Intuition and Rigor and Enriques's Quest

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In the preceding article we have seen that Enriques and, indeed, the whole Italian school of algebraic geometry in the first half of the twentieth century were frustrated by one glaring gap in their theory of algebraic surfaces. This magnificent theory answered essentially all the basic questions about algebraic surfaces and had been constructed using purely geometric tools. But one of its central theorems seemed to defy all their attempts to give it a geometric proof. It had been proven by analytic means by Poincaré with his theory of "normal functions",[1] so the theory was sound—but this approach was alien to their intuitions. It was much like the need for analysis in proving the prime number theorem before Selberg found his elementary proof. In my own education, I had assumed they were irrevocably stuck, and it was not until I learned of Grothendieck's theory of schemes and his strong existence theorems for the Picard scheme that I saw that a purely algebro-geometric proof was indeed possible. I say here "algebro-geometric", not "geometric", because the first requirement in moving ahead had been the introduction of new algebraic tools into the subject first by Zariski and Weil and subsequently by Serre and Grothendieck.

When Professors Babbitt and Goodstein wrote me about Enriques's work in the 1930s, I realized that the full story was more complex. As I see it now, Enriques must be credited with a nearly complete geometric proof *using, as did Grothendieck, higher order infinitesimal deformations.* In other words, he anticipated Grothendieck in understanding that the key to unlocking the Fundamental Theorem was understanding and manipulating geometrically higher order deformations. Let's be careful: he certainly had the correct ideas about infinitesimal geometry, though he had no idea at all how to make precise definitions. If you compare his ideas here with, for example, the way Leibniz described his calculus, the level of rigor is about the same. To use a fashionable word, his "yoga" of infinitesimal neighborhoods was correct, but basic parts of it needed some nontrivial algebra before they could ever be made into a proper mathematical theory.

Enriques himself realized that he did not have a clear definition of higher order infinitesimal deformations, and so he was uncertain what sort of arguments were permissible for "curves in a higher order neighborhood" of an actual curve on a surface. As we will see below, he had two lines of reasoning. One depended on being able to *add* infinitesimal points on a group, forming as it were $(n+m)\epsilon$ by adding $n\epsilon$ and $m\epsilon$. Another was based on an infinitesimal analog of Poincaré's normal function construction, which uses the Jacobian varieties of the level curves of a suitably general rational function f on the surface F (a so-called *Lefschetz pencil* on F). In this paper, I will use Enriques's 1936 paper in the Mathematical Seminar of Rome[2] to explain these two ideas. I will translate the key parts of this paper and add my commentary so the reader can see exactly what Enriques did and how it can be made into a fully rigorous argument with modern technique.

There is no exact way of assigning a percentage to the degree of completeness of an argument. Certainly Grothendieck's tools were needed, and

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[1] Sur les courbes tracées sur les surfaces algébriques, *in two parts:* Annales Écoles Normale Sup. ***29****, 1910, and* Sitzungsberichte Berlin Math. Gesellschaft ***10****, 1911. An excellent presentation can be found in O. Zariski's book* Algebraic Surfaces, *Chapter VII, section 5, published by Springer, 1934, and reprinted by Chelsea, 1948.*

[2] Rendiconti del seminario matematico della universita di Roma, *series 4, vol. 1, 1936, pp. 1-9.*

First definitions: It may be useful for nonalgebraic geometers to set the stage with the basic definitions. A *projective algebraic variety* V is the locus of zeros of a finite set of homogeneous polynomials in some projective space, which, moreover, is irreducible, i.e., not the union of two or more proper subsets of the same type. A *Zariski open set* in V is just the complement $V - \bigcup_i W_i$ of a set of subvarieties $W_i \subset V$. Such varieties are linked by *rational maps*, maps $f : V_1' \to V_2, V_1' \subset V_1$ a Zariski open set, given by rational functions f of the coordinates of V_1. (Think of x/y, which defines a rational map $(\mathbb{P}^2)' \to \mathbb{P}^1$ which cannot be defined at $x = y = 0$.) The main players are the *nonsingular* or *smooth* varieties, those which are locally transverse intersections of hypersurfaces with locally independent differentials.

Linear systems: The key structures that were studied on smooth varieties were their *linear systems*. If $H \subset V$ is the intersection of V with the hyperplane at infinity, think of the vector space L of affine coordinate functions, the span of $\{1, x_1, \ldots, x_n\}$. Note that almost all the functions in L have a simple pole along H. In general, we start with a *divisor* D on V which is just a linear combination $\sum_i n_i D_i$ of codimension 1 subvarieties $D_i \subset V$. Then we have the vector space $L(D)$ of rational functions f on V such that f has at most an n_i-fold pole at D_i if $n_i > 0$ and at least a $-n_i$-fold zero on D_i if $n_i < 0$. Alternately, one defines (f) to be the divisor of zeros and poles of a rational function f, assigning positive coefficients equal to the order of vanishing at all its zeros and similarly negative coefficients at poles. Using this language, we see that $L(D) = \{f \,|\, (f) + D \geq 0\}$. These $L(D)$ are the *complete linear systems*, and any of their linear subspaces are called *linear systems*. Computing the dimension of $L(D)$ is the concern of the class of theorems called "Riemann-Roch". The Italian geometers preferred to deal with the nonnegative divisors $(f) + D$ themselves. The set of these is written as $|D|$ (or $|D|_V$ if we need to make the ambient space explicit) and is just the projectivization of the vector space $L(D)$ (because $(f) + D = (g) + D$ only if f/g is a constant). To complete this list of standard definitions, we say two divisors D_1, D_2 are *linearly equivalent* (written $D_1 \equiv D_2$) if $D_1 - D_2 = (f)$ for some f and the *Picard group* $\mathrm{Pic}(F)$ is the group of divisors mod linear equivalence, called *divisor classes*.

they make the argument cleaner and more elegant. But Enriques's approach needs relatively few additional arguments to make it into a complete proof. He himself, with customary optimism, called it *pienamente rigorosa* ("fully rigorous")! Castelnuovo, on the other hand, when he comes to this argument in his edition of Enriques's posthumous book summarizing much of his lifetime's work, says more conservatively: "The section which follows has been left incomplete by the author and thus the argument which is developed has many gaps; however, it was thought appropriate to reproduce it because it contains ideas that perhaps, appropriately completed, will furnish the starting point for a systematization of the theory." We leave it to the reader to form his or her own judgment.

What Is This Fundamental Theorem?

