978-06911-399-44. (Reviewed in January 2011.)

The Numerati, by Stephen Baker. Houghton Mifflin, August 2008. ISBN-13: 978-06187-846-08. (Reviewed October 2009.)

Our Days Are Numbered: How Mathematics Orders Our Lives, by Jason Brown. Emblem Editions, April 2010. ISBN-13: 978-07710-169-74.

Perfect Rigor: A Genius and the Mathematical Breakthrough of the Century, by Masha Gessen. Houghton Mifflin Harcourt, November 2009. ISBN-13: 978-01510-140-64. (Reviewed in January 2011.)

Pioneering Women in American Mathematics: The Pre-1940 Ph.D.'s, by Judy Green and Jeanne LaDuke. AMS, December 2008. ISBN-13:978-08218-4376-5.

Plato's Ghost: The Modernist Transformation of Mathematics, by Jeremy Gray. Princeton University Press, September 2008. ISBN-13: 978-06911-361-03. (Reviewed February 2010.)

Probabilities: The Little Numbers That Rule Our Lives, by Peter Olofsson. Wiley, March 2010. ISBN-13: 978-04706-244-56.

Proofs from THE BOOK, by Martin Aigner and Günter Ziegler. Expanded fourth edition, Springer, October 2009. ISBN-13: 978-3-642-00855-9.

Pythagoras' Revenge: A Mathematical Mystery, by Arturo Sangalli. Princeton University Press, May 2009. ISBN-13: 978-06910-495-57. (Reviewed May 2010.)

Recountings: Conversations with MIT Mathematicians, edited by Joel Segel. A K Peters, January 2009. ISBN-13: 978-15688-144-90.

Roger Boscovich, by Radoslav Dimitric (Serbian). Helios Publishing Company, September 2006. ISBN-13: 978-09788-256-21.

The Shape of Inner Space: String Theory and the Geometry of the Universe's Hidden Dimensions, by Shing-Tung Yau (with Steve Nadis). Basic Books, September 2010. ISBN-13: 978-04650-202-32. (Reviewed in this issue.)

The Solitude of Prime Numbers, by Paolo Giordano. Pamela Dorman Books, March 2010. ISBN-13: 978-06700-214-82. (Reviewed September 2010.)

Solving Mathematical Problems: A Personal Perspective, by Terence Tao. Oxford University Press, September 2006. ISBN-13: 978-0-199-20560-8. (Reviewed February 2010.)

The Strangest Man, by Graham Farmelo. Basic Books, August 2009. ISBN-13: 978-04650-182-77.

Street-Fighting Mathematics: The Art of Educated Guessing and Opportunistic Problem Solving, by Sanjoy Mahajan. MIT Press, March 2010. ISBN-13: 978-0-262-51429-3.

Survival Guide for Outsiders: How to Protect Yourself from Politicians, Experts, and Other Insiders, by Sherman Stein. BookSurge Publishing, February 2010. ISBN-13: 978-14392-532-74.

Symmetry: A Journey into the Patterns of Nature, by Marcus du Sautoy. Harper, March 2008. ISBN: 978-00607-8940-4. (Reviewed in this issue.)

Symmetry in Chaos: A Search for Pattern in Mathematics, Art, and Nature, by Michael Field and Martin Golubitsky. Society for Industrial and Applied Mathematics, second revised edition, May 2009. ISBN-13: 978-08987-167-26.

Teaching Statistics Using Baseball, by James Albert. Mathematical Association of America, July 2003. ISBN-13: 978-08838-572-74. (Reviewed April 2010.)

What's Luck Got to Do with It? The History, Mathematics and Psychology of The Gambler's Illusion, by Joseph Mazur. Princeton University Press, July 2010. ISBN: 978-0-691-13890-9.

2010 Election Results

In the elections of 2010 the Society elected a vice president, a trustee, five members at large of the Council, three members of the Nominating Committee, and two members of the Editorial Boards Committee.

Vice President

Elected as the new vice president is **Barbara Lee Keyfitz** from Ohio State University. Term is three years (1 February 2011—31 January 2014).

Trustee

Elected as trustee is **William H. Jaco** from Oklahoma State University. Term is five years (1 February 2011—31 January 2016).

Members at Large of the Council

Elected as new members at large of the Council are

Matthew Ando from the University of Illinois, Urbana-Champaign

Estelle Basor from the American Institute of Mathematics

Patricia Hersh from North Carolina State University

Tara S. Holm from Cornell University

T. Christine Stevens from St. Louis University

Terms are three years (1 February 2011—31 January 2014).

Nominating Committee

Elected as new members of the Nominating Committee are

Richard A. Brualdi from the University of Wisconsin, Madison

Donal O'Shea from Mt. Holyoke College

Gunter Uhlmann from the University of Washington

Terms are three years (1 January 2011—31 December 2013).

Editorial Boards Committee

Elected as new members of the Editorial Boards Committee are

John R. Stembridge from the University of Michigan

Sergei K. Suslov from Arizona State University

Terms are three years (1 February 2011—31 January 2014).

2011 AMS Election

Nominations by Petition

Vice President or Member at Large

One position of vice president and member of the Council *ex officio* for a term of three years is to be filled in the election of 2011. The Council intends to nominate at least two candidates, among whom may be candidates nominated by petition as described in the rules and procedures.

Five positions of member at large of the Council for a term of three years are to be filled in the same election. The Council intends to nominate at least ten candidates, among whom may be candidates nominated by petition in the manner described in the rules and procedures.

Petitions are presented to the Council, which, according to Section 2 of Article VII of the bylaws, makes the nominations. The Council of 23 January 1979 stated the intent of the Council of nominating all persons on whose behalf there were valid petitions.

Prior to presentation to the Council, petitions in support of a candidate for the position of vice president or of member at large of the Council must have at least fifty valid signatures and must conform to several rules and operational considerations, which are described below.

Editorial Boards Committee

Two places on the Editorial Boards Committee will be filled by election. There will be four continuing members of the Editorial Boards Committee.

The President will name at least four candidates for these two places, among whom may be candidates nominated by petition in the manner described in the rules and procedures.

The candidate's assent and petitions bearing at least 100 valid signatures are required for a name to be placed on the ballot. In addition, several other rules and operational considerations, described below, should be followed.

Nominating Committee

Three places on the Nominating Committee will be filled by election. There will be six continuing members of the Nominating Committee.

The President will name at least six candidates for these three places, among whom may be candidates nominated by petition in the manner described in the rules and procedures.

The candidate's assent and petitions bearing at least 100 valid signatures are required for a name to be placed on the ballot. In addition, several other rules and operational considerations, described below, should be followed.

Rules and Procedures

Use separate copies of the form for each candidate for vice president, member at large, or member of the Nominating and Editorial Boards Committees.

1. To be considered, petitions must be addressed to Robert J. Daverman, Secretary, American Mathematical Society, Department of Mathematics, 1403 Circle Drive, University of Tennessee, 1534 Cumberland Avenue, Knoxville, TN 37996-1320, USA, and must arrive by 24 February 2011.
2. The name of the candidate must be given as it appears in the *Combined Membership List* (`www.ams.org/cml`). If the name does not appear in the list, as in the case of a new member or by error, it must be as it appears in the mailing lists, for example on the mailing label of the *Notices.* If the name does not identify the candidate uniquely, append the member code, which may be obtained from the candidate's mailing label or by the candidate contacting the AMS headquarters in Providence (amsmem@ams.org).
3. The petition for a single candidate may consist of several sheets each bearing the statement of the petition, including the name of the position, and signatures. The name of the candidate must be exactly the same on all sheets.
4. On the next page is a sample form for petitions. Petitioners may make and use photocopies or reasonable facsimiles.
5. A signature is valid when it is clearly that of the member whose name and address is given in the left-hand column.
6. The signature may be in the style chosen by the signer. However, the printed name and address will be checked against the *Combined Membership List* and the mailing lists. No attempt will be made to match variants of names with the form of name in the *CML.* A name neither in the *CML* nor on the mailing lists is not that of a member. (Example: The name Robert J. Daverman is that of a member. The name R. Daverman appears not to be.)
7. When a petition meeting these various requirements appears, the secretary will ask the candidate to indicate willingness to be included on the ballot. Petitioners can facilitate the procedure by accompanying the petitions with a signed statement from the candidate giving consent.

Nomination Petition

for 2011 Election

The undersigned members of the American Mathematical Society propose the name of

____

as a candidate for the position of (check one):

- ☐ **Vice President**
- ☐ **Member at Large of the Council**
- ☐ **Member of the Nominating Committee**
- ☐ **Member of the Editorial Boards Committee**

of the American Mathematical Society for a term beginning 1 February, 2012

Return petitions by 24 February 2011 to:
Secretary, AMS, Depatment of Mathematics, 1403 Circle Drive, University of Tennessee, Knoxville, TN 37996-1320 USA

Name and address (printed or typed)	
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Mathematics Calendar

February 2011

* 3–5 **From p-adic differential equations to arithmetic algebraic geometry on the occasion of Francesco Baldassarri's 60th birthday**, Università di Padova, Padova, Italy.

Description: The conference is organized on the occasion of the 60th birthday of Francesco Baldassarri, a leading figure in the fields of p-adic analysis and arithmetic algebraic geometry.

Invited speakers: The conference will include invited speakers working on related problems.

Information: `http://www.math.unipd.it/~ernesto/gruppo/p-adic2011/`.

March 2011

* 21–25 **Young Set Theory Workshop 2011**, A.-Z. Königswinter near Bonn, Germany.

Description: The goal of this workshop is to bring together postgraduates and postdocs in set theory in order to learn from senior researchers in the field, hear about the latest research, and to discuss research issues in small focused groups.

Tutorial speakers: Ali Enayat, Joel Hamkins, Slawomir Solecki, and Juris Steprans.

Postdoc speakers: Samuel Coskey, Dilip Raghavan, Assaf Rinot, Grigor Sargsyan, David Schrittesser, and Katherine Thompson.

Information: `http://www.math.uni-bonn.de/people/logic/events/young-set-theory-2011/`.

May 2011

* 5–7 **Workshop on Discrete, Tropical and Algebraic Geometry**, Goethe University, Frankfurt am Main, Germany.

Description: The goal of this workshop is to bring researchers together to discuss current developments in discrete, tropical and algebraic geometry, and also to attract younger people to these areas.

Information: `http://www.math.uni-frankfurt.de/geometry2011/`.

* 27–June 3 **12th Mathematical Theory in Fluid Mechanics**, Paseky/Kacov School, Kacov, Czech Republic.

Description: An activity of the Jindrich Necas Center for Mathematical Modeling, Charles University in Prague. The school will take place in Sport Hotel Kacov (60 kilometers south-east of Prague), in the beautiful countryside of "Posazavi".

Organizers: Eduard Feireisl (Prague, Czech Republic), Josef Malek (Prague, Czech Republic), Antonin Novotny (Toulon, France), Mirko Rokyta (Prague, Czech Republic), Michael Ruzicka (Freiburg, Germany), Vladimir Sverak (Minnesota, USA).

Main Lectures: Enrique Fernandez Cara (Universidad de Sevilla, Spain), The variable density Navier-Stokes equations and some related topics; Thierry Gallay (Université de Grenoble, France), Stability and interaction of vortices in two-dimensional viscous flows; K. R. Rajagopal (Texas A&M University, USA), Thermodynamics of rate type viscoelastic fluid models; Rolf Rannacher (Ruprecht-Karls-Universitat Heidelberg, Germany), Numerical simulation of viscous flow:

This section contains announcements of meetings and conferences of interest to some segment of the mathematical public, including ad hoc, local, or regional meetings, and meetings and symposia devoted to specialized topics, as well as announcements of regularly scheduled meetings of national or international mathematical organizations. A complete list of meetings of the Society can be found on the last page of each issue.

An announcement will be published in the *Notices* if it contains a call for papers and specifies the place, date, subject (when applicable), and the speakers; a second announcement will be published only if there are changes or necessary additional information. Once an announcement has appeared, the event will be briefly noted in every third issue until it has been held and a reference will be given in parentheses to the month, year, and page of the issue in which the complete information appeared. Asterisks (*) mark those announcements containing new or revised information.

In general, announcements of meetings and conferences carry only the date, title of meeting, place of meeting, names of speakers (or sometimes a general statement on the program), deadlines for abstracts or contributed papers, and source of further information. If there is any application deadline with respect to participation in the meeting, this fact should be noted. All communications on meetings and conferences in the mathematical sciences should be sent to the Editor of the *Notices* in care of the American Mathematical Society in Providence or electronically to `notices@ams.org` or `mathcal@ams.org`.

In order to allow participants to arrange their travel plans, organizers of meetings are urged to submit information for these listings early enough to allow them to appear in more than one issue of the *Notices* prior to the meeting in question. To achieve this, listings should be received in Providence **eight months** prior to the scheduled date of the meeting.

The complete listing of the Mathematics Calendar will be published only in the September issue of the *Notices*. The March, June/July, and December issues will include, along with new announcements, references to any previously announced meetings and conferences occurring within the twelve-month period following the month of those issues. New information about meetings and conferences that will occur later than the twelve-month period will be announced once in full and will not be repeated until the date of the conference or meeting falls within the twelve-month period.

The Mathematics Calendar, as well as Meetings and Conferences of the AMS, is now available electronically through the AMS website on the World Wide Web. To access the AMS website, use the URL: `http://www.ams.org/`.

discretization, optimization and stability; David Silvester (University of Manchester, United Kingdom), Finite elements and fast solvers for incompressible flow problems.

Information: More details and online registration can be found on: http://www.karlin.mff.cuni.cz/paseky-fluid/2011/.

* 31–June 3 **Poster of the 4th Chaotic Modeling and Simulation International Conference (CHAOS2011)**, Agios Nikolaos, Crete, Greece.

Information: Visit http://www.cmsim.org; email: secretariat@cmsim.org.

June 2011

* 5–12 **Symmetry and Perturbation Theory 2011**, Otranto, near Lecce, Italy.

Description: A new conference of the SPT series will be held June 5–12, 2011, in Otranto , near Lecce (Southern Italy). Information about previous SPT conferences is available at: http://www.sptspt.it. Talks will be by invitation only. Participation is open to anybody. We expect a poster session will be organized. The conference will be held at Hotel Daniela in Otranto, 40KM South to Lecce, http://www.grandhoteldaniela.it/. There will be a small registration fee but no financial support will be available to participants. Registration and/or inscription to the SPT mailing list is possible on the website http://www.sptspt.it/spt2011.

Organizers: Giuseppe Gaeta (Milano), Ferdinand Verhulst (Utrecht), Raffaele Vitolo (Lecce), Sebastian Walcher (Aachen). For matters concerning the conference, please use the e-mail address: spt@spt-spt.it.

Information: http://www.sptspt.it/spt2011.

* 6–8 **The International Conference on Numerical Analysis and Optimization (ICeMATH 2011)**, Universitas Ahmad Dahlan, Yogyakarta, Indonesia.

Description: On the occasion of the 50th anniversary of its founding celebration, Universitas Ahmad Dahlan (UAD) together with the collaboration of Journal KALAM have initiated The International Conference on Numerical Analysis and Optimization (ICeMATH 2011).

Information: http://icemath2011.uad.ac.id/wp/.