The easiest way to explain the Fundamental Theorem from First Principles is to recall some basic facts from the theory of algebraic curves. If C is a curve of genus g, you consider the set S_n of all unordered n-tuples of points on C (also called the symmetric nth power of C, or $C^n/(\text{symm.gp.})$). It is the set of positive divisors of degree n on C. Then if $n > 2g - 2$, S_n is very elegant space: it has a fibered structure with fibers that are the complete linear systems of divisors of degree n and hence projective spaces and base space an abelian variety, that is, a complex torus of dimension g, independent of n and called the Jacobian of C. Two n-tuples are in the same fiber if and only if they are linearly equivalent. This generalizes to a surface F (and to any higher dimensional variety) if we replace n-tuples of points by divisors D on F drawn from any "sufficiently ample" cohomology class.[3] Consider the set $\mathcal{D}$ of all such D, which is called a *complete continuous system* of divisors on F. It has a similar fibered structure. The fibers are again linear systems of divisors (as before, two divisors being in the same linear system if they are the zeros and poles of a rational function), and the base space is an abelian variety, the Picard variety of F, which is again independent of the choice of the divisors.

The hard question was: what is the dimension of the Picard variety? and, specifically, was it equal to the irregularity q? One way to approach this is to fix one member $C \in \mathcal{D}$ (which we can take to be an irreducible curve) and look at the intersections $A = D.C, D \in \mathcal{D}$. There is a number n such that all these are n-tuples of points on C, n being the self-intersection number $(C.C)$. Linearly equivalent divisors D will give linearly equivalent n-tuples on C, but inequivalent divisors will usually remain inequivalent. However, if we let an inequivalent divisor D approach C along a one-dimensional family of divisors D_t, then

[3] *E.g., a high enough multiple of a hyperplane section for some projective embedding of F is sufficiently ample.*

Invariants of surfaces: The main numerical invariant of an algebraic curve is its *genus* g, introduced by Riemann. This can be defined as the dimension of the vector space of rational 1-forms with no poles on a curve C, that is, a differential that can be written locally as $f\,dx$, where x is a local coordinate in $U \subset C$ and f is a rational function with no poles in U. One always defines the *canonical divisor class* K (up to linear equivalence) to be the divisor of zeros and poles of any rational 1-form, so we get $g = \dim L(K)$. In modern sheaf-theoretic terminology, $L(K) = \Gamma(\Omega^1_C)$. Topologically, g is also the number of handles in the surface defined by the set of complex points on the curve. A central question in the theory of algebraic surfaces was how to generalize the genus to higher dimension. Clebsch[4] defined the *geometric genus* p_g of a surface F as the dimension of the vector space of rational 2-forms with no poles, i.e., locally given as $f\,dx \wedge dy$ with f having no poles locally. Again this equals $\dim L(K)$ or $\Gamma(\Omega^2_F)$. This can be hard to compute, and soon after Cayley[5] found that the dimension of the space of rational 2-forms with mild poles was much easier to compute. Somewhat simplifying the history, it was found that there was a number p_a, the *arithmetic genus*, such that for sufficiently ample curves $C \subset F$ with genus $g(C)$, the dimension of the space of 2-forms with simple poles at C was $p_a + g(C)$. Now looking at the leading term in the pole of such a 2-form, one finds that it is naturally a 1-form on C called its *residue*. Since there are p_g independent 2-forms with no poles, Cayley's dimension $p_a + g(C)$ less p_g is the dimension of the space of residues and that is at most $g(C)$. So $p_a \le p_g$. The first surfaces investigated had $p_a = p_g$, and these were called *regular*. Then $q = p_g - p_a$ was defined as the *irregularity* of F. In cohomological terms, we now define p_a by $p_a + 1 = \chi(\mathcal{O}_F)$ and, knowing $p_g = h^2(\mathcal{O}_F)$, we have $q = h^1(\mathcal{O}_F)$.

the limit of the intersections $C.D_t$ will be an n-tuple, which always belongs to one and the same linear equivalence class, also denoted by $(C.C)$. Put formally, this means that if we let $\mathbb{P}$ = projectivization of the tangent space $T_C\mathcal{D}$ to $\mathcal{D}$ at its point defined by C, then $\mathbb{P}$ maps to a linear system of n-tuples on C, called the *characteristic linear system* of $\mathcal{D}$. The hypothesis was that this linear system was the complete linear system $|(C.C)|$ on C, and it was always referred to as *the completeness of the characteristic linear system of a complete continuous system.*

Once this was established, a small argument shows that the dimension of the Picard variety is indeed equal to the irregularity q of F. Let K_F be the canonical divisor on the surface F. Using residues, one finds that the divisor class $(C.C+K_F)$ is the canonical class K_C on C. Writing $|(C.C)|_C$ for the linear system of n-tuples on C given by the self-intersection, we use the Riemann-Roch theorem on C, which shows that

$$\dim |(C.C)|_C = (C.C) - g(C) + 1 + \dim |(K_F.C)|_C.$$

But $2g(C) - 2 = \deg(K_C) = (C.C + K_F)$, and a theorem of Severi shows that $\dim |(K_F.C)|_C = \dim|K_F|_F = p_g$ if C is sufficiently ample. Putting this together, we find $\dim|(C.C)|_C = \frac{(C.C-K_F)}{2} + p_g$. On the other hand, a slight generalization of Cayley's definition of p_a shows that for all sufficiently ample curves $C \subset F$, $\dim|C|_F = \frac{(C.C-K_F)}{2} + p_a$, so q is exactly the codimension of the trace of the linear system $|C|_F$ on F inside the linear system $|(C.C)|_C$ on C. But if the characteristic linear system of $\mathcal{D}$ is complete, this codimension equals the codimension of $|C|_F$ in $\mathcal{D}$, and this is the dimension of the Picard variety. We can, of course, rewrite this using cohomology exact sequences.[6]

Enriques's 1936 Paper

To the best of my knowledge, Enriques's first paper on his most successful approach to the Fundamental Theorem is the one with the title "Curve infinitamente vicine sopra una superficie algebrica", published in the *Rendiconti del Seminario Matematico della R. Universita di Roma*, 1936.[7] The basic idea in this paper is repeated in his note *Sulla proprietà caratteristica delle superficie algebriche irregolari* in the *Rendiconti della Accademia Nazionale dei Lincei*, 1938, and in Chapter 9 of his posthumous book *Le Superficie Algebriche*, published in 1949 with the editorial help of Castelnuovo (whose conservative evaluation we have quoted above).

As the title says, the paper is all about curves on an algebraic surface, but not actual curves. If $C \subset F$ is a curve in the usual sense, i.e., it is a subset defined by zeros of polynomials, then the paper concerns other "curves" $C_1, C_2, C_3, \ldots$ which he calls infinitely close to C in the neighborhood of order $1, 2, 3, \ldots$ (*" infinitamente vicine ad una C*

[4]Comptes Rendus de l'Acad. Fr., *vol. 67, 1868, p. 1238.*

[5]*Footnote on page 333 in* Le Superficie Algebriche.