* 6–10 **Faces of Geometry: 3-manifolds, Groups and Singularities**, Columbia University/Barnard College, New York, New York.

Description: This conference is in honor of Walter Neumann.

Information: http://www.math.columbia.edu/~walterfest.

* 6–10 **Toric geometry and applications**, Catholic University of Leuven, Leuven (Heverlee), Belgium.

Description: The goal of this meeting is to bring together researchers from a wide range of topics in toric geometry, such as: the geometry of toric varieties, effective aspects of toric geometry, toric geometry and mirror symmetry, tropical geometry and algebraic statistics, toric and arithmetic geometry. The research lectures on some of these topics will be preceded by an introductory lecture by an expert.

Information: http://wis.kuleuven.be/algebra/ToricConference/index.php.

* 7–10 **9th International Conference on Applied Cryptography and Network Security (ACNS 2011)**, Nerja, Malaga, Spain.

Deadlines: Submission: January 21, 2011, midnight PDT. Author notification: March 22, 2011. Final version: April 10, 2011.

Information: http://www.isac.uma.es/acns2011/.

* 12–18 **International Conference on Waves and Stability in Continuous Media WASCOM XVI**, Brindisi, Italy.

Main Topics: Discontinuity and shock waves, linear and nonlinear stability in fluid dynamics and solid mechanics, small parameter problems, kinetic theories and comparison with continuum models, propagation and nonequilibrium thermodynamics, group analysis and reduction techniques, numerical and technical applications.

Information: http://wascom.matematica.unisalento.it.

* 15–17 **2011 Usenix Annual Technical Conference (USENIX ATC '11)**, Portland Marriott, Downtown Waterfront, 1401 SW Naito Parkway, Portland, Oregon.

Description: USENIX Annual Tech has always been the place to present groundbreaking research and cutting-edge practices in a wide variety of technologies and environments. Join the community of programmers, developers, and systems professionals in sharing solutions and fresh ideas on topics including Linux, clusters, security, virtualization, system administration, and more. http://www.marriott.com/pdxor.

Information: http://www.usenix.org/event/atc11.

* 17–23 **The International Conference "Painleve equations and related topics"**, Euler Institute, Saint Petersburg, Russia.

Topics: General ODE equations, Painlevé equations and their generalizations, Painlevé property, discrete Painlevé equations, properties of solutions of all mentioned above equations, reductions of PDE to Painlevé equations and their generalization, ODE systems equivalent to Painlevé equations and their generalizations, applications of the equations and the solutions.

Itinerary: The time of the conference is the time of "white nights" in Saint Petersburg.

Information: http://www.pdmi.ras.ru/EIMI/2011/PC/index.html.

* 27–July 8 **Metric Measure Spaces: Geometric and Analytic Aspects**, Université de Montréal, Pavillon André-Aisenstadt, Montréal, Québec, Canada.

Description: In recent decades, metric-measure spaces have emerged as a fruitful source of mathematical questions in their own right, and as indispensable tools for addressing classical problems in geometry, topology, dynamical systems and partial differential equations. The purpose of the 2011 summer school is to lead young scientists to the research frontier concerning the analysis and geometry of metric-measure spaces, by exposing them to a series of mini-courses featuring leading researchers who will present both the state-of-the-art and the exciting challenges which remain.

Information: http://www.dms.umontreal.ca/~sms/Metric11/index_e.php.

* 30–July 2 **IDOTA—Integral and Differential Operators and their Applications**, Department of Mathematics, University of Aveiro, Aveiro, Portugal.

Description: IDOTA is an international conference in honour of Professor Stefan Samko on the occasion of his 70th birthday, organized by the Center for Research and Development in Mathematics and Applications of the University of Aveiro.

Topics: Integral and differential operators, variable exponent analysis, harmonic analysis, function spaces, nonlinear analysis, factorization theory, boundary value problems and PDEs.

Plenary speakers: Martin Costabel (Rennes, France); Roland Duduchava (Tbilisi, Georgia); George Hsiao (Delaware, USA); Vakhtang Kokilashvili (Tbilisi, Georgia); Carlos Pérez (Seville, Spain); Lars-Erik Persson (Luleå, Sweden); Ilya Spitkovsky (Williamsburg, USA); Vladimir Rabinovich (Mexico City, Mexico); Wolfgang Wendland (Stuttgart, Germany).

Organizers: A. Almeida, A. Caetano, L. Castro (Chairman), A. P. Nolasco, H. Rafeiro, E. Rocha, M. Rodrigues, S. Santos, F.-O. Speck.

Information: http://IDOTA2011.glocos.org.

July 2011

* 21–23 **The 7th IMT-GT International Conference on Mathematics, Statistics, and its Applications (ICMSA 2011)**, Bangkok, Thailand.

Description: The main objective of this conference is to provide a forum for researchers, educators, students and industries to exchange ideas, to communicate and discuss research findings and new advances in mathematics and statistics. To explore possible avenues to foster academic and student exchange, as well as scientific activities within the region. The conference will be a venue to communicate and discuss on mathematical and statistical problems faced by the industries.

Topics: Mathematics, applications of mathematics, statistics, operations research, mathematical education, and computer sciences.

Information: `http://icmsa2011.nida.ac.th`.

* 25–29 **IMA Special Workshop: Macaulay2**, Institute for Mathematics and its Applications (IMA), University of Minnesota, Minneapolis, Minnesota.

Description: This workshop will serve as a meeting point for users and developers of Macaulay2. Its goal is to bring together researchers who would like to improve or share their skills at writing packages and those interested in developing the corresponding mathematical algorithms. There will be groups working on packages for numerical algebraic geometry, algebraic statistics, and enumerative algebraic geometry, as well as participants developing new and old packages not necessarily related to these topics. For examples of possible projects see our past workshop topics at `http://wiki.macaulay2.com/Macaulay2`.

Information: `http://www.ima.umn.edu/2010-2011/SW7.25-29.11/`.

* 25–29 **Non-Associative Algebras and Related Topics**, Department of Mathematics, University of Coimbra, Coimbra, Portugal.

Description: The Theory of Non-Associative Algebras not only constitutes a well- developed and very active area of research but is also an intercurricular one, playing a central role in Mathematics and Physics. The aim of this conference is to bring together mathematicians and physicists interested in this field, with a central goal of increasing the quality of research, promoting interaction among researchers and discussing new directions for the future. The conference will last for five days and, besides a series of contributed sessions, there will also be five plenary talks and four courses. Contributions from all researchers in the area are welcome, including young researchers. The venue will be the historic University of Coimbra, founded in 1290—the first university in Portugal and one of the oldest in Europe.

Information: `http://cmup.fc.up.pt/cmup/nonassociative2011/index.html`.

August 2011

* 10–12 **USENIX Security 2011: 20th USENIX Security Symposium**, The Westin St. Francis, 335 Powell Street, San Francisco, California.

Description: The USENIX Security Symposium brings together researchers, practitioners, system administrators, system programmers, and others interested in the latest research, policy, and practical advances in the security of computer systems. It stands apart from the many other security conferences in the high level and vendor neutrality of its content.

Information: `http://www.usenix.org/events/sec11`.

* 14–20 **Special Functions and Orthogonal Polynomials of Lie Groups and their Applications**, Czech Technical University in Prague, Děčín, Czech Republic.

Aim: The aim of the conference is to present recent results on special functions and orthogonal polynomials associated to Lie groups and their application in physics, integrable systems etc.

Topics: Multivariate orthogonal polynomials, special functions associated to Lie groups and quantum groups, Lie groups and algebras, representation theory, orthogonal polynomials and special functions in applications, discretization of orthogonal polynomials and special functions and their applications.

Itinerary: Arrival day: August 14 (Sunday). Lecture days: August 15-19. Departure day: August 20 (Saturday).

Information: `http://www.imath.kiev.ua/~maryna/conf2011.html`.

* 17–20 **2011 Shanghai International Conference on Social Sciences (SICSS 2011)**, Shanghai, China.

Description: It is our great pleasure to invite you to submit your papers or abstracts to SICSS 2011. The submission deadline is March 15, 2011. Please go to: `http://www.shanghai-ic.org/` for further details. In this conference, all full papers whose authors have presented the papers in the conference will be considered for inclusion in one of several journals, depending on the outcome of the formal review process and the decision of the journal editor.

Important Dates: Deadline for Submission of the Abstract: March 15, 2011. Date for Notification of Acceptance/Rejection: April 15, 2011. Final Manuscript Due: May 15, 2011.

Information: `http://www.shanghai-ic.org/`.

September 2011

* 19–25 **ICNAAM2011 Symposium: Semigroups of Linear Operators and Applications**, G-Hotels, Halkidiki, Greece.

Information:

Description: The 2nd Symposium on "Semigroups of Linear Operators and Applications" brings together researchers from all over the world to present new results in the theory of linear operators and its applications. Besides scheduling talks from established mathematicians, we will give opportunity to junior researchers to present their works.

Topics: Groups and semigroups of linear operators, one-parameter semigroups and linear evolution equations, Markov semigroups and applications to diffusion processes, Schrödinger and Feynman-Kac semigroups, operator sine and cosine functions, C-semigroups, integrated semigroups,diffusion processes,diffusion processes and Stochastic analysis on manifolds, selfadjoint operator theory in quantum theory, dynamic lattice systems.

Information: `http://www.fih.upt.ro/personal/dan.lemle/Lemle_Simpozion_2011.htm`.

December 2011

* 4–9 **LISA'11: 25th Large Installation System Administration Conference**, Sheraton Boston Hotel at 39 Dalton St., Boston, Massachusetts.

Description: System administrators of all specialties and levels of expertise meet at LISA to exchange ideas, sharpen old skills, learn new techniques, debate current issues, and meet colleagues, vendors, and friends. Talks, presentations, posters, WiPs, and BoFs address a wide range of administration specialties, including system, network, storage, and security administration on a variety of platforms including Linux, BSD, Solaris, and OS X.

Information: `http://www.usenix.org/events/lisa11`.

The following new announcements will not be repeated until the criteria in the next to the last paragraph at the bottom of the first page of this section are met.

July 2012

* 30–August 3 **Iwasawa 2012**, Heidelberg University, Heidelberg, Germany.

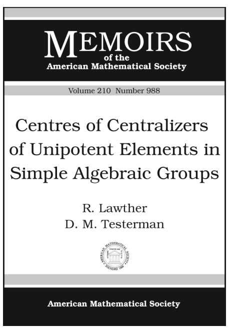
Description: The Mathematics Center Heidelberg (MATCH) will host the conference "Iwasawa 2012" which is the fifth conference in a biannual series. The conference aims to provide a platform to present and discuss the latest developments of research in the area of Iwasawa theory.

Information: `http://www.mathi.uni-heidelberg.de/~iwasawa2012/`.

New Publications Offered by the AMS

To subscribe to email notification of new AMS publications, please go to http://www.ams.org/bookstore-email.

Algebra and Algebraic Geometry



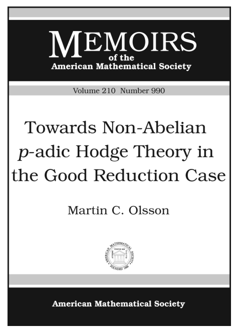
Centres of Centralizers of Unipotent Elements in Simple Algebraic Groups

R. Lawther, *Girton College, University of Cambridge, England*, and **D. M. Testerman**, *École Polytechnique Federale de Lausanne, Switzerland*

Contents: Introduction; Notation and preliminary results; Reduction of the problem; Classical groups; Exceptional groups: Nilpotent orbit representatives; Associated cocharacters; The connected centralizer; A composition series for the Lie algebra centralizer; The Lie algebra of the centre of the centralizer; Proofs of the main theorems for exceptional groups; Detailed results; Bibliography.

Memoirs of the American Mathematical Society, Volume 210, Number 988

February 2011, 188 pages, Softcover, ISBN: 978-0-8218-4769-5, LC 2010046991, 2010 *Mathematics Subject Classification:* 20G15, 20G41, **Individual member US$49.80**, List US$83, Institutional member US$66.40, Order code MEMO/210/988



Towards Non-Abelian p-adic Hodge Theory in the Good Reduction Case

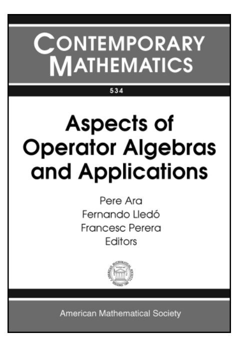
Martin C. Olsson, *University of California, Berkeley, CA*

Contents: Introduction; Review of some homotopical algebra; Review of the convergent topos; Simplicial presheaves associated to isocrystals; Simplicial presheaves associated to smooth sheaves; The comparison theorem; Proofs of 1.7–1.13; A base point free version; Tangential base points; A generalization; Appendix A. Exactification; Appendix B. Remarks on localization in model categories; Appendix C. The coherator for algebraic stacks; Appendix D. $\widetilde{B}_{\mathrm{cris}}(V)$-admissible implies crystalline; Bibliography.

Memoirs of the American Mathematical Society, Volume 210, Number 990

February 2011, 157 pages, Softcover, ISBN: 978-0-8218-5240-8, LC 2010046756, 2010 *Mathematics Subject Classification:* 14-XX; 11-XX, **Individual member US$46.20**, List US$77, Institutional member US$61.60, Order code MEMO/210/990

Analysis



Aspects of Operator Algebras and Applications

Pere Ara, *Universitat Autònoma de Barcelona, Spain*, **Fernando Lledó**, *University Carlos III de Madrid, Spain*, and **Francesc Perera**, *Universitat Autònoma de Barcelona, Spain*, Editors

This volume contains survey papers on the theory of operator algebras based on lectures given at the "Lluís Santaló" Summer School of the Real Sociedad Matemática Española, held in July 2008 at the Universidad Internacional Menéndez Pelayo, in Santander (Spain).

Topics in this volume cover current fundamental aspects of the theory of operator algebras, which have important applications such as:

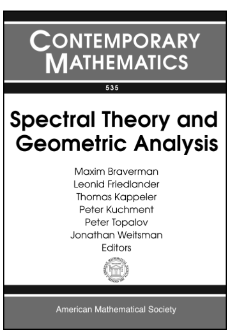
- K-Theory, the Cuntz semigroup, and classification for C^*-algebras
- Modular theory for von Neumann algebras and applications to quantum field theory
- Amenability, hyperbolic groups, and operator algebras.

The theory of operator algebras, introduced in the thirties by J. von Neumann and F. J. Murray, was developed in close relationship with fundamental aspects of functional analysis, ergodic theory, harmonic analysis, and quantum physics. More recently, this field has shown many other fruitful interrelations with several areas of mathematics and mathematical physics.

Contents: **P. Ara**, **F. Perera**, and **A. S. Toms**, K-theory for operator algebras. Classification of C*-algebras; **F. Lledó**, Modular theory by example; **D. Guido**, Modular theory for the von Neumann algebras of local quantum physics; **N. P. Brown**, The symbiosis of C*- and W*-algebras; **P. Ara**, **F. Lledó**, and **F. Perera**, Appendix: Basic definitions and results for operator algebras.