[6]*Incidentally, I'd like to dispel the misconception that Italian algebraic geometry had nothing to do with cohomology of sheaves. Higher cohomology groups were implicit in most of their work but were always treated indirectly with geometric tools. For instance, the result that H^2 of the sheaf of 2-forms on the projective plane was one-dimensional is equivalent to the classical Cayley-Bacharach theorem.*

[7]*I am very grateful to Dr. Pier Vittorio Ceccherini for his help in obtaining a copy of this article, which is hard to find in the United States, and to Francine Laporte for a great deal of help with the translation.*

nell'intorno del 1^o *ordine, del* 2^o *ordine e così via"*). He also calls these curves $C_1, C_2, \ldots$ "successive" to C, and we follow this for lack of a better English word.

To make sense of this paper we have to have some idea of what he meant by "infinitely close". The problem is that Enriques was so thoroughly a geometer that he avoided ever using an equation of any sort, and it is hard to make sense of infinitely close things without some equations. Most easily, one can put a parameter t into the defining equations of the curve, replacing $f(x_1, \ldots, x_m) = 0$ or simply as $f(x) = 0$, by $f(x) + tf_1(x) + t^2 f_2(x) + \cdots$ and calculate mod t^n for some n. But you can read his papers and never know that algebraic varieties had to be defined by polynomials! However, in two places in small asides he gives some clues about what he meant. These are places where he refers to auxiliary varieties parameterizing divisors. Here is the first from §2 of his paper dealing with infinitely close "groups" of points on curves, that is, positive 0-cycles of some degree m, $G_m = \sum_{i=1}^m P_i \in S_m$ (P_i not necessarily distinct):

> And first of all, let us note that the groups of points G_m (made up of m points), infinitely close to a given group on a curve, can be properly defined using differential expressions and conditions, as elements or "points" of the variety representing the groups of m points of the curve.
> *E anzitutto osserviamo che i gruppi di punti G_m (costituti di m punti) infinitamente vicini ad un gruppo dato, sopra una curva, riusciranno bene definiti mediante espressioni e condizioni differenziali, siccome elementi o ⟨⟨punti⟩⟩ della varietà rappresentiva dei gruppi di m punti della curva.*

This variety "representing groups" is nothing but the mth symmetric power of the curve C. He is reducing the study of infinitely close complex objects on the curve C to the study of infinitely close points on the auxiliary parameterizing space. And again from §3 he says:

> For our aims it suffices to consider curves successive to C on linear branches (within the space that has C, C_1, C_2, etc., for "points").
> *E pel nostro scopo basta limitarsi a considerare curve successive alle C su rami lineari (entro l'ente che ha per ⟨⟨punti⟩⟩ le C, C_1, C_2 ecc.)*

Here he restricts himself to successive infinitely close points along a one-dimensional arc in an auxiliary space, one that parametrizes effective divisors on a surface—which he assumes is readily constructed. We are left with the issue: what does he mean by the "point" infinitely close of order n to a real point on a curve or on an analytic branch?

Algebraic geometry had introduced "infinitely near" points on all varieties as points on any variety obtained by blowing up the original variety a certain number of times. Given an analytic branch on the variety X, one can indeed blow up one point $P \in X$. At a point P on a surface where x, y are local coordinates, the blowup will be the closure of the graph of x/y. This will replace P by a projective line whose points correspond to tangent directions, the projectivization of $T_P F$. Then we can blow up the new limit point of the branch after "lifting" it to the first blowup, and continue to do this n times. The resulting infinitely near points on some auxiliary parameterizing variety is one approach to making sense of Enriques's concept of infinitely close curves on a surface. There was also a tradition of studying Puiseux series, power series in fractional exponents to describe curves, and then the terms of order n give another approach to higher order neighborhoods. But after Grothendieck's work, I think the best approach is clearly to define an infinitely close point of order n on a branch to be its unique subscheme whose ring of functions is isomorphic to $(k[t]/(t^{n+1}))$. On the whole variety, the infinitely close points of order n are then the set of all such subschemes on different branches. In my notes below, I will use this approach. Such an infinitely close point is essentially the same as what in differential geometry are called n-jets on a manifold.

Enriques, I believe, thought of a sequence of true points $P_0, P_1, \ldots, P_n$, *equally spaced* on an analytic arc, and then imagined (as Leibniz might) that P_1 and hence all the rest approached P_0. Then the "limit" of P_n is the point of order n successive to P_0. I think we all recognize that there is a common intuitive meaning to the idea that on a linear branch, there is something like a unique infinitely close point of order n at each usual point. However, as we'll see, it gets sticky when you begin to play games with this concept and don't actually have a precise definition.

The Translation, Part I, and Enriques's First Argument

Now I will translate the key sections of this paper interspersed with commentary to make the article more accessible. I show Enriques's words in italics and my commentary in sans-serif typeface to distinguish the two voices.

In the demonstration of the characteristic property of irregular surfaces with geometric and numerical (=arithmetic) *genera p_g and p_a, that is that they contain continuous systems of curves formed of $\infty^{p_g - p_a}$ inequivalent* (that is, not linearly equivalent) *linear systems, one needs to count the number of curves of the continuous system that are* *infinitely close to a given curve K and to show that they cut on K the complete characteristic system. But to the demonstration that I gave in my note in the Academy of Bologna in 1904 and that at first was accepted by all, or rather restated with slight modifications, the objection was raised that curves infinitely close to K (whose existence result from compatibility of superabundance conditions) do not necessarily lead to the existence of continuous series of curves containing them.*

The "complete characteristic system" is the linear system that I denoted by $L_K((K.K))$ or, in sheaf-theoretic terms, by $\Gamma(\mathcal{O}_K(K^2))$ or written projectively by $|(K.K)|_K$. In this translation, I have retained his symbol K, although now K is usually reserved for the canonical divisor class. This is the best place to describe what he had done in 1904, namely, he showed that for every positive divisor $G \in |(K.K)|_K$, he could construct a curve K_1 infinitely close to K of first order which intersects K in G. His method was to take a second sufficiently ample divisor $\widetilde{K}$ and look at the curves E in $|K + \widetilde{K}|_F$ that pass through the intersection points $K \cap \widetilde{K}$. He first showed that these curves E intersect K, after discarding the points $K \cap \widetilde{K}$, in the complete characteristic system. He then argued that if such an E is infinitely close to $K + \widetilde{K}$ in the first order, E effectively has a double point infinitely close to each point of $K \cap \widetilde{K}$ and then, by a Riemann surface argument, this E must continue to split into two pieces $K_1 + \widetilde{K}_1$. This K_1 is the infinitely close curve to K that we want. This argument is correct. For instance, one can use the exact sequence:

$$0 \longrightarrow \mathcal{O}_F(\widetilde{K}) \longrightarrow m_{K\cap\widetilde{K}}\mathcal{O}(K+\widetilde{K}) \longrightarrow \mathcal{O}_K((K.K)) \longrightarrow 0$$

to verify the first point. The second point follows because if you deform a double point $u.v = 0$ by a curve through $u = v = 0$, the result is $u.v + t.(au + bv) = 0$ and modulo t^2, this equals $(u + tb).(v + ta) = 0$. And finally, if resolving a set of double points disconnects a curve, the same holds for any deformation in which the double points persist.

Now, however, we would argue directly that $\mathcal{O}_K((K.K))$ was the normal sheaf to K and thus its sections define first-order deformations. But Enriques's approach is perfectly correct.

Next he addresses the reasons that his old argument was incomplete by making some very general observations about the pitfalls of drawing global conclusions from infinitesimal facts.

To explain this doubt: if one defines, for example, a curve as the intersection of two or more surfaces, one cannot say that on the curve so defined one always has (one) *single point infinitely close to a given point; one will have instead ∞^1 if the given curve is a line of contact of the defining surfaces: in this case there are no further successive points infinitely close a point taken in this neighborhood of a proper point of the curve but outside its tangent.*

What he is saying is that if the surfaces meet transversely along the curve, at each point of the curve there would be only one infinitesimal direction in which you can move, namely along the curve; on the other hand, if the surfaces are tangent, you can start to move infinitesimally away from the curve in many directions, but unless they are tangent to an even higher order, one cannot extend this path to second order and stay inside the intersection unless you moved along the curve. In the language of schemes, the intersection of surfaces tangent along a curve is an everywhere nonreduced scheme, generically with the square of its nilpotent ideal equal to (0). The "Zariski tangent space" has dimension two, but there are higher order infinitely close points only for the one direction tangent to the reduced curve.

Re-examining the same question in my "Lessons on the classification of surface", edited by L. Campedelli, I observed, however, that the conclusions of my treatment would keep their validity if one admitted that curves infinitely close to a given curve on a surface had an effective existence and that one could operate on them as on finite curves, by adding and subtracting. Thus, letting C_1 be a curve infinitely close to C and inequivalent to it, the operation $+C_1 - C$, successively repeated, serves to define, in the neighborhood of any curve K whatsoever, a series of infinitely close curves $K_1, K_2, K_3, \ldots$ belonging to a suitably high order neighborhood and this leads to the conclusion that this K should belong to a continuous nonlinear series in which that K_1 would be close to K.

It remained however to justify the intuitive truth: that one can effectively operate on infinitely close curves on a surface by addition and subtraction. And this is precisely the aim of the present note.

A Modern Version of This Intuitive Argument

The argument immediately above is the core of Enriques's first argument for the theorem using infinitely close curves (but, as he says, not of the more refined "proof" that follows). When I read this, it sounded a bit far-out. But then, putting it carefully into the language of schemes, I found it could be given quite a complete and elegant modern formulation.

Let me paraphrase his idea like this: suppose $t \mapsto \phi(t) \in A$ is a 1-parameter subgroup of a Lie group A (such as the Picard variety of the surface F). Then start with a point $\phi(\epsilon) \in A$ where ϵ is first taken to be *positive*, not infinitesimal. Then simply adding, using the group law in A, gets you back $\phi(n\epsilon)$ for all n. Now suppose ϵ is infinitesimal, passing from finite numbers to infinitesimals as in Leibniz's treatment of calculus. Then $\phi(\epsilon)$ lies in what Enriques is calling the first-order neighborhood of the identity $e \in A$. Operating with the group law on A, he wants to automatically generate something like $\phi(n\epsilon)$ which is to live in the nth-order neighborhood of e. In particular, his difference $C_1 - C$ is like an infinitely close point of first order $\phi(\epsilon)$ in the Picard variety of F and "adding", he wants to generate the higher order "points" of the Picard variety, hence the higher order infinitesimal deformations K_n of K.

Nowadays we can use the language of schemes to make this precise. A point on A in the nth-order neighborhood of e is just a morphism of $\mathrm{Spec}(k[t]/(t^{n+1}))$ to A whose set-theoretic image is the point e. So does it make sense that from a $k[t]/(t^2)$-valued point, we can use the group law alone to get a $k[t]/(t^{n+1})$-valued point?

In fact, this is true in characteristic 0, and a simple algebraic argument provides rigorous support for Enriques's Leibnizian treatment of infinitesimals. The key is this purely algebraic fact:

Proposition. *If the characteristic of k is 0, then the subring of*

$$k[t_1, \dots, t_n]/(t_1^2, \dots, t_n^2)$$

of elements invariant under permutations of the t_i is isomorphic to $k[s]/(s^{n+1})$ where $s = t_1 + \cdots + t_n$.

The reason is that on the one hand the invariant subring is generated, as usual, by the elementary symmetric polynomials in the t_i and on the other hand:

$$(t_1 + \cdots + t_n)^k \equiv k! \cdot k^{th} \text{ elem.symm.polyn.}(t_1, \dots, t_n) \mod (t_1^2, \dots, t_n^2).$$

In characteristic zero, we can divide by $k!$; hence the proposition follows.

Now given an infinitely close point of first order $\phi : \mathrm{Spec}(k[t]/(t^2)) \longrightarrow A$, we get the following n-fold summation by adding via the group law on A:

$$\phi \circ p_1 + \cdots + \phi \circ p_n : \mathrm{Spec}(k[t_1, \dots, t_n]/(t_1^2, \dots, t_n^2)) \longrightarrow A$$

and, by commutativity, the pullback of all functions on A are permutation invariant; hence this map factors through $\mathrm{Spec}(k[s]/(s^{n+1}))$. This is the infinitely close point of order n.

In his example, we would say that he wants to *add* a divisor class on $F \times \mathrm{Spec}(k[t]/(t^2))$, trivial on F, to itself to get a divisor class on $F \times \mathrm{Spec}(k[s]/(s^{n+1}))$. Suppose the divisor class is defined by a 1-cocycle $\{1 + tf_{\alpha,\beta}\}$. Then adding this to itself n-times gives the divisor class on $F \times \mathrm{Spec}(k[t_1, \dots, t_n]/(t_1^2, \dots, t_n^2))$ given by

$$\begin{aligned}\prod_{i=1}^{n}(1 + t_i f_{\alpha,\beta}) &= \sum_{k=0}^{n} \left(k^{th} \text{ elem.symm.polyn.in } t_i\right) \cdot f_{\alpha,\beta}^k \\ &= \sum_{k=0}^{n} \frac{s^k}{k!} f_{\alpha,\beta}^k.\end{aligned}$$

Thus this n-fold added divisor class is defined over the subring $k[s]/(s^{n+1})$ and the divisor class is defined by the 1-cocycle given by the truncated exponential. This, I think, is the precise meaning behind Enriques's assertion that *adding* first-order deformations defines higher deformations—of course only in characteristic zero. Enriques certainly did not know such an argument, but this at least confirms that his intuition was completely sound—and, as Hartshorne remarked to me, was based on a characteristic zero world without his knowing it.