Contemporary Mathematics, Volume 534

February 2011, 168 pages, Softcover, ISBN: 978-0-8218-4905-7, LC 2010030561, 2010 *Mathematics Subject Classification:* 43A07, 46L05, 46L06, 46L10, 46L35, 46L60, 46L80, 47L30, 47L90, 81T05, **AMS members US$55.20**, List US$69, Order code CONM/534



Spectral Theory and Geometric Analysis

Maxim Braverman, *Northeastern University, Boston, MA*, **Leonid Friedlander**, *University of Arizona, Tucson, AZ*, **Thomas Kappeler**, *University of Zürich, Switzerland*, **Peter Kuchment**, *Texas A&M University, College Station, TX*, and **Peter Topalov** and **Jonathan Weitsman**, *Northeastern University, Boston, MA*, Editors

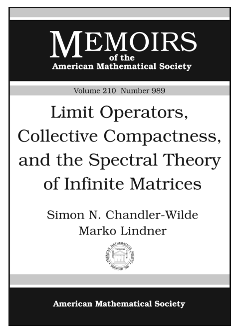
This volume contains the proceedings of the conference on Spectral Theory and Geometric Analysis, held at Northeastern University, Boston, MA, from July 29–August 2, 2009, which honored Mikhail Shubin on his 65th birthday.

The papers in this volume cover important topics in spectral theory and geometric analysis such as resolutions of smooth group actions, spectral asymptotics, solutions of the Ginzburg–Landau equation, scattering theory, Riemann surfaces of infinite genus, tropical mathematics and geometric methods in the analysis of flows in porous media, and artificial black holes.

Contents: **P. Albin** and **R. Melrose**, Resolution of smooth group actions; **E. Aulisa**, **A. Ibragimov**, and **M. Toda**, Geometric methods in the analysis of non-linear flows in porous media; **G. Eskin**, Artificial black holes; **B. Helffer** and **Y. A. Kordyukov**, Semiclassical spectral asymptotics for a two-dimensional magnetic Schrödinger operator: The case of discrete wells; **R. O. Hryniv**, **Y. V. Mykytyuk**, and **P. A. Perry**, Sobolev mapping properties of the scattering transform for the Schrödinger equation; **V. Ivrii**, Local spectral asymptotics for 2d-Schrödinger operators with strong magnetic field near the boundary; **T. Kappeler**, **P. Lohrman**, and **P. Topalov**, On normalized differentials on families of curves of infinite genus; **A. Larrain-Hubach**, **S. Rosenberg**, **S. Scott**, and **F. Torres-Ardila**, Characteristic classes and zeroth order pseudodifferential operators; **G. L. Litvinov**, Tropical mathematics, idempotent analysis, classical mechanics and geometry; **J. J. Perez**, A transversal Fredholm property for the $\bar{\partial}$-Neumann problem on G-bundles; **T. Tzaneteas** and **I. M. Sigal**, Abrikosov lattice solutions of the Ginzburg-Landau equations.

Contemporary Mathematics, Volume 535

March 2011, 213 pages, Softcover, ISBN: 978-0-8218-4948-4, LC 2010037838, 2010 *Mathematics Subject Classification:* 30F30, 35P20, 35P25, 35Q35, 35S05, 58E15, 58J40, 58J50, **AMS members US$63.20**, List US$79, Order code CONM/535



Limit Operators, Collective Compactness, and the Spectral Theory of Infinite Matrices

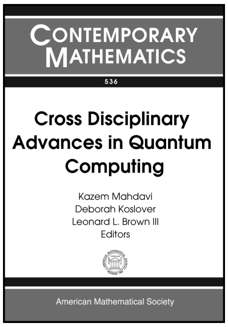
Simon N. Chandler-Wilde, *University of Reading, England*, and **Marko Lindner**, *Technical University of Chemnitz, Germany*

Contents: Introduction; The strict topology; Classes of operators; Notions of operator convergence; Key concepts and results; Operators on $\ell^p(\mathbb{Z}^N, U)$; Discrete Schrödinger operators; A class of integral operators; Some open problems; Bibliography; Index.

Memoirs of the American Mathematical Society, Volume 210, Number 989

February 2011, 111 pages, Softcover, ISBN: 978-0-8218-5243-9, LC 2010046758, 2010 *Mathematics Subject Classification:* 47A53, 47B07; 46N20, 46E40, 47B37, 47L80, **Individual member US$42**, List US$70, Institutional member US$56, Order code MEMO/210/989

Applications



Cross Disciplinary Advances in Quantum Computing

Kazem Mahdavi, **Deborah Koslover**, and **Leonard L. Brown III**, *University of Texas at Tyler, TX*, Editors

This volume contains a collection papers, written by physicists, computer scientists, and mathematicians, from the Conference on Representation Theory, Quantum Field Theory, Category Theory, and Quantum Information Theory, which was held at the University of Texas at Tyler from October 1–4, 2009.

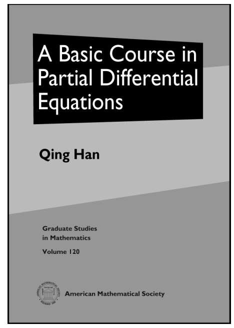
Quantum computing is a field at the interface of the physical sciences, computer sciences and mathematics. As such, advances in one field are often overlooked by practitioners in other fields. This volume brings together articles from each of these areas to make students, researchers and others interested in quantum computation aware of the most current advances. It is hoped that this work will stimulate future advances in the field.

Contents: **A. D. Ballard** and **Y.-S. Wu**, Cartan decomposition and entangling power of braiding quantum gates; **G. Chen**, **V. Ramakrishna**, and **Z. Zhang**, A unified approach to universality for three distinct types of 2-qubit quantum computing devices; **S. Bravyi**, Efficient algorithm for a quantum analogue of 2-SAT; **H. E. Brandt**, Quantum computational curvature and Jacobi fields; **L. H. Kauffman**, A quantum model for the Jones polynomial, Khovanov homology and generalized simplicial homology; **L. H. Kauffman** and **D. E. Radford**, Oriented quantum algebras and coalgebras, invariants of oriented 1-1 tangles, knots and links; **P. Benioff**, Space and time lattices in frame fields of quantum representations of real and complex numbers.

Contemporary Mathematics, Volume 536

March 2011, 152 pages, Softcover, ISBN: 978-0-8218-4975-0, LC 2010045181, 2010 *Mathematics Subject Classification:* 81P68, 81-01, 81-02, 81-06, **AMS members US$47.20**, List US$59, Order code CONM/536

Differential Equations



A Basic Course in Partial Differential Equations

Qing Han, *University of Notre Dame, IN*

This is a textbook for an introductory graduate course on partial differential equations. Han focuses on linear equations of first and second order. An important feature of his treatment is that the majority of the techniques are applicable more generally. In particular, Han emphasizes a priori estimates throughout the text, even for those equations that can be solved explicitly. Such estimates are indispensable tools for proving the existence and uniqueness of solutions to PDEs, being especially important for nonlinear equations. The estimates are also crucial to establishing properties of the solutions, such as the continuous dependence on parameters.

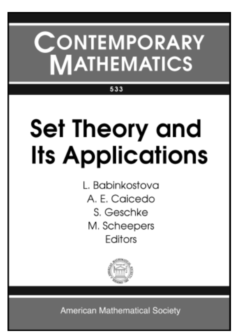
Han's book is suitable for students interested in the mathematical theory of partial differential equations, either as an overview of the subject or as an introduction leading to further study.

Contents: First-order differential equations; An overview of second-order PDEs; Laplace equations; Heat equations; Wave equations; First-order differential systems; Epilogue; Bibliography; Index.

Graduate Studies in Mathematics, Volume 120

March 2011, approximately 297 pages, Hardcover, ISBN: 978-0-8218-5255-2, LC 2010043189, 2010 *Mathematics Subject Classification:* 35-01, **AMS members US$50.40**, List US$63, Order code GSM/120

Logic and Foundations



Set Theory and Its Applications

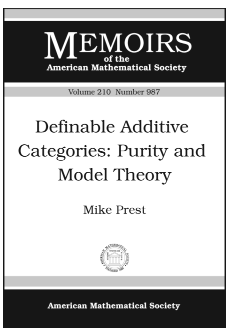
L. Babinkostova and **A. E. Caicedo**, *Boise State University, ID*, **S. Geschke**, *University of Bonn, Germany*, and **M. Scheepers**, *Boise State University, ID*, Editors

This book consists of several survey and research papers covering a wide range of topics in active areas of set theory and set theoretic topology. Some of the articles present, for the first time in print, knowledge that has been around for several years and known intimately to only a few experts. The surveys bring the reader up to date on the latest information in several areas that have been surveyed a decade or more ago. Topics covered in the volume include combinatorial and descriptive set theory, determinacy, iterated forcing, Ramsey theory, selection principles, set-theoretic topology, and universality, among others. Graduate students and researchers in logic, especially set theory, descriptive set theory, and set-theoretic topology, will find this book to be a very valuable reference.

Contents: M. Džamonja, Some positive results in the context of universal models; **G. Gruenhage**, A survey of D-spaces; **M. Hrušák**, Combinatorics of filters and ideals; **R. Ketchersid**, More structural consequences of AD; **L. D. R. Kočinac**, α_1-selection principles and games; **A. Rinot**, Jensen's diamond principle and its relatives; **J. Roitman**, Paracompactness and normality in box products: Old and new; **F. D. Tall**, Some problems and techniques in set-theoretic topology; **B. Tsaban**, Menger's and Hurewicz's problems: Solutions from "the book" and refinements; **A. E. Caicedo** and **R. Ketchersid**, A trichotomy theorem in natural models of AD^+; **S. Geschke**, The coinitialities of Efimov spaces; **E. Gruenhut** and **S. Shelah**, Uniforming n-place functions on well founded trees; **B. D. Miller**, A classical proof of the Kanovei-Zapletal canonization; **A. Rosłanowski** and **S. Shelah**, Lords of the iteration.

Contemporary Mathematics, Volume 533

February 2011, 307 pages, Softcover, ISBN: 978-0-8218-4812-8, LC 2010030559, 2010 *Mathematics Subject Classification:* 03C55, 03E15, 03E17, 03E35, 03E60, 46L05, 54A20, 54A25, 54D20, 91A44, **AMS members US$84**, List US$105, Order code CONM/533



Definable Additive Categories: Purity and Model Theory

Mike Prest, *University of Manchester, England*

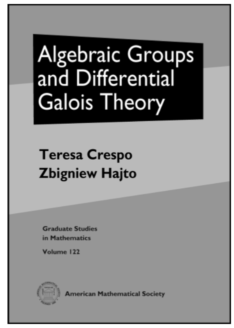
This item will also be of interest to those working in algebra and algebraic geometry.

Contents: Introduction; Preadditive and additive categories; Preadditive categories and their ind-completions; The free abelian category of a preadditive category; Purity; Locally coherent categories; Localisation; Serre subcategories of the functor category; Conjugate and dual categories; Definable subcategories; Exactly definable categories; Recovering the definable structure; Functors between definable categories; Spectra of definable categories; Definable functors and spectra; Triangulated categories; Some open questions; Model theory in finitely accessible categories; pp-Elimination of quantifiers; Ultraproducts; Pure-injectives and elementary equivalence; Imaginaries and finitely presented functors; Elementary duality; Hulls of types and irreducible types; Interpretation functors; Stability; Ranks; Bibliography.

Memoirs of the American Mathematical Society, Volume 210, Number 987

February 2011, 109 pages, Softcover, ISBN: 978-0-8218-4767-1, LC 2010046770, 2010 *Mathematics Subject Classification:* 03C60; 03C52, 16D90, 18C35, 18E05, 18E35, 18E10, **Individual member US$42**, List US$70, Institutional member US$56, Order code MEMO/210/987

Number Theory



Algebraic Groups and Differential Galois Theory

Teresa Crespo, *Universitat de Barcelona, Spain*, and **Zbigniew Hajto**, *Jagiellonian University, Kraków, Poland*

Differential Galois theory has seen intense research activity during the last decades in several directions: elaboration of more general theories, computational aspects, model theoretic approaches, applications to classical and quantum mechanics as well as to other mathematical areas such as number theory.

This book intends to introduce the reader to this subject by presenting Picard-Vessiot theory, i.e. Galois theory of linear differential equations, in a self-contained way. The needed prerequisites from algebraic geometry and algebraic groups are contained in the first two parts of the book. The third part includes Picard-Vessiot extensions, the fundamental theorem of Picard-Vessiot theory, solvability by quadratures, Fuchsian equations, monodromy group and Kovacic's algorithm. Over one hundred exercises will help to assimilate the concepts and to introduce the reader to some topics beyond the scope of this book.

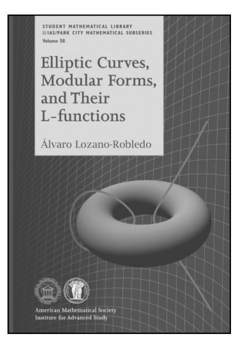
This book is suitable for a graduate course in differential Galois theory. The last chapter contains several suggestions for further reading encouraging the reader to enter more deeply into different topics of differential Galois theory or related fields.

This item will also be of interest to those working in algebra and algebraic geometry.

Contents: *Algebraic geometry:* Affine and projective varieties; Algebraic varieties; *Algebraic groups:* Basic notions; Lie algebras and algebraic groups; *Differential Galois theory:* Picard-Vessiot extensions; The Galois correspondence; Differential equations over $\mathbb{C}(z)$; Suggestions for further reading; Bibliography; Index.

Graduate Studies in Mathematics, Volume 122

April 2011, approximately 232 pages, Hardcover, ISBN: 978-0-8218-5318-4, LC 2010044378, 2010 *Mathematics Subject Classification:* 12H05, 13B05, 14A10, 17B45, 20G15, 34M35, 68W30, **AMS members US$42.40**, List US$53, Order code GSM/122



Elliptic Curves, Modular Forms, and Their L-functions

Álvaro Lozano-Robledo, *University of Connecticut, Storrs, CT*

Many problems in number theory have simple statements, but their solutions require a deep understanding of algebra, algebraic geometry, complex analysis, group representations, or a combination of all four. The original simply stated problem can be obscured in the depth of the theory developed to understand it. This book is an introduction to some of these problems, and an overview of the theories used nowadays to attack them, presented so that the number theory is always at the forefront of the discussion.

Lozano-Robledo gives an introductory survey of elliptic curves, modular forms, and L-functions. His main goal is to provide the reader with the big picture of the surprising connections among these three families of mathematical objects and their meaning for number theory. As a case in point, Lozano-Robledo explains the modularity theorem and its famous consequence, Fermat's Last Theorem. He also discusses the Birch and Swinnerton-Dyer Conjecture and other modern conjectures. The book begins with some motivating problems and includes numerous concrete examples throughout the text, often involving actual numbers, such as 3, 4, 5, $\frac{3344161}{747348}$, and $\frac{2244035177043369699245575130906674863160948472041}{8912332268928859588025535178967163570016480830}$.

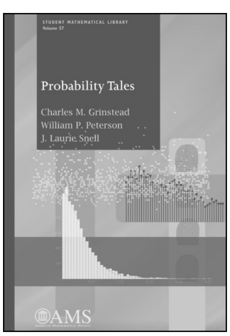
The theories of elliptic curves, modular forms, and L-functions are too vast to be covered in a single volume, and their proofs are outside the scope of the undergraduate curriculum. However, the primary objects of study, the statements of the main theorems, and their corollaries are within the grasp of advanced undergraduates. This book concentrates on motivating the definitions, explaining the statements of the theorems and conjectures, making connections, and providing lots of examples, rather than dwelling on the hard proofs. The book succeeds if, after reading the text, students feel compelled to study elliptic curves and modular forms in all their glory.