The Translation, Part II, Analysis of Divisors on Curves

Let us take the steps of the argument in the case of groups of points and linear series (linear series on a curve are the same as linear systems) *on a curve.*

One will observe firstly that the ordinary manner of adding and subtracting series on a curve falls apart when one treats infinitely close groups or series. Thus if, among the curves of a certain order that pass through a certain group G (of points on some curve), *one considers those that are tangent to the basic curve at the points of G, one then subtracts from the series cut out not a group infinitely close to G but the group G itself.*

From that one should not conclude that as a result infinitely close groups of points or linear systems on a given curve do not have a real existence and that—in every respect—one cannot operate on them by addition and subtraction: the law of continuity operates in the field of algebra and so one must rather admit a priori that, with appropriate considerations, these entities and the operations we are dealing with will succeed in being properly justified. And first of all, we note that groups of points G_m (made up of m points), infinitely close to a given group on a curve, will be properly defined using differential expressions and conditions, as elements or "points" of the variety representing the groups of m points of the curve. (The variety representing m-tuples is just the mth symmetric power of the curve which we called S_m above. I think that when he talks here of differential conditions, Enriques foreshadows the basic idea behind nonreduced schemes.)

Let us suppose that the curve has genus p and let us consider, in particular, the Jacobian variety

V that represents the groups of p points, G, of that curve. (As Enriques knew well, the pth symmetric power of the curve is not exactly the Jacobian but it is birational to it, that is, they are the same on Zariski open sets. In fact, on the Zariski open set of "nonspecial" p-tuples, those which do not move in positive dimensional linear systems $|G_p|$, the two are the same and for special p-tuples, there is still a map from S_p to the Jacobian V but it is many-to-one, blowing down linear systems to points. Using such p-tuples on C to describe almost all of the Jacobian was used by Poincaré and will be used by Enriques in a very crucial way below.)

It is known that this V has a continuous group ∞^p of commuting birational transformations that one defines on the curve using the sum of a difference $G_1 - G$ of two groups. (He is just saying that the group of divisor classes, that is, divisors mod linear equivalence, is a *group*, hence the Jacobian is a group as well as a projective variety, that is, an abelian variety. But he prefers to use the birationally equivalent model given by the pth symmetric power, and here the group law is only defined on a Zariski open set and, more precisely, if G', G, and G_1 are three groups of p points, then for almost all G', $G' + G_1 - G$ will be linearly equivalent to a unique p-tuple G'', hence $G' \to G''$ defines a birational map on p-tuples but not an everywhere defined morphism.)

Now, if the second group G_1 comes infinitely close to the first, the operation $+G_1 - G$ does not cease describing a transformation of V, more precisely (it is) an infinitesimal transformation that is a generator of the group in the sense of Sophus Lie. By means of this infinitesimal transformation, the group of points G gives rise to an analytic series of transformations by which will be properly defined the groups $G_2, G_3, \ldots$, successive to G_1 which fall in the second, then in the third neighborhood of G and so forth. (Enriques's "analytic series" are what we call 1-parameter subgroups $\phi(t)$ in the Jacobian. In the complex torus representation of the Jacobian, they are just straight lines through the origin. $C_1 - C$ defines a tangent vector at the origin, hence a specific ϕ, which, considered to order n, are his higher order infinitesimals on the Jacobian and define what he calls $G_2, G_3, \ldots$.).

After that it is clear how one treats infinitely close complete linear systems g^r_m on a curve (The notation g^r_m simply means a complete linear system of m-tuples of dimension r. It is "nonspecial" if $r = n - p$, the generic case whenever $n \geq p$.)*: at least in the simplest case of a nonspecial series, their sum and difference can be reduced to the sum and difference of the groups of p points that one gets from them as residuals of m − p fixed points. If one wants to operate on special series, it is convenient to extend them by adding fixed points so they become nonspecial.*

But it's enough to restrict ourselves to this: given on a curve K a nonspecial series $g^r_n = g$ and a series $g_1 = (g^r_m)$ infinitely close to it (defined, as we've said, on the representing variety (the representing variety being the symmetric power. In fact, all he's going to use is the basic case of nonspecial p-tuples, or g^0_p which, as we said, gives a Zariski open subset of the Jacobian.)*, there is determined a continuous (analytic) series of series and in it the series successive to g_1, namely $g_2, g_3, \ldots$ neighboring g in the neighborhoods of second, third, etc., order.*

In place of the operation $+G_1 - G$, one can equally carry out on groups of p points of the curve (or on its nonspecial series of a given order) the inverse operation: $+G - G_1$. This defines, starting from G, a continuous (analytic) series of groups of p points complementary to $G_1, G_2, G_3, \ldots$ that we can denote $\overline{G}_1, \overline{G}_2, \overline{G}_3, \ldots$ where in general one has

$$\overline{G}_i \equiv 2G - G_i.$$

To make this as concrete as possible, I think what Enriques has in mind is that you start at some nonspecial p-tuple of points $G = \sum_{i=1}^{p} P_i$ on C (so that the pth symmetric power and the Jacobian are locally isomorphic near G). You can take any p-tuple G_1 in its first-order neighborhood, and this defines a tangent vector to the Jacobian at the point defined by G. Then, knowing that the Jacobian is a complex torus, you get a 1-parameter subgroup of the Jacobian. In the pth symmetric power this gives power series $P_i(t)$ with $P_i = P_i(0)$ so that

$$t \mapsto G(t) = P_1(t) + \cdots + P_p(t)$$

represents a 1-parameter subgroup of the Jacobian. This was classical stuff, the then century-old theory of abelian functions if you take as its beginning Abel's 1829 paper in Crelle's Journal. One could write down these power series using abelian functions. Then his G_ℓ and $\overline{G}_\ell$ are "points" on this analytic branch of order ℓ for $t > 0$ and $t < 0$, respectively.

The "Pencil" Construction of Poincaré and Enriques

Instead of beginning with a translation of Enriques's second argument for the Fundamental Theorem, which follows and certainly has some gaps, it seems better to first present a modern variant that will show that his argument is fundamentally sound and that will make explicit its links to the normal function technique of Poincaré. Then the reader can see why Enriques chose a somewhat different, more classical route but one that unfortunately ran into some difficulties. The core of Enriques's second argument introduces an idea that originated in Poincaré's analytic approach: the use of a suitably general *pencil* on the surface

F so as to construct curves on F by sweeping out finite groups on the curves of the pencil.

What is a pencil? Simply put, one takes any nonconstant rational function f on F (e.g., one of its coordinate functions x_i) and considers the one-dimensional family of level curves $f = t$ (including $f = \infty$, the poles of f): call this C_t. All the curves C_t are linearly equivalent and form a projective line in the linear system $|C_0|$ (which, as we said, is a projective space). The function f will not be defined at finite set points $\{P_1, \dots, P_d\}$ where all the curves C_t intersect: these are called the *base points* of the pencil. One can also consider the closure of the graph of f in $F \times \mathbb{P}^1$: this will be a surface F^* mapping to F by a birational map in which the base points P_i have each been replaced by projective lines called the exceptional curves E_i. In the generic case, the curves C_t meet transversely at each P_i and F^* is smooth and is the standard surface obtained by blowing up each P_i. Now f becomes an everywhere defined map $f^* : F^* \to \mathbb{P}^1$ with disjoint fibers C_t.

Poincaré and after him Enriques used the Jacobian varieties J_t of each curve C_t. (When C_t is singular, one uses the so-called "generalized Jacobian" of C_t.) The dimension of J_t is the genus of C_t, which we denote by p. We may set up a one-to-one correspondence between the points of the Jacobian J_t and the divisor classes of degree 0 on the curve C_t, and this correspondence will be set up by a "universal" divisor D_t on $C_t \times J_t$, i.e., $D_t.(C_t \times \{a\})$ represents the divisor class corresponding to a. In the classical approach, D_t is readily defined using abelian functions. It is then convenient to glue the J_t together, forming a variety $\mathcal{J}$ of dimension $p + 1$, which maps to $\mathbb{P}^1$ with fibers the individual Jacobians J_t. Then the union of all the products $C_t \times J_t$ forms a variety $F^* \times_{\mathbb{P}^1} \mathcal{J}$ of dimension $p + 2$ and the D_t's glue together to one big divisor $\mathcal{D}$ on this product.[8]

Taking some sufficiently ample curve D on F, Enriques's old 1904 argument constructed for him $q = p_g - p_a$ independent infinitely close curves D_1 in the first-order neighborhood of D. More precisely, he took the complete characteristic series $|(D.D)|_D$ and for any $a \in L_D(D.D)$, he defines an infinitely close curve $D_1^{(a)}$ in the first-order neighborhood of D. He wants to prolong these to infinitely close curves of higher order as the key step in showing that dim(Pic)= q. It is more natural today to consider the difference $D_1^{(a)} - D$ as defining a divisor on the scheme $F \times \mathrm{Spec}(k[t]/(t^2))$, which is trivial on F itself. (I think it is correct to say that although the Italian school was well aware that one could form divisors with negative coefficients, they strongly preferred to deal with positive divisors, and hence were averse to this step.)

We can intersect both D and $D_1^{(a)}$ with all the members of the pencil C_t. If m is the intersection number $(D.C_t)$, we get groups of m points $G_t^{(m)} = D.C_t$ and infinitely close groups $G_t^{(a,m)} = D_1^{(a)}.C_t$ on each C_t. As Enriques pointed out above, quoting Lie, the difference $G_t^{(a,m)} - G_t^{(m)}$ defines a tangent vector $v_t^{(a)}$ at 0 to the Jacobian variety J_t. All the tangent vectors $v_t^{(a)}$ together form a vector field to $\mathcal{J}$ along the zero-section of $\mathcal{J}$ over $\mathbb{P}^1$. In the language of schemes, such a vector field is the same as a morphism $f_1^{(a)} : \mathrm{Spec}(k[t]/(t^2)) \times \mathbb{P}^1 \to \mathcal{J}$ with $f_1^{(a)}(\mathrm{Spec}(k[t]/(t^2)) \times \{t\}) \subset J_t$. Note that $D_1^{(a)} - D$ can be recovered from the universal divisor $\mathcal{D}$ using the morphism $f_1^{(a)}$, that is, $D_1^{(a)} - D$ (lifted to F^*) is just the "pullback" of $\mathcal{D}$ via the morphism $1_{F^*} \times f_1^{(a)} : F^* \times \mathrm{Spec}(k[t]/(t^2)) \to F^* \times_{\mathbb{P}^1} \mathcal{J}$ (possibly up to adding some multiple of C_t).

Now J_t is known to be a complex torus. So through any tangent vector at the origin such as $v_t^{(a)}$, there is a straight line, that is, a one-parameter subgroup of J_t. Truncating this at nth order, we get canonical morphisms $\mathrm{Spec}(k[t]/(t^{n+1})) \to J_t$ for each t extending the vector $v_t^{(a)}$. It is clear that these vary at least analytically as t varies. But in fact, they fit together into an *algebraic* map $f_n^{(a)} : \mathrm{Spec}(k[t]/(t^{n+1})) \times \mathbb{P}^1 \to \mathcal{J}$ with $f_n(\mathrm{Spec}(k[t]/(t^{n+1})) \times \{t\}) \subset J_t$. This is an easy consequence of the elementary fact that meromorphic functions on $\mathbb{P}^1$ are all rational,[9] using crucially the fact that the construction works for all t with no exceptions. Finally we can define the sought-for infinitely close $D_n^{(a)}$ of higher order (as a divisor class) as D plus the pullback of $\mathcal{D}$ via $1_{F^*} \times f_n^{(a)}$. Using the Riemann-Roch theorem, we can show it is represented by a positive divisor.

How does this connect to Poincaré's argument? Although he came to it from a completely different route, he used the noninfinitesimal points on the one-parameter subgroups $\exp(s.v_t^{(a)}), s \in \mathbb{R}$, to construct global divisor classes on F. Without tracing all the links, let us just say that he constructed an explicit basis of 1-forms $\{\omega_t^{(k)}\}$ simultaneously on all but one of the curves C_t such that the vector fields $v_t^{(a)}$ have constant inner product zero with all of them, zero with $p - q$, and arbitrary constants with the remaining q. Moreover, fixing one of the base points x_0 of the pencil, D_t is given in the classical way by the divisor

[8] *Technical aside: any of the exceptional curves gives a section of F over $\mathbb{P}^1$ that serves to "rigidify" the relative Picard functor. Anyway, standard abelian functions define it, too.*

[9] *The most general results of this type were given by Serre in his famous "GAGA" paper "Géométrie algébrique et géométrie analytique",* Annales de l'Institut Fourier, *1956.*

$(\sum_{i=1}^{p} x_i) - p.x_0$ on C_t defined over $\{c_k\} \in J_t$ by:

$$\left(\sum_{i=1}^{i=p} \int_{x_0 \in C_t}^{x_i \in C_t} \omega_t^{(k)}\right) = c_k.$$

My guess is that Enriques knew he was dealing with the same approach as Poincaré to a certain extent, though he may not have realized how close his infinitely close curves $D_1^{(a)}$ were to Poincaré's basis of 1-forms.

The Translation, Part III, the Use of a Pencil

Section 3 of his paper deals with representing the surface F as a branched cover of the projective plane $\mathbb{P}^2$ and considering infinitely close curves on F via their images in $\mathbb{P}^2$. We have quoted above one sentence in which he alludes to parameterizing curves (and divisors) on a surface F by some auxiliary variety. Otherwise the section does not seem to add very much, and we omit it.

Section 4 deals with the injectivity of the map from the group of divisors on F mod linear equivalence (the Picard group of F) to the group of divisors mod linear equivalence on a curve K in F, given by intersection with K. In particular, he asserts that for suitable K, a nonzero infinitesimal divisor class $C_1 - C$ should have nonzero intersection with K. This is closely related to the lemma of Enriques-Severi that, in cohomological terms, asserts that $H^1(\mathcal{O}(-K)) = (0)$ if K is a sufficiently ample divisor. It was proven to modern standards by Zariski in 1951 in the *Annals of Math.*, volume 55, and treated cohomologically in Serre's fundamental paper "Faisceaux Algébriques Cohérents", *Annals of Math.*, volume 61, 1955. I omit this section, too.