Contents: Introduction; Elliptic curves; Modular curves; Modular forms; L-functions; PARI/GP and Sage; Complex analysis; Projective space; The p-adic numbers; Parametrization of torsion structures; Bibliography; Index.

Student Mathematical Library, Volume 58

March 2011, 193 pages, Softcover, ISBN: 978-0-8218-5242-2, LC 2010038952, 2010 *Mathematics Subject Classification:* 14H52, 11G05; 11F03, 11G40, **AMS members US$29.60**, List US$37, Order code STML/58

Probability and Statistics



Probability Tales

Charles M. Grinstead, *Swarthmore College, PA*, **William P. Peterson**, *Middlebury College, VT*, and **J. Laurie Snell**, *Dartmouth College, Hanover, NH*

This book explores four real-world topics through the lens of probability theory. It can be used to supplement a standard text in probability or statistics. Most elementary textbooks present the basic theory and then illustrate the ideas with some neatly packaged examples. Here the authors assume that the reader has seen, or is learning, the basic theory from another book and concentrate in some depth on the following topics: streaks, the stock market, lotteries, and fingerprints. This extended format allows the authors to present multiple approaches to problems and to pursue promising side discussions in ways that would not be possible in a book constrained to cover a fixed set of topics.

To keep the main narrative accessible, the authors have placed the more technical mathematical details in appendices. The appendices can be understood by someone who has taken one or two semesters of calculus.

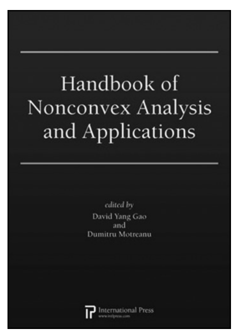
Contents: Streaks; Modeling the stock market; Lotteries; Fingerprints; Answers to John Haigh's lottery questions; Bibliography; Index.

Student Mathematical Library, Volume 57

April 2011, approximately 246 pages, Softcover, ISBN: 978-0-8218-5261-3, LC 2010038517, 2010 *Mathematics Subject Classification:* 60-01, 62-01, **AMS members US$33.60**, List US$42, Order code STML/57

New AMS-Distributed Publications

Analysis



Handbook of Nonconvex Analysis and Applications

David Yang Gao, *University of Ballarat, Victoria, Australia*, and **Dumitru Motreanu**, *University of Perpignan, France*, Editors

Nonconvex analysis is a rapidly developing, multi-disciplinary field of research, encompassing theoretical analysis in mathematical modelling of natural systems, bifurcation and chaos in dynamical systems, finite deformation theory, nonlinear partial differential equations, global optimization, calculus of variation, numerical methods, and scientific computations. The field of nonconvex analysis has undergone considerable development in a remarkably short time—with extensive applications to theoretical physics, material science, modern mechanics, complex systems, and scientific computations.

This volume consists of thirteen chapters written by notable experts in the field, addressing essential recent developments in nonconvex analysis and its applications and keeping a balance between major areas of theory, methods, and applications. Each chapter provides an illuminating exposition of state-of-the-art approaches to a specific topic, with discussions of the central contributions, and pointers to some basic references. A variety of topics regarding nonconvex analysis and its applications are discussed: nonconvex variational principles; comparison principles; nonlinear eigenvalue problems; critical point theory; boundary value problems; topological methods, including Morse theory; nonlinear elliptic equations; evolution problems; difference equations; inequality problems; geometric properties of functions and spaces; and applications in mechanics.

This handbook will serve as a much-needed reference work for the dynamic and ever-growing field of nonconvex analysis and its applications.

A publication of International Press. Distributed worldwide by the American Mathematical Society.

Contents: G. Bonanno and **P. Candito**, Nonlinear difference equations through variational methods; **S. Carl** and **D. Motreanu**, Sub-supersolution method for multi-valued elliptic and evolution problems; **G. Colombo** and **L. Thibault**, Prox-regular sets and applications; **L. Gasinski** and **N. S. Papageorgiou**, Multiplicity of solutions for nonlinear elliptic equations with combined nonlinearities; **A. Kristály** and **N. S. Papageorgiou**, Study of some semilinear elliptic problems on $\mathbb{R}^n$ via variational methods; **V. K. Le** and **K. Schmitt**, Equations and inequalities in Orlicz–Sobolev spaces: Selected topics; **R. Livrea** and **S. A. Marano**, Non-smooth critical point theory; **S. Migórski**, Evolution hemivariational inequalities with applications; **K. Perera**, Morse theory and applications to variational problems; **E. Rentsen**, Quasiconvex optimization and its applications; **B. Ricceri**, Nonlinear eigenvalue problems; **A. Szulkin** and **T. Weth**, The method of Nehari manifold; **Z. Zhang**, Solutions for elliptic problems with precise sign information.

International Press

November 2010, 680 pages, Hardcover, ISBN: 978-1-57146-200-8, 2010 *Mathematics Subject Classification:* 35Jxx, 35Kxx, 47Jxx, 47Nxx, 49Jxx, 49Mxx, 58Exx, 58Jxx, 90Cxx, **AMS members US$71.20**, List US$89, Order code INPR/93

Classified Advertisements

Positions available, items for sale, services available, and more

FLORIDA

FLORIDA INTERNATIONAL UNIVERSITY
Department of Mathematics and Statistics

Florida International University (FIU) is a multi-campus public research university located in Miami, a vibrant, international city. FIU offers more than 180 baccalaureate, master's, professional, and doctoral degree programs to over 42,000 students. As one of South Florida's anchor institutions, FIU is worlds ahead in its local and global engagement and is committed to finding solutions to the most challenging problems of our times.

The Department of Mathematics and Statistics at FIU invites applications for three positions beginning Fall 2011:

1. Applied Mathematics: Open Rank. 2. Applied Mathematics: Assistant Professor. 3. Mathematics Education: Instructor/Assistant/Associate Professor.

Qualifications for the positions at assistant professor level or higher include Ph.D. in mathematics and outstanding record in research and teaching. Established record of funded research and successful Ph.D. student supervisions is a plus.

Duties for the third position include curriculum development and teaching innovation `http://scholar.google.com/scholar?q=curriculum+development+and+teaching++innovation&hl=en&as_sdt=0&as_vis=1&oi=scholart` in a variety of undergraduate courses including college algebra. Senior candidates with a Ph.D. and experience in math education research are also welcome to apply. A master's degree or higher in mathematics is required for the position at instructor level.

To apply, send an application letter, curriculum vita and at least three letters of reference to:

Recruitment Committee
Department of Mathematics and Statistics
Florida International University
MMC
11200 S.W. 8th Street
Miami, Florida 33199

Review of applications will start on February 01, 2011, and will continue until the positions are filled. For more information, please visit `http://www.fiu.edu` and `http://casgroup.fiu.edu/mathstatistics`.

FIU is a member of the State University System of Florida and is an Equal Opportunity, Equal Access Affirmative Action Employer.

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HAWAII

THE UNIVERSITY OF HAWAII
Department of Mathematics

The University of Hawaii, College of Natural Sciences, Department of Mathematics invites applications for two permanent, full-time, tenure-track positions as assistant professors of mathematics, to begin August 1, 2011, pending authorization to fill and availability of funds.

Duties and Responsibilities: teach graduate and undergraduate courses, direct graduate research, establish an international reputation for excellent research.

A complete application will include a curriculum vitae; a list publications; copies of transcripts; a statement of research done and planned; and four confidential letters of recommendation, at least one of which must address teaching. Upon hire, official transcripts will be required.

Electronic application at: `http://www.mathjobs.org` is preferred. Applications and supporting documents may also be mailed to:

Hiring Committee
Department of Mathematics
University of Hawaii

Suggested uses for classified advertising are positions available, books or lecture notes for sale, books being sought, exchange or rental of houses, and typing services.

The 2010 rate is $3.25 per word. No discounts for multiple ads or the same ad in consecutive issues. For an additional $10 charge, announcements can be placed anonymously. Correspondence will be forwarded.

Advertisements in the "Positions Available" classified section will be set with a minimum one-line headline, consisting of the institution name above body copy, unless additional headline copy is specified by the advertiser. Headlines will be centered in boldface at no extra charge. Ads will appear in the language in which they are submitted.

There are no member discounts for classified ads. Dictation over the telephone will not be accepted for classified ads.

Upcoming deadlines for classified advertising are as follows: March 2011 issue–December 28, 2010; April 2011 issue–January 30, 2011; May 2011 issue–February 28 2011; June/July 2011 issue–April 28, 2011; August 2011 issue–May 27, 2010; September 2011 issue–June 28, 2011.

U.S. laws prohibit discrimination in employment on the basis of color, age, sex, race, religion, or national origin. "Positions Available" advertisements from institutions outside the U.S. cannot be published unless they are accompanied by a statement that the institution does not discriminate on these grounds whether or not it is subject to U.S. laws. Details and specific wording may be found on page 1373 (vol. 44).

Situations wanted advertisements from involuntarily unemployed mathematicians are accepted under certain conditions for free publication. Call toll-free 800-321-4AMS (321-4267) in the U.S. and Canada or 401-455-4084 worldwide for further information.

Submission: Promotions Department, AMS, P.O. Box 6248, Providence, Rhode Island 02940; or via fax: 401-331-3842; or send email to `classads@ams.org`. AMS location for express delivery packages is 201 Charles Street, Providence, Rhode Island 20904. Advertisers will be billed upon publication.

2565 The Mall
Honolulu, HI 96822

Review of applications will begin Nov. 7, 2010. Applications will be accepted until the positions are filled.

Minimum Qualifications: Ph.D. from a college or university of recognized standing in mathematics or equivalent, evidence of research and/or scholarly achievement, evidence of teaching excellence, poise and good address for meeting and conferring with others. Desirable qualifications: postdoctoral experience, ability to enhance the department's reputation for excellence in both research and the teaching of graduate and undergraduate courses, external grant support for research.

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KANSAS

KANSAS STATE UNIVERSITY
Department of Mathematics

Applications are invited for a Visiting Assistant Professorship commencing August 7, 2011. This will be an annual appointment with the possibility of two subsequent one-year appointments depending on performance, funding, and need of services. A Ph.D. in mathematics or a Ph.D. dissertation accepted with only formalities to be completed is required by the time of appointment. The department seeks candidates whose research interests are in geometry. The successful candidate should have strong research credentials as well as strong accomplishments or promise in teaching, and should value working with colleagues and students from diverse backgrounds. Applicants must submit the following: A letter of application, curriculum vita, outline of teaching philosophy, a statement of research objectives, and four letters of reference, at least one of which addresses the applicant's teaching ability or potential. All application materials must be submitted electronically via `http://www.mathjobs.org`. Screening of applications begins December 17, 2010, and continues until position is closed. Kansas State University is an Equal Opportunity Employer and actively seeks diversity among its employees and encourages applications from women and minorities. A background check is required.

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VIRGINIA

UNIVERSITY OF VIRGINIA
Department of Mathematics

The Department of Mathematics at the University of Virginia invites applications for a Whyburn Instructorship beginning August 25, 2011. This position carries a three-year appointment. Preference will be given to candidates who have received their Ph.D. within the last three years. Candidates must have a Ph.D. by the date of hire, an outstanding research record, and demonstrated teaching success.

Preference will be given to researchers working in an area of ANALYSIS covered by the department. In the cover letter, it will be very helpful to indicate which members of our department are closest to YOUR research interests. See `http://artsandsciences.virginia.edu/mathematics/research/researchguide/index.html`.

To apply, please submit the following required documents electronically through `www.MathJobs.org`: A cover letter, an AMS Standard Cover Sheet, a curriculum vitae, a publication list, a description of research, and a statement about teaching interests and experience. The applicant must also have four letters of recommendation submitted, of which one letter must support the applicant's effectiveness as a teacher.

In addition, all candidates are required to complete the Candidate Profile through the University of Virginia's employment system, which is `Jobs@UVA` (`http://jobs.virginia.edu`); posting number 0606116. Your application process will not be complete until all required documents are available on `MathJobs`, and you receive a confirmation number for your Candidate Profile from `Jobs@UVA`.

Priority consideration will be given to applications received by November 15, 2010; however, the position remains open to applications until filled.

Additional information about this position and our department is available on our website: `http://artsandsciences.virginia.edu/mathematics/`.

Women and members of underrepresented groups are encouraged to apply. The University of Virginia is an Affirmative Action/Equal Opportunity Employer and is strongly committed to building diversity within its community.

For more information about the position or institution/company: `http://artsandsciences.virginia.edu/mathematics/aboutus/employment/index.html`.

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CHINA

EAST CHINA NORMAL UNIVERSITY
Center for Partial Differential Equations
Applicants for Postdoctoral Positions

The successful candidates are expected to be young researchers with Ph.D. degrees in mathematics or related areas, with a strong research record in at least one of the following areas: analysis/computation/modeling. More information about the positions as well as the introduction of the Center are available at: `http://postdoctor.ecnu.edu.cn/details.aspx?id=31`.

000111

TURKEY

KOÇ UNIVERSITY
Department of Mathematics
Istanbul, Turkey
Faculty Positions

Applications and nominations are invited for faculty positions in all areas of mathematics. Appointment will be effective September 1, 2011. Koç University, founded in 1993, is a private Turkish institution of higher education, committed to the pursuit of excellence in both research and teaching. Its aim is to provide a world-class education to a highly select group of students in a liberal arts setting. The medium of instruction is English. Currently, Koç University offers BA/BS degrees in the College of Sciences, Administrative Sciences and Economics, Social Sciences and Humanities, and Engineering, as well as the School of Medicine, Law School, and the School of Health Sciences. Koç University also offers MA/MS and Ph.D. degree programs in many areas including mathematics. Successful candidates are expected to have a strong research and publication record. Information about the faculty and their research activities is available in `http://math.ku.edu.tr`.

The compensation package is competitive. All information on candidates will be kept confidential.

Letter, vitae, and names of references should be sent to:

Prof. Dr. Alphan Sennaroglu
Dean, College of Sciences
Koç University, Rumelifeneri Yolu
34450 Sarýyer-Istanbul, Turkey;
email: `smartens@ku.edu.tr`;
phone: (90-212) 338-1401;
fax: (90-212) 338-1188

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Meetings & Conferences of the AMS

IMPORTANT INFORMATION REGARDING MEETINGS PROGRAMS: AMS Sectional Meeting programs do not appear in the print version of the *Notices*. However, comprehensive and continually updated meeting and program information with links to the abstract for each talk can be found on the AMS website. See `http://www.ams.org/meetings/`. Final programs for Sectional Meetings will be archived on the AMS website accessible from the stated URL and in an electronic issue of the *Notices* as noted below for each meeting.

Statesboro, Georgia

Georgia Southern University

March 12–13, 2011

Saturday–Sunday

Meeting #1068

Southeastern Section
Associate secretary: Matthew Miller
Announcement issue of *Notices*: January 2011
Program first available on AMS website: January 27, 2011
Program issue of electronic *Notices*: March 2011
Issue of Abstracts: Volume 32, Issue 2

Deadlines

For organizers: Expired
For consideration of contributed papers in Special Sessions: Expired
For abstracts: January 20, 2011

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Jason A. Behrstock, Lehman College (CUNY), *The quasi-isometric classification of 3-manifold groups.*

Gordana Matic, University of Georgia, *Title to be announced.*

Jeremy T. Tyson, University of Illinois at Urbana-Champaign, *Sobolev mappings into metric spaces.*

Brett D. Wick, Georgia Institute of Technology, *The corona problem.*

Special Sessions

Advances in Biomedical Mathematics (Code: SS 4A), **Yangbo Ye**, University of Iowa, and **Jiehua Zhu**, Georgia Southern University.

Advances in Optimization (In honor of Florian Potra's 60th birthday) (Code: SS 20A), **Goran Lesaja**, Georgia Southern University.

Algebraic Geometry (Code: SS 18A), **Jing Zhang**, State University of New York at Albany, **Roya Beheshti Zavareh**, Washington University in St Louis, and **Qi Zhang**, University of Missouri at Columbia.

Algebraic and Geometric Combinatorics (Code: SS 13A), **Drew Armstrong**, University of Miami, and **Benjamin Braun**, University of Kentucky.

Applied Combinatorics (Code: SS 2A), **Hua Wang**, Georgia Southern University, **Miklos Bona**, University of Florida, and **Laszlo Szekely**, University of South Carolina.

Categorical Topology (Code: SS 9A), **Frederic Mynard**, Georgia Southern University, and **Gavin Seal**, EPFL, Lausanne.

Control Systems and Signal Processing (Code: SS 14A), **Zhiqiang Gao**, Cleveland State University, **Frank Goforth**, Georgia Southern University, **Thomas Yang**, Embry-Riddle Aeronautical University, and **Yan Wu**, Georgia Southern University.

Dynamic Equations on Time Scales with Applications (Code: SS 17A), **Billur Kaymakcalan**, Georgia Southern University, and **Bonita Lawrence**, Marshall University.

Fractals and Tilings (Code: SS 3A), **Ka-Sing Lau**, The Chinese University of Hong Kong, **Sze-Man Ngai**, Georgia Southern University, and **Yang Wang**, Michigan State University.

Geometric Group Theory (Code: SS 7A), **Xiangdong Xie**, Georgia Southern University, **Jason A. Behrstock**, Lehman College, CUNY, and **Denis Osin**, Vanderbilt University.

Geometric Mapping Theory in Euclidean and Non-Euclidean Spaces (Code: SS 11A), **Jeremy Tyson**, University of Illinois at Urbana-Champaign, **David A. Herron**, University of Cincinnati, and **Xiagdong Xie**, Georgia Southern University.

Harmonic Analysis and Applications (Code: SS 5A), **Dmitriy Bilyk**, University of South Carolina, **Laura De Carli**, Florida International University, **Alex Stokolos**, Georgia Southern University, and **Brett Wick**, Georgia Institute of Technology.

Harmonic Analysis and Partial Differential Equations (Code: SS 1A), **Paul A. Hagelstein**, Baylor University, **Alexander Stokolos**, Georgia Southern University, **Xiaoyi Zhang**, IAS Princeton and University of Iowa, and **Shijun Zheng**, Georgia Southern University.

Homological Methods in Commutative Algebra (Code: SS 6A), **Alina C. Iacob**, Georgia Southern University, and **Adela N. Vraciu**, University of South Carolina.

Low Dimensional Topology and Contact and Symplectic Geometry (Code: SS 19A), **Gordana Matic**, University of Georgia, and **John Etnyre**, Georgia Institute of Technology.

Matrix Theory and Numerical Linear Algebra (Code: SS 8A), **Richard S. Varga**, Kent State University, and **Xiezhang Li**, Georgia Southern University.

Nonlinear Analysis of PDEs (Code: SS 15A), **Ronghua Pan**, Georgia Institute of Technology, **Tristan Roy**, Institiute for Advanced Study, and **Shijun Zheng**, Georgia Southern University.

Set-theoretic Topology (Code: SS 16A), **Frederic Mynard**, Georgia Southern University, and **Peter Nyikos**, University of South Carolina.

Sparse Data Representations and Applications (Code: SS 10A), **Alexander Petukhov** and **Alex Stokolos**, Georgia Southern University, **Ahmed Zayed**, DePaul University, and **Inna Kozlov**, Holon Institute of Technology, Department of Computer Science.

Symplectic and Poisson Geometry (Code: SS 12A), **Yi Lin**, Georgia Southern University, **Alvaro Pelayo**, Washington University, St. Louis, and **Francois Ziegler**, Georgia Southern University.

Iowa City, Iowa

University of Iowa

March 18–20, 2011

Friday–Sunday

Meeting #1069

Central Section
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: January 2011
Program first available on AMS website: February 5, 2011
Program issue of electronic *Notices*: March 2011
Issue of Abstracts: Volume 32, Issue 2

Deadlines

For organizers: Expired
For consideration of contributed papers in Special Sessions: Expired
For abstracts: January 25, 2011

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Mihai Ciucu, Indiana University, *Title to be announced.*

David Damanik, Rice University, *Title to be announced.*

Chiu-Chu Liu, Columbia University, *Title to be announced.*

Kevin B. Ford, University of Illinois Urbana-Champaign, *Prime chains, arithmetic functions and branching random walks.*

Special Sessions

Algebraic Combinatorics (Code: SS 19A), **Mihai Ciucu**, Indiana University.

Algebraic K-Theory and Homotopy Theory (Code: SS 8A), **Teena Gerhardt**, Michigan State University, and **Daniel Ramras**, New Mexico State University.

Analytic and Algebraic Number Theory (Code: SS 5A), **Ling Long**, Iowa State University, and **Yangbo Ye**, University of Iowa.

Commutative Ring Theory (Code: SS 6A), **Daniel D. Anderson**, University of Iowa, and **David F. Anderson**, University of Tennessee Knoxville.

Computational Medical Imaging (Code: SS 21A), **Jun Ni** and **Lihe Wang**, University of Iowa.

Geometric Commutative Algebra and Applications (Code: SS 7A), **David Anderson**, University of Washington, and **Julianna Tymoczko**, University of Iowa.

Global and P-adic Representation Theory (Code: SS 3A), **Muthukrishnan Krishnamurthy**, **Philip Kutzco**, and **Yangbo Ye**, University of Iowa.

Graph Theory (Code: SS 17A), **Maria Axenovich**, **Lale Ozkahya**, and **Michael Young**, Iowa State University.

History of Mathematics (Code: SS 13A), **Colin McKinney**, Bradley University.

Modelling, Analysis and Simulation in Contact Mechanics (Code: SS 1A), **Weimin Han**, University of Iowa, and **Mircea Sofonea**, University of Perpignan.

Nonlinear Partial Differential Equations (Code: SS 20A), **Hongjie Dong**, Brown University, and **Dong Li**, **Lihe Wang**, and **Xiaoyi Zhang**, University of Iowa.

Numerical Analysis and Scientific Computing (Code: SS 14A), **Kendall E. Atkinson**, **Bruce P. Ayati**, **Weimin Han**, **Laurent O. Jay**, **Suely Oliveira**, and **David Stewart**, University of Iowa.

Recent Advances in Hyperbolic and Kinetic Problems (Code: SS 15A), **Tong Li**, University of Iowa, and **Hailing Liu**, Iowa State University.

Recent Developments in Nonlinear Evolution Equations (Code: SS 4A), **Yinbin Deng**, Central China Normal University, **Yong Yu** and **Yi Li**, University of Iowa, and **Shuangjie Peng**, Central China Normal University.

Recent Developments in Schubert Calculus (Code: SS 9A), **Leonardo Mihalcea**, Baylor University.

Representations of Algebras (Code: SS 2A), **Frauke Bleher**, University of Iowa, and **Calin Chindris**, University of Missouri.

Spectral Theory (Code: SS 12A), **David Damanik**, Rice University, and **Christian Remling**, University of Oklahoma.

Stochastic Processes with Applications to Mathematical Finance (Code: SS 18A), **Igor Cialenco**, Illinois Institute of Technology, and **José E. Figueroa-López**, Purdue University.

Thin Position (Code: SS 11A), **Jesse Johnson**, Oklahoma State University, and **Maggie Tomova**, University of Iowa.

Topological Problems in Molecular Biology (Code: SS 16A), **Isabel K. Darcy**, University of Iowa, **Stephen D. Levene**, University of Texas at Dallas, and **Jonathan Simon**, University of Iowa.

Universal Algebra and Order (Code: SS 10A), **John Snow**, Concordia University, **Jeremy Alm**, Illinois College, **Clifford Bergman**, Iowa State University, and **Kristi Meyer**, Wisconsin Lutheran College.

Worcester, Massachusetts

College of the Holy Cross

April 9–10, 2011

Saturday–Sunday

Meeting #1070

Eastern Section
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: February 2011
Program first available on AMS website: March 10, 2011
Program issue of electronic *Notices*: April 2011
Issue of Abstracts: Volume 32, Issue 3

Deadlines

For organizers: Expired
For consideration of contributed papers in Special Sessions: Expired
For abstracts: February 15, 2011

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Vitaly Bergelson, Ohio State University, *Title to be announced.*

Kenneth M. Golden, University of Utah, *Title to be announced.*

Walter D. Neumann, Columbia University, *What does a complex surface really look like?*

Natasa Sesum, University of Pennsylvania, *Title to be announced.*

Special Sessions

Celestial Mechanics (Code: SS 16A), **Glen R. Hall**, Boston University, and **Gareth E. Roberts**, College of the Holy Cross.

Combinatorial Representation Theory (Code: SS 14A), **Cristina Ballantine**, College of the Holy Cross, and **Rosa Orellana**, Dartmouth College.

Combinatorics of Coxeter Groups (Code: SS 19A), **Dana C. Ernst**, Plymouth State University, and **Matthew Macauley**, Clemson University.

Complex Analysis and Banach Algebras (Code: SS 1A), **John T. Anderson**, College of the Holy Cross, and **Alexander J. Izzo**, Bowling Green State University.

Computability Theory and Applications (Code: SS 18A), **Brooke Andersen**, Assumption College.

Dynamics of Rational Systems of Difference Equations with Applications (Code: SS 3A), **M. R. S. Kulenovic** and **O. Merino**, University of Rhode Island.

Geometric and Topological Problems in Curvature (Code: SS 17A), **Megan Kerr** and **Stanley Chang**, Wellesley College.

Geometry and Applications of 3-Manifolds (Code: SS 13A), **Abhijit Champanerkar** and **Ilya Kofman**, College of Staten Island, CUNY, and **Walter Neumann**, Barnard College, Columbia University.

Geometry of Nilpotent Lie Groups (Code: SS 11A), **Rachelle DeCoste**, Wheaton College, **Lisa DeMeyer**, Central Michigan University, and **Maura Mast**, University of Massachusetts-Boston.

History and Philosophy of Mathematics (Code: SS 5A), **James J. Tattersall**, Providence College, and **V. Frederick Rickey**, United States Military Academy.

Interactions between Dynamical Systems, Number Theory, and Combinatorics (Code: SS 9A), **Vitaly Bergelson**, The Ohio State University, and **Dmitry Kleinbock**, Brandeis University.

Mathematical and Computational Advances in Interfacial Fluid Dynamics (Code: SS 15A), **Burt S. Tilley**, Worcester Polytechnic Institute, and **Lou Kondic**, New Jersey Institute of Technology.

Mathematics and Climate (Code: SS 8A), **Kenneth M. Golden**, University of Utah, **Catherine Roberts**, College of the Holy Cross, and **MaryLou Zeeman**, Bowdoin College.

Modular Forms, Elliptic Curves, L-functions, and Number Theory (Code: SS 20A), **Sharon Frechette** and **Keith Ouellette**, College of the Holy Cross.

New Trends in College and University Faculty Engagement in K–12 Education (Code: SS 21A), **Jennifer Beineke**, Western New England College, and **Corri Taylor**, Wellesley College.

Number Theory, Arithmetic Topology, and Arithmetic Dynamics (Code: SS 10A), **Michael Bush**, Smith College, and **Farshid Hajir**, University of Massachusetts, Amherst.

Physically Inspired Higher Homotopy Algebra (Code: SS 4A), **Thomas J. Lada**, North Carolina State University, and **Jim Stasheff**, University of North Carolina, Chapel Hill.

Random Processes (Code: SS 7A), **Andrew Ledoan**, Boston College, and **Steven J. Miller** and **Mihai Stoiciu**, Williams College.

The Algebraic Geometry and Topology of Hyperplane Arrangements (Code: SS 6A), **Graham Denham**, University of Western Ontario, and **Alexander I. Suciu**, Northeastern University.

Topics in Partial Differential Equations and Geometric Analysis (Code: SS 12A), **Maria-Cristina Caputo**, University of Arkansas, and **Natasa Sesum**, Rutgers University.

Topological, Geometric, and Quantum Invariants of 3-manifolds (Code: SS 2A), **David Damiano**, College of the Holy Cross, **Scott Taylor**, Colby College, and **Helen Wong**, Carleton College.

Undergraduate Research (Code: SS 22A), **David Damiano**, College of the Holy Cross, **Giuliana Davidoff**, Mount Holyoke College, **Steve Levandosky**, College of the Holy Cross, and **Steven J. Miller**, Williams College.

Accommodations

Participants should make their own arrangements directly with the hotel of their choice as early as possible. Special discounted rates have been negotiated with the first three hotels listed below. Rates quoted do not include the combined Massachusetts state sales and occupancy tax of 14.45%. Participants must state that they are with the **American Mathematical Society (AMS) Meeting at the College of the Holy Cross** to receive the discounted rate. The AMS is not responsible for rate changes or for the quality of the accommodations. **Hotels have varying cancellation and early checkout penalties; be sure to ask for details when making your reservation**.

Hilton Garden Inn, 35 Major Taylor Blvd., Worcester, MA 01608, 508-753-5700. (`www.hiltongardeninn.com`) Rates are US$104 single/double occupancy room. Amenities offered include parking, complimentary wired and wireless Internet, complimentary remote printing to the business center, in-room refrigerator and microwave oven, an indoor pool, fitness room and a restaurant on-site; this hotel is located approximately 3–4 miles from the campus. Cancellation and early check-out policies vary; be sure to check when you make your reservation. **The deadline for reservations at this rate is March 9, 2011**. *NOTE: Major Taylor Blvd. was Worcester Center Blvd. and may not be in all GPS systems or found in Mapquest at this time, it is advised to use 35 Worcester Center Blvd. when mapping.*

Courtyard by Marriott, 72 Grove Street, Worcester, MA 01605, 508-363-0300. (`www.marriott.com`) Rates are US$79 for a single and US$89 for a double occupancy room. Amenities include free parking and complimentary wireless Internet, an indoor pool, exercise room, fireplace lounge, and an on-site restaurant serving breakfast. This hotel will also offer a very limited shuttle service to the campus for the meeting—this service is subject to change or cancellation, please ask upon check-in; approximately 3.8 miles from the campus. Cancellation and early check-out policies vary; be sure to check on these policies when you make your reservation. **The deadline for reservations at this rate is March 8, 2011.**

Holiday Inn Express Hotel & Suites, 10–12 Johnson Street, Auburn, MA 01501, 508-832-2500. (`www.ichotelsgroup.com`) Rates are approximately US$99 per night for single/double occupancy room for the dates of this meeting. Amenities include complimentary parking, continental breakfast, complimentary newspaper, business center, and fitness room; this hotel is located approximately 4–5 miles from the campus. Cancellation and early check-out policies vary; be sure to check on these policies when you make your reservation. **The deadline for reservations at this rate is March 18, 2011.**

Additional hotels in the area that may have a number of rooms available are listed below; however, no rooms were specifically reserved for participants at these hotels.