Below I will translate section 5, in which a pencil of curves on the surface F is introduced so that he can extend a curve $C \subset F$ to infinitesimal neighborhoods by extending the divisors $C.K_t$ on each K_t.[10]

On the surface F we can choose two linear systems of regular (irreducible) curves $|C|$ and $|K|$ in such a way that the curves C cut nonspecial series on the curves K; it suffices to suppose $|C|$ sufficiently ample with respect to $|K|$, as, for example, by assuming it contains a multiple of $|K|$.

This assumed, let C and C_1 be two nonequivalent infinitely close curves, which certainly exist if the surface is irregular ($p_a < p_g$). By the theorem demonstrated in the preceding section, C and C_1 will cut inequivalent groups of m points $G = G^{(m)}$ and $G_1 = G_1^{(m)}$ on K, which will define two different complete nonspecial series g and g_1; consequently, thanks to the operation $+g_1 - g$, one will construct a continuous (analytic) series of inequivalent linear series in which one finds the series (always nonspecial) $g_2, g_3, \ldots$ infinitely close to g in neighborhoods of second, third, etc., order.

One chooses on a K a group G^n of the characteristic series and in this a point A. (From the context in the previous omitted section, it is clear that he is choosing here a *pencil* $\langle K_1, K_2\rangle$ from the linear system $|K|$ with base points $G^n = K_1 \cap K_2$ plus a specific base point $A \in G^n$. Also the number π introduced below is the genus of K. In what follows, he will make constructions on arbitrary curves K in this pencil and then take their loci as the curve varies in the pencil. To do this he wants canonical groups on each K_t, not just groups given up to linear equivalence. So he constructs next groups of degree π which—if nonspecial—are unique up to linear equivalence.)

The point A counted $m - \pi$ times will determine a group G^m of the series g (This is the unique group in the linear system g of the form:

$$G^m = (m - \pi)A + G^\pi, \ G^\pi = \pi \text{ further points.}$$

This constructs a *canonical* group of points G^π representing the divisor class. Some argument is needed to check that G^π is indeed nonspecial.) *and similarly a group G_1 of m points of the series g_1 and then a group of g_2 and so on. The loci* (as K varies in the pencil) *of the groups G_k^m defined in this way will be curves $L, L_1, L_2, \ldots$ of the same order, passing a certain number i times through the points of G^n and $m - \pi + i$ times through A. In fact, it's easy to check that if L touches a particular K of the pencil with base G^n, so that the corresponding generator G^m has on this K a point coinciding with a point of G^n, the same thing happens for L_1 and for L_2, etc.*

There is a major gap in his argument here as he doesn't make precise in any way what "the loci of the groups G_k^m" means when the groups are infinitely close. The locus L of the groups of ordinary points G^m on the members of the pencil, that is, the case $k = 0$, is clearly a good algebraic curve, but what are the loci when $k \geq 1$?

As an aside, I want to explain the technical point of where the integer i comes from. It is easier to follow if as above we blow up on F the base points G^n, giving a surface F^*, which is now fibered over $\mathbb{P}^1$ by the pencil of curves K_s. We can certainly consider L^*, defined as the locus of G^π on all the fibers K in the blown-up surface. Moreover, if A^* is the exceptional curve that is the blowup of the point A, from the definition of L^*, we will have a linear equivalence

$$L^* + (m - \pi)A^* \equiv C + iK \text{ for some } i.$$

Since $(C.A^*) = 0$, $(L^*.A^*) = i + m - \pi$, which is why he says that L has an $m - \pi + i$-fold point at A.

This is a reasonable construction showing that the locus of the ordinary groups G^m is an ordinary

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[10] *I'd like to thank Michael Artin for his help in understanding this argument of Enriques and especially for pointing out the problem of special groups on some members of the pencil.*

curve L. *BUT* Enriques claims without any discussion that this works for the infinitely close groups $G_1, G_2, \ldots$, sweeping them out to infinitely close curves $L_1, L_2, \ldots$. Here he is assuming that an operation that works for ordinary curves also works for the infinitely close ones. But, worse than that, he has defined the infinitely close groups G_k^π on each K_s in the pencil, successive to G^π, by analytic means, and he needs infinitely close algebraic curves L_k successive to L. To be as concrete as possible, if $G^m = \sum_{i=1}^m P_i(s)$ on the curve K_s, then we can imagine that the infinitely close group G_k^m is defined by the kth-order terms in t of points given by power series $P_i(s,t)$, $1 \le i \le m$, for $s \in \mathbb{P}^1 - S$, $|t| < c(s)$ but allowing for some finite set S of possibly "bad" points where either (a) L has a branch point so the P_i's interchange, (b) G^m is special, or (c) the curve K_s is singular. Enriques needs to define G_k^m for all s in order to prove he has an infinitely close *algebraic* locus. This problem was apparently raised by B. Segre. Enriques discusses this criticism in his later 1938 memoir "Sulla proprietà caratteristica delle superficie algebriche irregolari" (*Rendiconti della Accademia dei Lincei*, volume 27, pp. 493-498). He asserts here (p. 497) that this extension is truly algebraic. Actually I don't think (a) or (c) is a real problem, but (b) certainly is. It is not clear (to me) whether for generic pencils there will be curves K_s where G^π is special. Enriques addressed this briefly in his final book *Le Superficie Algebriche*, p. 336, pointing out that the set of special divisor classes of degree π has codimension two but not saying why this locus can be avoided by the curves in a generic pencil. Such points s may mean that the infinitely close curves L_k must be viewed as deformations of L *plus* a sum of special fibers K_{s_i} and showing that the whole mess is algebraic is not simple. What Enriques missed here is that everything is simpler if you use divisor classes of degree zero instead of positive divisors of degree π and use the existence of a universal divisor on the Jacobian as sketched in the previous section.

With the preceding construction we have defined a curve L (belonging to the linear system $|C + iK|$ and curves $L_1, L_2, L_3, \ldots$ infinitely close to it in the neighborhoods of order $1, 2, 3, \ldots$ as far as one wants, whose real existence is thus demonstrated.

That shows that the curve L_1 infinitely close to L is close to L in a continuous ∞^1 series of inequivalent curves; and since L_1 is substantially an arbitrary curve infinitely close to L, inequivalent to it, this proves that the linear system $|L|$ belongs to a continuous system $\{L\}$ that has as characteristic series the complete characteristic series on the curve L.