Hampton Inn, 736 Southbridge Street,Auburn, MA 01501, 774-221-0055. (`www.hamptoninn.com`) Rates are approximately US$109–129 for single/double occupancy room for the dates of this meeting. Amenities include complimentary parking, free continental breakfast, business center, and fitness room; approximately 5.5 miles from the campus. Cancellation and early check-out policies vary; be sure to check when you make your reservation.

Beechwood Hotel, 363 Plantation Street, Worcester, MA 01605, 508-754-5789. (`www.beechwoodhotel.com`) Rate is approximately US$179 per night for a single/double occupancy room. Amenities include complimentary parking, complimentary continental breakfast, free newspapers, free wireless high speed Internet, fitness center, and a fine dining restaurant ; approximately 3.9 miles from the campus. Cancellation and early check-out policies vary; be sure to check on these policies when you make your reservation.

Travel

Worcester is approximately 53 miles/one-hour drive from Logan International Airport (BOS) in Boston, MA; 50 miles/one-hour drive from T. F. Green Airport (PVD) in Rhode Island; and approximately 68 miles/one-hour-and-fifteen-minutes drive from Bradley International Airport (BDL) in Windsor Locks, Connecticut.

By Air: The College of the Holy Cross is located in central Massachusetts in the city of Worcester. There are four airports that are convenient for travel to and from this campus.

Boston's Logan International Airport (BOS) (www.massport.com/logan) is approximately one hour from Worcester. Shuttle service from Logan is available through **Knight's Limousine Service** (www.knightsairportlimo.com) (800-227-7005) with rates starting at US$44 one way for a shared van service, **Worcester Airport Limousine (**www.wlimo.com) (800-343-1369) with rates starting at US$59 one-way for a shared van, and **Boston Town Car (**www.bostontowncar.com) (888-765-LIMO) fare is approximately US$130 for a private town car. Please call in advance for reservations.

Rhode Island's T. F. Green Airport (PVD) (www.pvdairport.com) is approximately one hour from Worcester. Shuttle service from T. F. Green is available through **Knight's Limousine Service** (www.knightsairportlimo.com) (800-227-7005) with rates starting at US$54 one way for a shared van, **Worcester Airport Limousine (**www.wlimo.com) (800-343-1369) with rates starting at US$72 one way for a shared van. Please call in advance for reservations.

Connecticut's Bradley International Airport (BDL) (www.bradleyairport.com) is located in Windsor Locks, CT, approximately one hour from Worcester. Shuttle service from Bradley is available through **Worcester Airport Limousine** (www.Wlimo.com) (800-660- 0992) with varying rates for shared van service or private sedan.

Worcester Airport (ORH) (www.massport.com/hanscom-worcester-airports), is approximately a 15-minute car ride. Taxi service is available.

Estimated cab fares from Logan Airport to Holy Cross were approximately US$142 one way and approximately US$118 one way from T. F. Green Airport at the time that this announcement was published.

By Train:

Amtrak train service is available to and from Worcester. The train station is located five minutes (by car) from campus. Taxi service is available. Schedule and fare information is available on the Amtrak website at www.amtrak.com.

Commuter rail service between Worcester and Boston is provided by Amtrak and the Massachusetts Bay Transit Authority [MBTA]. Search Amtrak schedules and fares online using the search terms WOR [Worcester] and BOS [Boston]. MBTA schedules and fares are available on the Worcester Commuter Station section of the MBTA Web site. The MBTA Commuter Rail stop in Worcester is located on Shrewsbury Street. The stop is at the end of the Worcester/Framingham line. Fares on the commuter rail system are approximately US$8.25 one way plus additional subway T-fare when necessary. For fare information, maps, and schedules, call 617-222-3200 or visit the MBTA Commuter Rail website: http://www.mbta.com/schedules_and_maps/rail/.

By Bus:

Peter Pan (www.peterpanbus.com) and **Greyhound** (www.greyhound.com) bus services are available to and from Worcester. The bus station is located five minutes from campus. Taxi service is available. For fares and schedule information on Greyhound, call 800-231-2222. For fares and schedule information on Peter Pan, call 800-343- 9999.

Directions to Hogan Campus Center:

From the East and West (Boston, Hartford, and New York): Massachusetts Turnpike I-90. Take Exit 10 to I-290 East toward Worcester. Take Exit 11 (College Square/Southbridge Street) off I-290. Cross over to the right lane immediately after coming off the ramp to College Square. Take the first right (before the traffic light) onto College Street. Go up the hill and enter the last gate on the left, Gate 7. The Hogan Campus Center is the second building on the left with the large silver cross on it. Visitor parking is to the right of the Campus Center.

From the North:

I-495 South to I-290 West, Exit 25B. From I-290 West take Exit 11 (College Square, Southbridge Street). Bear left coming off the ramp onto Southbridge Street. Take the first right (before the traffic light) onto College Street. Go up the hill and enter the last gate on the left, Gate 7. The Hogan Campus Center is the second building on the left with the large silver cross on it. Visitor parking is to the right of the Campus Center.

From the Southeast:

I-495 North to Massachusetts Turnpike I-90. Take Exit 10 to I-290 East toward Worcester. Take Exit 11 (College Square/Southbridge Street) off I-290. Cross over to the right lane immediately after coming off the ramp to College Square. Take the first right (before the traffic light) onto College Street. Go up the hill and enter the last gate on the left, Gate 7. The Hogan Campus Center is the second building on the left with the large silver cross on it. Visitor parking is to the right of the Campus Center.

From the South (Providence, Rhode Island):

Route 146 North to Exit 12, McKeon Road/College Square. Take left off ramp and follow signs for I-290 West/College Square. Left at traffic light on McKeon Road. Follow overpass through several sets of lights and proceed straight up hill. Take left after fire station into Holy Cross on Loyola Road. Follow left curve which becomes McCarthy Lane. Continue uphill to upper campus. Visitor parking is straight ahead, last parking lot.

Car Rental

Hertz is the official car rental company for the meeting. To reserve your special meeting rates, please provide your CV#, **04N30001**, to your corporate travel department, or your travel agent, when making reservations. Details on rates and general instructions can be found at http://www.ams.org/amsmtgs/04N30001.wor11-1.pdf.

You can make reservations online at http://link.hertz.com/link.html?id=23051&LinkType=HZLK&TargetType=Homepage&ret_url=www.ams.org or call Hertz directly:

- In the U.S. and Canada: 800-654-2240
- Other: 405-749-4434

At the time of reservation, the meeting rates will be automatically compared to other Hertz rates and you'll be quoted the best comparable rate available.

Local Information and Parking

Hotels are all in the same general location, between 3 to 5 miles driving distance from the College of the Holy Cross Campus. There is ample free parking at each of the hotels. All day parking (no fee) is available for visitors adjacent to the Hogan Campus Center. Visitor parking is located at number 45 on the campus map. A campus map can be found at `http://www.holycross.edu/directions/camp_map.html`. Information about the College of the Holy Cross Department of Mathematics and Computer Science may be found at `http://academics.holycross.edu/mathcs`. Please watch the website available at `http://www.ams.org/meetings/sectional/sectional.html` for additional information on this meeting. Please visit the Holy Cross website at `http://www.holycross.edu/` for additional information on the campus.

Local Transportation

Taxi Services:
Two taxi services provide transportation in the Worcester area: **Yellow Cab**, 508-754- 3211, and **Red Cab**, 508-792-9999 & 508-756-9000.
Bus Service:
The **Worcester Regional Transit Authority** [RTA] provides bus service around the city of Worcester. Holy Cross is served by routes 10 and 42. Bus schedules and maps are available on the RTA website at `www.theRTA.com`.

Weather

Weather in Central Massachusetts in early April is quite variable. The average high temperature is approximately 55 degrees and the average low is approximately 40 degrees. Visitors should be prepared for inclement weather and check weather forecasts in advance of their arrival.

Registration and Meeting Information

The registration desk will be open Saturday, April 9, 7:30 a.m.–4:00 p.m., and Sunday, April 10, 8:00 a.m.–noon. Fees are US$50 for AMS members; US$70 for nonmembers; and US$5 for students, unemployed mathematicians, and emeritus members. **Fees are payable on-site via cash, check, or credit card.**

The registration and exhibit areas will be located on the on the first floor of Smith Laboratories. Special Sessions will be held in the Integrated Science Complex (Smith Laboratories and Beaven, Haberlin, O'Neil, and Swords Halls) and also in Stein Hall. The Invited Addresses will be held in the Hogan Campus Center.

There will be a **conference banquet** taking place on April 9 from 6:00 p.m.–8:00 p.m. in the Hogan Campus Center. It will be an international buffet dinner with a cost of US$30 per person including tax and tip. There will also be a cash bar. Attendance will be limited to 100 people and guests may reserve a seat in advance. **The deadline for purchasing tickets to this function is March 25, 2011**, two weeks prior to the start of the meeting. Payment should be made by check made payable to the College of the Holy Cross and mailed to: AMS Banquet Reservation/ Department of Mathematics and Computer Science, Attn: Ms. Gail Haddad, College of the Holy Cross, 1 College St., Worcester, MA 01610.

There will also be a **reception** sponsored by the College of the Holy Cross Department of Mathematics and Computer Science for student presenters, participants, and their advisors. It will take place between 1:00 p.m. and 2:00 p.m. on Saturday, April 9 in room 320, Hogan Campus Center.

A block of tickets has been reserved for a performance of "Breaking the Code", Saturday, April 9, 8:00 p.m. at the Central Square Theater. "Breaking the Code" is Hugh Whitemore's play about Alan Turing, and this is a production of the `Catalyst Collaborative@MIT`, the science theater collaboration between the Underground Railway Theater and MIT. Performances are enhanced by post-show conversations featuring renowned scientists and mathematicians. Central Square Theater is located in Cambridge, MA, and can be reached by car. The driving distance is approximately 43 miles and the driving time is under one hour on the Massachusetts Turnpike. The group rate for conference participants is US$25. To attend the performance at the group rate, tickets must be paid for in advance by check. The check should be made out to "College of the Holy Cross" and with the comment "Breaking the Code", please also indicate the number of tickets requested. **Checks must be received by Friday, March 18, 2010, to obtain the group rate**. Please mail checks to: Ms. Gail Hadad, Attn: Breaking the Code, Department of Mathematics and Computer Science, College of the Holy Cross, 1 College St., Worcester, MA 01610.

Other Activities

Book Sales:
Stop by the on-site AMS bookstore and review the newest titles from the AMS, enjoy up to 25% off all AMS publications, or take home an AMS t-shirt! Complimentary coffee will be served courtesy of AMS Membership Services.
AMS Editorial Activity:
An acquisitions editor from the AMS book program will be present to speak with prospective authors. If you have a book project that you would like to discuss with the AMS, please stop by the book exhibit.

Information for International Participants

Visa regulations are continually changing for travel to the United States. Visa applications may take from three to four months to process and require a personal interview, as well as specific personal information. International participants should view the important information about traveling to the U.S. found at `http://sites.nationalacademies.org/pga/biso/visas/` and `http://travel.state.gov/visa/visa_1750.html`. If you need a preliminary conference invitation in order to secure a visa, please send your request to `dls@ams.org`.

If you discover you do need a visa, the National Academies website (see above) provides these tips for successful visa applications:

* Visa applicants are expected to provide evidence that they are intending to return to their country of residence. Therefore, applicants should provide proof of "binding"

or sufficient ties to their home country or permanent residence abroad. This may include documentation of the following:

- family ties in home country or country of legal permanent residence
- property ownership
- bank accounts
- employment contract or statement from employer stating that the position will continue when the employee returns;

* Visa applications are more likely to be successful if done in a visitor's home country than in a third country;

* Applicants should present their entire trip itinerary, including travel to any countries other than the United States, at the time of their visa application;

* Include a letter of invitation from the meeting organizer or the U.S. host, specifying the subject, location and dates of the activity, and how travel and local expenses will be covered;

* If travel plans will depend on early approval of the visa application, specify this at the time of the application;

* Provide proof of professional scientific and/or educational status (students should provide a university transcript).

This list is not to be considered complete. Please visit the websites above for the most up-to-date information.

Las Vegas, Nevada

University of Nevada

April 30–May 1, 2011

Saturday–Sunday

Meeting #1071

Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: February 2011
Program first available on AMS website: March 17, 2011
Program issue of electronic *Notices*: April 2011
Issue of Abstracts: Volume 32, Issue 3

Deadlines

For organizers: Expired
For consideration of contributed papers in Special Sessions: Expired
For abstracts: March 8, 2011

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Elizabeth Allman, University of Alaska, *Evolutionary trees and phylogenetics: An algebraic perspective.*

Danny Calegari, California Institute of Technology, *Stable commutator length in free groups.*

Hector Ceniceros, University of California Santa Barbara, *Immersed boundaries in complex fluids.*

Tai-Ping Liu, Stanford University, *Hilbert's sixth problem.*

Special Sessions

Advances in Modeling, Numerical Analysis and Computations of Fluid Flow Problems (Code: SS 2A), **Monika Neda**, University of Nevada, Las Vegas.

Computational Algebra, Groups and Applications (Code: SS 7A), **Benjamin Fine**, Fairfield University, **Gerhard Rosenberger**, University of Hamburg, Germany, and **Delaram Kahrobaei**, City University of New York.

Discrete Dynamical Systems in Graph Theory, Combinatorics, and Geometry (Code: SS 15A), **Eunjeong Yi** and **Cong X. Kang**, Texas A&M University at Galveston.

Extremal Combinatorics (Code: SS 6A), **Jozsef Balogh**, University of California San Diego, and **Ryan Martin**, Iowa State University.

Flow-Structure Interaction (Code: SS 9A), **Paul Atzberger**, University of California Santa Barbara.

Geometric Group Theory and Dynamics (Code: SS 12A), **Matthew Day**, **Danny Calegari**, and **Joel Louwsma**, California Institute of Technology, and **Andy Putnam**, Rice University.

Geometric PDEs (Code: SS 1A), **Matthew Gursky**, Notre Dame University, and **Emmanuel Hebey**, Université de Cergy-Pontoise.

Knots, Surfaces and 3-manifolds (Code: SS 18A), **Stanislav Jabuka**, **Swatee Naik**, and **Chris Herald**, University of Nevada, Reno.

Lie Algebras, Algebraic Transformation Groups and Representation Theory (Code: SS 16A), **Andrew Douglas** and **Bart Van Steirteghem**, City University of New York.

Multilevel Mesh Adaptation and Beyond: Computational Methods for Solving Complex Systems (Code: SS 4A), **Pengtao Sun**, University of Nevada, Las Vegas, and **Long Chen**, University of California Irvine.

Nonlinear PDEs and Variational Methods (Code: SS 11A), **David Costa** and **Hossein Tehrani**, University of Nevada, Las Vegas, and **Zhi-Qiang Wang**, Utah State University.