The Translation, Part IV, "Algebraization"

Now Enriques comes to the final key idea in his argument, the use of reducible curves in order to extract continuous nonlinear systems of curves from within linear systems:

But whoever looks at the demonstration with critical eyes, as is advisable with reasoning of this nature, will ask not only for the explicit proof that truly the L_1 that we constructed is an arbitrary member of the system of inequivalent curves infinitely close to L, but also that L_1 and then $L_2, L_3, \ldots$ are effectively curves infinitely close to L in the sense that we defined in the section "The Translation, Part I", and Enriques's First Argument, since indeed the construction of these curves $L_1, L_2, L_3, \ldots$ appears to be something different from that definition.

(The question is why the series of higher order infinitely close deformations of L is contained in an *algebraic* family of curves. Today we would only need to cite the existence of the Hilbert scheme. If we have deformations of a curve L to arbitrarily high order, there has to be a component of the Hilbert scheme giving not merely infinitesimal deformations but global ones, thus defining a continuous system of curves containing L and L_1. Enriques, however, found an elementary way to do this:)

To respond to the doubt so raised, one considers the linear system of the sum $|L+C|$ and inside it one considers the curves infinitely close to a reducible curve: they cut the L_1 that we have constructed in as many points as they cut L and so containing L_1 as a component imposes the same number of conditions (the dimension of a special series of the same order plus one): one must conclude that among the curves infinitely close to $L + C$ in the given system, there are curves made up of L_1 (defined as always as a curve infinitely close to L in the sense of the section "The Translation, Part I", and Enriques's First Argument) and a $\overline{C}_1$ infinitely close to C. Concerning this $\overline{C}_1$, one can say that it cuts a group $\overline{G}_1$ on K of a series complementary to that defining the group G_1, the section of C_1 (which was an arbitrary inequivalent curve in the neighborhood of C): indeed, designating with G the group $(C.K)$, one has:

$$G_1 + \overline{G}_1 = 2G.$$

As a consequence, $\overline{C}_1$ is, like C_1, an arbitrary curve among those inequivalent neighboring C: since if one takes $\overline{C}_1$ in the place of C_1, one finds C_1 in the place of $\overline{C}_1$.

Now the reasoning which precedes extends to all the curves infinitely close in neighborhoods of higher order. Among the curves of the linear system $|L + C|$, infinitely close $L_1 + \overline{C}_1$ (i.e., in the second-order neighborhood of $L + C$), one will find reducible curves that contain as a component the L_2, constructed above, and another component $\overline{C}_2$ neighboring $\overline{C}_1$ and successive to C. And continuing, one will find curves infinitely close to C belonging to neighborhoods of appropriate heights that extend $\overline{C}_1$, that is—as has been said—to an *arbitrary inequivalent curve infinitely close C. This shows that these infinitely close curves which cut the complete characteristic series on C are curves belonging to an effective continuous series, and thus that the linear system $|C|$ is contained in a continuous system $\{C\}$ that has on C a complete characteristic system and so is made up of $\infty^{p_g-p_a}$ inequivalent systems. q.e.d.*

The key observation is that the family L_k of deformations of L constructed earlier to all infinitesimal orders can be algebraicized to a family of ordinary curves on F by considering the family of *reducible* curves in the linear system $|L+C|$. He argues that this linear system contains curves of the form $L_k + C_k$ and these must lie in an algebraic family of linearly equivalent curve $L_t + C_t$. Taking either the system $\{L_t\}$ or the system $\{C_t\}$, one sees that the dimension of the Picard variety is indeed $p_g - p_a$.

Summary

Where should we place Enriques if we seek to summarize how algebraic geometry developed in the twentieth century? That he built a comprehensive theory of algebraic surfaces and their classification by what we now call Kodaira dimension is clear. But he is also a transition figure between the age of the classical geometry of varieties and linear systems and the modern period of schemes and cohomology. This transition was not marked at first by the discovery of new theorems but rather by the creation of whole new vocabulary and the toolkit that went along with this. Transitions of this kind may often look as if they appear out of nowhere, but this is rarely the case. Many of the ideas that came to full flower in the 1960s were "in the air" before then. The obstacle to their creation was one of naming, of admitting that it's going to be easier to understand some circle of ideas if you make some vague thing you're working with into a tangible object—reifying something dimly seen.

An example from the topic of this paper is that the characteristic linear system of a linear system on a surface F, if it is not enlarged to a complete continuous system, is incomplete. That is, given a curve C on a surface F and a divisor D on F, it was clear to all classical geometers that there were rational functions on C with poles bounded by $(D.C)$ that did not lift to rational functions on F with poles bounded by D. But the idea that you should give a name (i.e., H^1) to the cokernel:

$$\{\text{fcns. on } C, \text{ poles at } (D.C)\} \; mod \; \{\text{restrictions of fcns. on } F, \text{ poles at } D\}$$

was simply not the sort of thing they ever considered. Naming such cokernels came out of algebraic topology and was transplanted into algebraic geometry by Serre. This immediately systematized large areas of classical geometry.

Enriques's particular insight, however, was that he saw that there was a calculus of infinitesimal deformations of subvarieties. Although he gave these names, they remained in a limbo, without substance, *because he did not think of what it meant to have a function on them.* Grothendieck realized that functions on such objects should be rings with nilpotent elements, and this gave life to these infinitesimal deformations. He reified them as ringed spaces, and with the word Spec.

But reification never happens in a vacuum: there has to be a clear need for it, an intuition that has leaped ahead of the available tools. This Enriques had. The proof discussed above is a wonderful example of how, before the new system is invented, an ingenious mind can limn out what the new structure should look like.

So far, we have been emphasizing Grothendieck's theory of schemes and his existence theorems for the Hilbert and Picard schemes that make the Fundamental Theorem seem extremely easy. But we also know more today because Zariski and Weil introduced the parallel world of varieties over fields of finite characteristic. In this world, the Fundamental Theorem that $q = p_g - p_a$ is *false*: many varieties in characteristic p have a nonreduced Picard scheme. This allows us to trace the ideas behind the various attempts to prove the Fundamental Theorem to see where they use the essential hypothesis that the characteristic of the field is zero.

If we study Enriques's intuitive proof and try to make sense of it, as we did in the section "The Translation, Part I", and Enriques's First Argument, the characteristic zero hypothesis comes in through the use of the power series for the exponential function, since that requires dividing by $n!$. More generally the exponential function is the key ingredient in the theorem that all group schemes in characteristic zero are reduced. But Enriques's proof above, in the section "The Translation, Part III, the Use of a Pencil", was different. He used instead the well-established theory of the Jacobian variety. Being a complex torus, it had straight lines through the origin that define 1-parameter subgroups in every direction. This also does not hold in characteristic p: for instance, there are no 1-parameter subgroups of the formal two-dimensional multiplicative group with transcendental slope. The idea of using such subgroups is very ingenious.

In short, Enriques was a visionary. And, remarkably, his intuitions never seemed to fail him (unlike those of Severi, whose extrapolations of known theories were sometimes quite wrong). Mathematics needs such people—and perhaps, with string theory, we are again entering another age in which intuitions run ahead of precise theories.