Partial Differential Equations Modeling Fluids (Code: SS 5A), **Quansen Jiu**, Capital Normal University, Beijing, China, and **Jiahong Wu**, Oklahoma State University.

Recent Advances in Finite Element Methods (Code: SS 3A), **Jichun Li**, University of Nevada, Las Vegas.

Recent Developments in Stochastic Partial Differential Equations (Code: SS 8A), **Igor Cialenco**, Illinois Institute of Technology, and **Nathan Glatt-Holtz**, Indiana University, Bloomington.

Set Theory (Code: SS 14A), **Douglas Burke** and **Derrick DuBose**, University of Nevada, Las Vegas.

Special Session in Arithmetic Dynamics (Code: SS 17A), **Arthur Baragar**, University of Nevada, Las Vegas, and **Patrick Ingram**, University of Waterloo.

Special Session on Computational and Mathematical Finance (Code: SS 13A), **Hongtao Yang**, University of Nevada, Las Vegas.

Topics in Modern Complex Analysis (Code: SS 10A), **Zair Ibragimov**, California State University, Fullerton, **Zafar Ibragimov**, Urgench State University, and **Hrant Hakobyan**, Kansas State University.

Accommodations

Participants should make their own arrangements directly with a hotel of their choice as early as possible. Special rates have been negotiated with the hotels listed below. Rates quoted do not include sales tax of 12%. The AMS is not responsible for rate changes or for the quality of the accommodations. When making a reservation, participants should state that they are with the **American Mathematical Society (AMS) Meeting at the University of Nevada (UNLV)** group. Cancellation and early checkout policies vary; be sure to check when you make your reservation.

Hyatt Place Las Vegas, 4520 Paradise Road, Las Vegas, NV 89169; phone: 702-369-3366; fax: 702-369-0009. Rates begin at US$94 single/double and includes complimentary Wi-Fi, free parking, complimentary continental breakfast and fitness center. The hotel is a short walk away from the Math Department—directly across the street. (`www.lasvegas.place.hyatt.com`) Use booking code: **G-AMS1** for our special AMS rate. Cancellation and early checkout policies vary; be sure to check when you make your reservation. **The deadline for reservations is April 1, 2011.**

Fairfield Inn Las Vegas Airport, 3850 South Paradise Road, Las Vegas, NV 89169; phone: 702-791-0899; fax: 702-791-2705. Rates are US$89 single/double plus tax. The Fairfield Inn offers complimentary airport shuttle, complimentary breakfast buffet for overnight guests, and complimentary high speed Internet access. The hotel is a half mile away from the Math Department. Cancellation and early checkout policies vary; be sure to check when you make your reservation. **The deadline for reservations is April 1, 2011.** Go to this link—`http://www.marriott.com/hotels/travel/lasfi-fairfield-inn-las-vegas-airport/?toDate=5/2/11&groupCode=amsamsa&fromDate=4/28/11&app=resvlink` or use group code: **amsamsa**.

Embassy Suites Hotel Las Vegas Airport, 4315 Swenson Street, Las Vegas, NV 89119; phone: 702-795-2800; fax: 702-795-1520. Rates begin at US$119 single/double plus sales tax. Use code **L-AMS** fo get the AMS rates. (`http://embassysuites1.hilton.com/en_US/es/hotel/LASESES-Embassy-Suites-Las-Vegas-Nevada/index.do`) The hotel is a good walk (1/4 mile) away from the Math Department. Cancellation and early checkout policies vary; be sure to check when you make your reservation. **The deadline for reservations is April 1, 2011.**

Hard Rock Hotel, 4455 Paradise Road, Las Vegas, NV 8910; phone: 800-693-ROCK. The hotel is a short walk away from the Math Department—across the street. (`www.HardRockHotel.com`) All rooms are subject to availability. We were not able to confirm a group rate for this property. We are providing this information as a convenience to our meeting guests.

Food Service

The following dining options are available on-campus: Dining Commons, Einstein Bros Bagels, Jamba Juice, Metro Pizza, Panda Express, Starbucks, Subway, Taco Bell, The Coffee Bean & Tea Leaf.

Nearby off-campus options include:

To the west, off Swenson & Flamingo Road: Bahama Breeze, Cozymel's Mexican Grill, Firefly on Paradise, Hamada of Japan, P. F. Chang's, Roy's Hawaiian Fusion.

To the east, off Maryland Pkwy & Tropicana Avenue: Boardhouse Serious Sandwiches, Café Rio, Einstein Bros Bagels, In-N-Out, Jimmy Johns Gourmet Sandwiches, N&N Oriental, Sushi Boy Desu. A more detailed list and map will be provided at registration.

Also, University of Nevada, Las Vegas, is located 1.5 miles east of Las Vegas Blvd (The Strip), which offers an array of dining options housed in hotel/casino atmospheres.

Local Information

Please visit the websites maintained by the University of Nevada, Las Vegas, at `http://www.unlv.edu/` and the Department of Mathematical Sciences at `http://sciences.unlv.edu/Mathematics/`. For Las Vegas tourism information, visit `http://www.visitlasvegas.com` or `http://www.lvchamber.com/`.

Parking

Parking Lot N and Black Lot (with parking garage) are closest to Classroom Building Complex Auditorium. A parking permit is not required for Saturday and Sunday. If you are parking on campus on Friday, a pass is required. Please visit the website maintained by Parking and Transportation Services at `http://parking.unlv.edu/`. To view a campus parking map, please visit `http://maps.unlv.edu/PDF/main-parking-color-name.pdf`. To view a campus building map, please visit `http://maps.unlv.edu/PDF/main-building-color-name.pdf`.

Travel

By air:

If you are flying into Las Vegas, **McCarran International Airport (LAS)** (`www.mccarran.com`) would be your choice of arrival airports. You will find taxi services are available outside Baggage Claim for the short drive to the University of Nevada, Las Vegas. Depending on the hotel you choose, a shuttle bus may be available; please ask hotel clerk when making your reservation. McCarran Airport is extremely close to UNLV campus, as well as Las Vegas Blvd.

Driving directions to campus from the airport: Head North on Swenson Street. Turn right on Tropicana Avenue. Turn left at Wilbur to access Black Lot and parking garage near Thomas & Mack Center. Classroom Building Complex Auditorium is just east of Thomas & Mack. No parking permit is required for Saturday and Sunday. If you are parking on campus on Friday, a pass is required.

If you are driving:

Traveling on Interstate 15: Take exit 37 for Tropicana Ave. Turn east onto Tropicana Ave. (traveling towards Las Vegas Blvd.) Turn left at Wilbur St., into parking lot/garage near Thomas & Mack Center.

Traveling on U.S. Highway 93: Take exit 68 for Tropicana Ave. toward McCarran Airport/LV Blvd./UNLV. Turn west onto Tropicana Ave. Turn right at Wilbur St., into parking lot/garage near Thomas & Mack Center.

Traveling on U.S. Highway 95: U.S. Highway 95 merges onto Interstate 15 at exit 76B. U.S. Highway 95 joins U.S. Highway 93, after exit 76B (I-15 connector). Follow directions as listed above.

Car Rental

Hertz is the official car rental company for the meeting. To reserve your special meeting rates, please provide your CV#, **04N30001**, to your corporate travel department, or your travel agent, when making reservations. Details on rates and general instructions can be found at `http://www.ams.org/amsmtgs/04N30001.las11.pdf`.

You can make reservations online at `http://link.hertz.com/link.html?id=23051&LinkType=HZLK&TargetType=Homepage&ret_url=www.ams.org` or call Hertz directly:

·In the U.S. and Canada: 800-654-2240 or 405-749-4434.

At the time of reservation, the meeting rates will be automatically compared to other Hertz rates and you'll be quoted the best comparable rate available.

Weather

At the end of April/beginning of May, Las Vegas generally experiences an average high temperature of 83 degrees, with lows around 58 degrees. Precipitation is uncommon.

Registration and Meeting Information

The registration desk will be located in the lobby of the Carol C. Harter Classroom Building Complex and will be open from 7:30 a.m. to 4:00 p.m. on Saturday, and 7:30 a.m. to 12:00 p.m. on Sunday. All talks will take place in the Classroom Building Complex.

Registration Fees: Fees are US$50 for AMS or CMS members, US$70 for nonmembers; and US$5 for students/unemployed/emeritus, payable on site by cash, check, or credit card.

The University of Nevada, Las Vegas, Classroom Building Complex Auditorium is located on the west side of campus off Harmon Avenue. The Department of Mathematical Sciences' phone number is (702) 895-3567. The fax number is (702) 895-4343. Emails can be directed to `math@unlv.edu`.

Other Activities

Book Sales:
Stop by the on-site AMS bookstore and review the newest titles from the AMS, enjoy up to 25% off all AMS publications, or take home an AMS t-shirt! Complimentary coffee will be served courtesy of AMS Membership Services.

AMS Editorial Activity:
An acquisitions editor from the AMS book program will be present to speak with prospective authors. If you have a book project that you would like to discuss with the AMS, please stop by the book exhibit.

Information for International Participants

Visa regulations are continually changing for travel to the United States. Visa applications may take from three to four months to process and require a personal interview, as well as specific personal information. International participants should view the important information about traveling to the U.S. found at `http://sites.nationalacademies.org/pga/biso/visas/` and `http://travel.state.gov/visa/visa_1750.html`. If you need a preliminary conference invitation in order to secure a visa, please send your request to `dls@ams.org`.

If you discover you do need a visa, the National Academies website (see above) provides these tips for successful visa applications:

* Visa applicants are expected to provide evidence that they are intending to return to their country of residence. Therefore, applicants should provide proof of "binding" or sufficient ties to their home country or permanent residence abroad. This may include documentation of the following:

- family ties in home country or country of legal permanent residence
- property ownership
- bank accounts
- employment contract or statement from employer stating that the position will continue when the employee returns;

* Visa applications are more likely to be successful if done in a visitor's home country than in a third country;

* Applicants should present their entire trip itinerary, including travel to any countries other than the United States, at the time of their visa application;

* Include a letter of invitation from the meeting organizer or the U.S. host, specifying the subject, location and dates of the activity, and how travel and local expenses will be covered;

* If travel plans will depend on early approval of the visa application, specify this at the time of the application;

* Provide proof of professional scientific and/or educational status (students should provide a university transcript).

This list is not to be considered complete. Please visit the websites above for the most up-to-date information.

Ithaca, New York

Cornell University

September 10–11, 2011

Saturday – Sunday

Meeting #1072

Eastern Section
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: June 2011
Program first available on AMS website: July 28, 2011
Program issue of electronic *Notices*: September 2011
Issue of Abstracts: Volume 32, Issue 4

Deadlines

For organizers: February 10, 2011
For consideration of contributed papers in Special Sessions: May 24, 2011
For abstracts: July 19, 2011

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Mladen Bestvina, University of Utah, *Title to be announced.*

Nigel Higson, Pennsylvania State University, *Title to be announced.*

Gang Tian, Princeton University, *Title to be announced.*

Katrin Wehrheim, Massachusetts Institute of Technology, *Title to be announced.*

Special Sessions

Difference Equations and Applications (Code: SS 1A), **Michael Radin**, Rochester Institute of Technology.

Parabolic Evolution Equations of Geometric Type (Code: SS 4A), **Xiaodong Cao**, Cornell University, and **Bennett Chow**, University of California, San Diego.

Partial Differential Equations of Mixed Elliptic-Hyperbolic Type and Applications (Code: SS 3A), **Marcus Khuri**, Stony Brook University, and **Dehua Wang**, University of Pittsburgh.

Set Theory (Code: SS 2A), **Paul Larson**, Miami University, Ohio, **Justin Moore**, Cornell University, and **Ernest Schimmerling**, Carnegie Mellon University.

Special Session in Symplectic Geometry and Topology (Code: SS 5A), **Tara Holm**, Cornell University, and **Katrin Wehrheim**, M.I.T.

Winston-Salem, North Carolina

Wake Forest University

September 24–25, 2011

Saturday – Sunday

Meeting #1073

Southeastern Section
Associate secretary: Matthew Miller
Announcement issue of *Notices*: June 2011
Program first available on AMS website: August 11, 2011
Program issue of electronic *Notices*: September 2011
Issue of Abstracts: Volume 32, Issue 4

Deadlines

For organizers: February 24, 2011
For consideration of contributed papers in Special Sessions: June 7, 2011
For abstracts: August 2, 2011

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Benjamin B. Brubaker, Massachusetts Institute of Technology, *Title to be announced.*

Shelly Harvey, Rice University, *Title to be announced.*

Allen Knutson, Cornell University, *Title to be announced.*

Seth M. Sullivant, North Carolina State University, *Title to be announced.*

Lincoln, Nebraska

University of Nebraska-Lincoln

October 14–16, 2011

Friday–Sunday

Meeting #1074

Central Section
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: August 2011
Program first available on AMS website: September 1, 2011
Program issue of electronic *Notices*: October 2011
Issue of Abstracts: Volume 32, Issue 4

Deadlines

For organizers: March 14, 2011
For consideration of contributed papers in Special Sessions: June 28, 2011
For abstracts: August 23, 2011

The scientific information listed below may be dated. For the latest information, see `www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Lewis Bowen, Texas A&M University, *Title to be announced.*

Emmanuel Candes, Stanford University, *Title to be announced* (Erdős Memorial Lecture).

Alina Cojocaru, University of Illinois at Chicago, *Title to be announced.*

Michael Zieve, University of Michigan, *Title to be announced.*

Special Sessions

Association Schemes and Related Topics (Code: SS 1A), **Sung Y. Song**, Iowa State University, and **Paul Terwilliger**, University of Wisconsin, Madison.

Quantum Groups and Representation Theory (Code: SS 2A), **Jonathan Kujawa**, University of Oklahoma, and **Natasha Rozhkovskaya**, Kansas State University.

Salt Lake City, Utah

University of Utah

October 22–23, 2011

Saturday–Sunday

Meeting #1075

Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: August 2011
Program first available on AMS website: September 8, 2011
Program issue of electronic *Notices*: October 2011
Issue of Abstracts: Volume 32, Issue 4

Deadlines

For organizers: March 22, 2011
For consideration of contributed papers in Special Sessions: July 5, 2011
For abstracts: August 30, 2011

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Graeme Milton, University of Utah, *Title to be announced*.

Lei Ni, University of California San Diego, *Title to be announced*.

Igor Pak, University of California Los Angeles, *Title to be announced*.

Monica Visan, University of California Los Angeles, *Title to be announced*.

Special Sessions

Geometric Evolution Equations and Related Topics. (Code: SS 2A), **Andrejs Treibergs**, University of Utah, Salt Lake City, **Lei Ni**, University of California, San Diego, and **Brett Kotschwar**, Arizona State University.

Special Session on Geometric, Combinatorial, and Computational Group Theory (Code: SS 1A), **Eric Freden**, Southern Utah University, and **Eric Swenson**, Brigham Young University.

Port Elizabeth, Republic of South Africa

Nelson Mandela Metropolitan University

November 29–December 3, 2011

Tuesday–Saturday

Meeting #1076

First Joint International Meeting between the AMS and the South African Mathematical Society.
Associate secretary: Matthew Miller
Announcement issue of Notices: June 2011
Program first available on AMS website: Not applicable
Program issue of electronic *Notices*: Not applicable
Issue of Abstracts: Not applicable

Deadlines

For organizers: February 23, 2011
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/internmtgs.html.

Invited Addresses

Mark J. Ablowitz, University of Colorado, *Title to be announced*.

James Raftery, University of Kwazulu Natal, *Title to be announced*.

Daya Reddy, University of Cape Town, *Title to be announced*.

Peter Sarnak, Princeton University, *Title to be announced*.

Robin Thomas, Georgia Institute of Technology, *Title to be announced*.

Amanda Weltman, University of Cape Town, *Title to be announced*.

Boston, Massachusetts

John B. Hynes Veterans Memorial Convention Center, Boston Marriott Hotel, and Boston Sheraton Hotel

January 4–7, 2012

Wednesday–Saturday
Joint Mathematics Meetings, including the 118th Annual Meeting of the AMS, 95th Annual Meeting of the Math-

ematical Association of America, annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association for Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).

Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: October 2011
Program first available on AMS website: November 1, 2011
Program issue of electronic *Notices*: January 2012
Issue of Abstracts: Volume 33, Issue 1

Deadlines

For organizers: April 1, 2011
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

Honolulu, Hawaii

University of Hawaii

March 3–4, 2012

Saturday–Sunday
Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: March 2012
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of Abstracts: To be announced

Deadlines

For organizers: August 3, 2011
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

Tampa, Florida

University of South Florida

March 10–11, 2012

Saturday–Sunday
Southeastern Section
Associate secretary: Matthew Miller
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: March 2012
Issue of Abstracts: To be announced

Deadlines

For organizers: August 10, 2011
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

Washington, District of Columbia

George Washington University

March 17–18, 2012

Saturday–Sunday
Eastern Section
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: March 2012
Issue of Abstracts: To be announced

Deadlines

For organizers: August 17, 2011
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

Lawrence, Kansas

University of Kansas

March 30–April 1, 2012

Friday–Sunday
Central Section
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: April 2012
Issue of Abstracts: To be announced

Deadlines

For organizers: To be announced
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

New Orleans, Louisiana

Tulane University

October 13–14, 2012

Saturday–Sunday
Southeastern Section
Associate secretary: Matthew Miller
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: October 2012
Issue of Abstracts: To be announced

Deadlines

For organizers: January 13, 2012
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

San Diego, California

San Diego Convention Center and San Diego Marriott Hotel and Marina

January 9–12, 2013

Wednesday–Saturday
Joint Mathematics Meetings, including the 119th Annual Meeting of the AMS, 96th Annual Meeting of the Mathematical Association of America, annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association for Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: October 2012
Program first available on AMS website: November 1, 2012
Program issue of electronic *Notices*: January 2012
Issue of Abstracts: Volume 34, Issue 1

Deadlines

For organizers: April 1, 2012
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

Ames, Iowa

Iowa State University

April 27–28, 2013

Saturday–Sunday
Central Section
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: April 2013
Issue of Abstracts: To be announced

Deadlines

For organizers: September 27, 2012
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Special Sessions

Operator Algebras and Topological Dynamics (Code: SS 1A), **Andrejs Treibergs**, University of Utah, Salt Lake City, **Lei Ni**, University of California, San Diego, and **Brett Kotschwar**, Arizona State University.

Alba Iulia, Romania

June 27–30, 2013

Thursday–Sunday
First Joint International Meeting of the AMS and the Romanian Mathematical Society, in partnership with the "Simion Stoilow" Institute of Mathematics of the Romanian Academy.
Associate secretary: Robert J. Daverman
Announcement issue of *Notices*: To be announced
Program first available on AMS website: Not applicable
Program issue of electronic *Notices*: Not applicable
Issue of Abstracts: Not applicable

Deadlines

For organizers: To be announced
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

Baltimore, Maryland

Baltimore Convention Center, Baltimore Hilton, and Marriott Inner Harbor

January 15–18, 2014

Wednesday–Saturday
Joint Mathematics Meetings, including the 120th Annual Meeting of the AMS, 97th Annual Meeting of the Mathematical Association of America, annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association for Symbolic Logic, with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).
Associate secretary: Matthew Miller
Announcement issue of *Notices*: October 2013
Program first available on AMS website: November 1, 2013
Program issue of electronic *Notices*: January 2013
Issue of Abstracts: Volume 35, Issue 1

Deadlines

For organizers: April 1, 2013
For consideration of contributed papers in Special Sessions: To be announced
For abstracts: To be announced

San Antonio, Texas

Henry B. Gonzalez Convention Center and Grand Hyatt San Antonio

January 10–13, 2015

Saturday–Tuesday

Joint Mathematics Meetings, including the 121st Annual Meeting of the AMS, 98th Annual Meeting of the Mathematical Association of America, annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic, with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).

Associate secretary: Steven H. Weintraub

Announcement issue of *Notices*: October 2014

Program first available on AMS website: To be announced

Program issue of electronic *Notices*: January 2015

Issue of Abstracts: Volume 36, Issue 1

Deadlines

For organizers: April 1, 2014

For consideration of contributed papers in Special Sessions: To be announced

For abstracts: To be announced

Seattle, Washington

Washington State Convention & Trade Center and the Sheraton Seattle Hotel

January 6–9, 2016

Wednesday–Saturday

Joint Mathematics Meetings, including the 122nd Annual Meeting of the AMS, 99th Annual Meeting of the Mathematical Association of America, annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic, with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).

Associate secretary: Michel L. Lapidus

Announcement issue of *Notices*: October 2015

Program first available on AMS website: To be announced

Program issue of electronic *Notices*: January 2016

Issue of Abstracts: Volume 37, Issue 1

Deadlines

For organizers: April 1, 2015

For consideration of contributed papers in Special Sessions: To be announced

For abstracts: To be announced

Atlanta, Georgia

Hyatt Regency Atlanta and Marriott Atlanta Marquis

January 4–7, 2017

Wednesday–Saturday

Joint Mathematics Meetings, including the 123rd Annual Meeting of the AMS, 100th Annual Meeting of the Mathematical Association of America, annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic, with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).

Associate secretary: Georgia Benkart

Announcement issue of *Notices*: October 2016

Program first available on AMS website: To be announced

Program issue of electronic *Notices*: January 2017

Issue of Abstracts: Volume 38, Issue 1

Deadlines

For organizers: April 1, 2016

For consideration of contributed papers in Special Sessions: To be announced

For abstracts: To be announced

San Diego, California

San Diego Convention Center

January 10–13, 2018

Wednesday–Saturday

Associate secretary: Matthew Miller

Announcement issue of *Notices*: October 2017

Program first available on AMS website: To be announced

Program issue of electronic *Notices*: To be announced

Issue of Abstracts: To be announced

Deadlines

For organizers: April 1, 2017

For consideration of contributed papers in Special Sessions: To be announced

For abstracts: To be announced

Meetings and Conferences of the AMS

Associate Secretaries of the AMS

Western Section: Michel L. Lapidus, Department of Mathematics, University of California, Surge Bldg., Riverside, CA 92521-0135; e-mail: `lapidus@math.ucr.edu`; telephone: 951-827-5910.

Central Section: Georgia Benkart, University of Wisconsin-Madison, Department of Mathematics, 480 Lincoln Drive, Madison, WI 53706-1388; e-mail: `benkart@math.wisc.edu`; telephone: 608-263-4283.

Eastern Section: Steven H. Weintraub, Department of Mathematics, Lehigh University, Bethlehem, PA 18105-3174; e-mail: `steve.weintraub@lehigh.edu`; telephone: 610-758-3717.

Southeastern Section: Matthew Miller, Department of Mathematics, University of South Carolina, Columbia, SC 29208-0001, e-mail: `miller@math.sc.edu`; telephone: 803-777-3690.

The Meetings and Conferences section of the *Notices* gives information on all AMS meetings and conferences approved by press time for this issue. Please refer to the page numbers cited in the table of contents on this page for more detailed information on each event. Invited Speakers and Special Sessions are listed as soon as they are approved by the cognizant program committee; the codes listed are needed for electronic abstract submission. For some meetings the list may be incomplete. **Information in this issue may be dated. Up-to-date meeting and conference information can be found at** `www.ams.org/meetings/`**.**

Meetings:

Important Information Regarding AMS Meetings

Potential organizers, speakers, and hosts should refer to page 100 in the January 2011 issue of the *Notices* for general information regarding participation in AMS meetings and conferences.

Abstracts

Speakers should submit abstracts on the easy-to-use interactive Web form. No knowledge of LaTeX is necessary to submit an electronic form, although those who use LaTeX may submit abstracts with such coding, and all math displays and similarily coded material (such as accent marks in text) must be typeset in LaTeX. Visit `http://www.ams.org/cgi-bin/abstracts/abstract.pl`. Questions about abstracts may be sent to `abs-info@ams.org`. Close attention should be paid to specified deadlines in this issue. Unfortunately, late abstracts cannot be accommodated.

Conferences: (see `http://www.ams.org/meetings/` for the most up-to-date information on these conferences.)

February 17–21, 2011: AAAS Meeting in Washington, DC (Please see `www.aaas.org/meetings` for more information.)

June 12–July 2, 2011: MRC Research Communities, Snowbird, Utah. (Please see `http://www.ams.org/amsmtgs/mrc.html` for more information.)

July 4–7, 2011: von Neumann Symposium on Multimodel and Multialgorithm Coupling for Multiscale Problems, Snowbird, Utah. (Please see `http://www.ams.org/meetings/amsconf/symposia/symposia-2011` for more information.)

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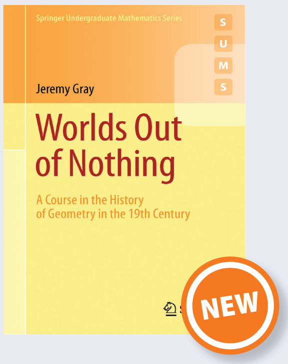
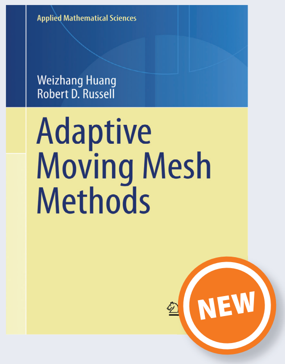
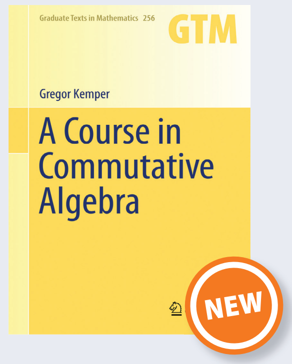


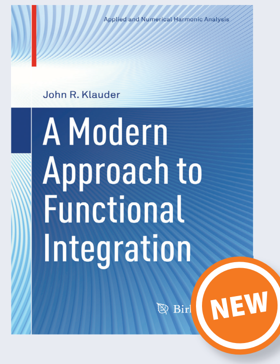
2010. XXIV, 400 p. 71 illus. (Springer Undergraduate Mathematics Series) Softcover
ISBN 978-0-85729-059-5
► **$49.95**



2011. XVII, 432 p. 121 illus., 7 in color. (Applied Mathematical Sciences, Volume 174) Hardcover
ISBN 978-1-4419-7915-5
► **$49.95**



2011. XI, 246 p. (Graduate Texts in Mathematics, Volume 256) Hardcover
ISBN 978-3-642-03544-9
► **$59.95**



2011. XVI, 282 p. 9 illus. (Applied and Numerical Harmonic Analysis) Hardcover
ISBN 978-0-8176-4790-2
► **$59.95**

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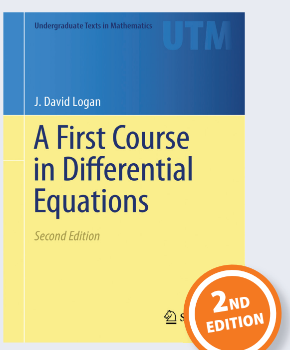
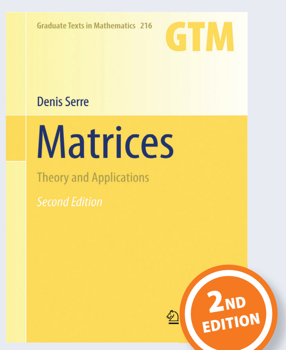
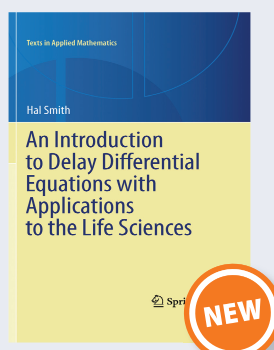


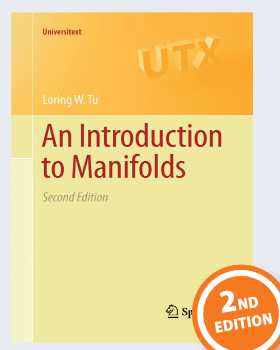
2nd ed. 2011. XVIII, 386 p. 82 illus. (Undergraduate Texts in Mathematics) Hardcover
ISBN 978-1-4419-7591-1
► **$74.95**



2nd Edition 2010. XIV, 289 p. 2 illus. (Graduate Texts in Mathematics, 216) Hardcover
ISBN 978-1-4419-7682-6
► **$74.95**



2011. XII, 177 p. (Texts in Applied Mathematics, Volume 57) Hardcover
ISBN 978-1-4419-7645-1
► **$49.95**



2nd Edition. 2011. XVIII, 410 p. 125 illus., 1 in color. (Universitext) Softcover
ISBN 978-1-4419-7399-3
► **$59.95**

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American Mathematical Society

AMS Sectional Meetings – Spring 2011

MAR 12-13

Georgia Southern University, Statesboro, GA

March 12-13

Georgia Southern University, Statesboro, GA

Invited Addresses by **Jason A. Behrstock**, Lehman College (CUNY); **Gordana Matic**, University of Georgia; **Jeremy T. Tyson**, University of Illinois at Urbana-Champaign; and **Brett D. Wick**, Georgia Institute of Technology

MAR 18-20

University of Iowa, Iowa City, IA

March 18-20

University of Iowa, Iowa City, IA

Invited Addresses by **Mihai Ciucu**, Indiana University; **David Damanik**, Rice University; **Kevin B. Ford**, University of Illinois at Urbana-Champaign; and **Chiu-Chu Liu**, Columbia University

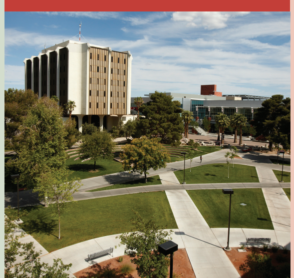
APR 9-10

College of the Holy Cross, Worcester, MA

April 9–10

College of the Holy Cross, Worcester, MA

Invited Addresses by **Vitaly Bergelson**, Ohio State University; **Kenneth M. Golden**, University of Utah; **Walter D. Neumann**, Columbia University; and **Natasa Sesum**, University of Pennsylvania



APR 30-MAY 1

University of Nevada, Las Vegas, NV

April 30-May 1

University of Nevada, Las Vegas, NV

Invited Addresses by **Elizabeth Allman**, University of Alaska; **Danny Calegari**, California Institute of Technology; **Hector Ceniceros**, University of California Santa Barbara; and **Tai-Ping Liu**, Stanford University

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