The Community of Math Teachers, from Elementary School to Graduate School

Sybilla Beckmann

Why should mathematicians be interested and involved in pre-K–12 mathematics education? What are the benefits of mathematicians working with school teachers and mathematics educators?[1] I will answer these questions from my perspective of research mathematician who became interested in mathematics education, wrote a book for prospective elementary teachers, and taught sixth-grade math a few years ago. I think my answers may surprise you because they would have surprised me not long ago.

It's Interesting!

If you had told me twenty-five years ago, when I was in graduate school studying arithmetic geometry, that my work would shift toward improving pre-K–12 mathematics education, I would have told you that you were crazy. Sure, I would have said, that is important work, it's probably hard, and somebody needs to do it, but it doesn't sound very interesting. Much to my surprise, this is the work I am now fully engaged in. It's hard, and I believe what I'm doing is useful to improving education, but most surprising of all is how interesting the work is.

Yes, I find it interesting to work on improving pre-K–12 math! And in retrospect, it's easy to see how it could be interesting. Math at every level is beautiful and has a wonderful mixture of intricacy, big truths, and surprising connections. Even preschool math is no exception.

Consider this connection between preschool math and number theory. Young children play with pattern tile sets that consist of the shapes shown in Figure 1. Playing with these shapes,

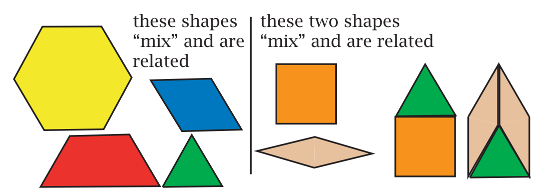

Figure 1. Pattern tiles that young children play with.

children discover that some of them can be put together to make others (e.g., three triangles fit together to make the trapezoid) but that the squares and thin rhombuses are different. In fact, shapes that are made without the squares and thin rhombuses, such as the shape in Figure 2, can never be made in a different way using the squares or thin rhombuses. Why not? Because the square root of three is irrational! The square and thin rhombus have rational area (in terms of square inches), but the other shapes' areas are rational multiples of the square root of three.

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[1] *A note on terminology: By "mathematician" I mean individuals in mathematics departments at colleges and universities who teach mathematics courses and who have done research in math. By "teacher" I mean individuals who teach within pre-kindergarten through grade 12. By "mathematics educator" I mean individuals who teach mathematics methods courses, professional development seminars, or workshops or who supervise or coordinate math teaching or curricula in schools and who have done research in mathematics education. I acknowledge that these categories are neither exhaustive among mathematics professionals nor mutually exclusive, that the descriptions of these categories should be viewed as somewhat fuzzy and approximate, and that the names of these categories are not fully descriptive.*

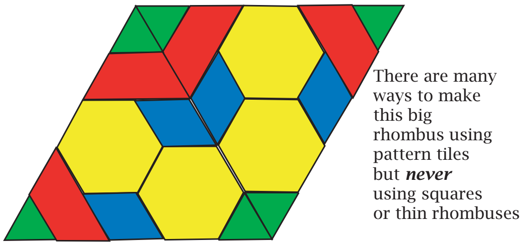
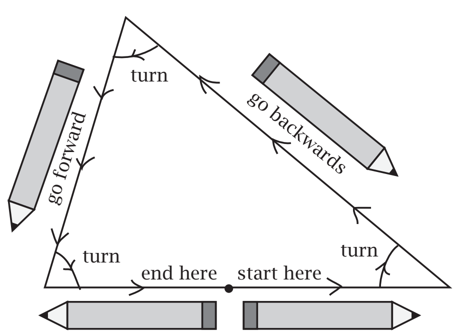


Figure 2. A shape made from pattern tiles without using the square or thin rhombus.

Of course it is interesting to find connections between elementary math and more advanced math (such as my example with the pattern tiles, which delighted me to discover). We can discover these connections without ever interacting with children, their teachers, or with mathematics educators. But what I have learned from mathematics educators is how interesting it is to find out how students—our own students in college classes as well as younger students in school—think about mathematical ideas. I've always enjoyed teaching but before I interacted with mathematics educators, I didn't realize it would be both a useful teaching tool and also *interesting* to find out how my students were thinking about the math I was trying to teach them. In retrospect, this lack of awareness is surprising. Most (all?) mathematicians enjoy talking to each other about math to find out how others are approaching problems and thinking about ideas. But all humans are capable of mathematical thought. Why not delight in it at every level? From the four-year-old who realizes that 8 + 9 is 17 because she knows 8 + 8 is 16 and so 8 + 9 must be one more, to the prospective middle-grades teachers in my geometry class this semester who devised the argument for why the sum of the angles in a triangle is 180° that is sketched in Figure 3, students can come up with ways to solve problems that we might not have thought of ourselves.



Going all the way around the triangle, the pencil turned a half-turn, which was the sum of the angles in the triangle.

Figure 3. An explanation for why the sum of the angles in a triangle is 180°.

One surprise in listening to how students think about math is to find that insightful ideas can come even from students who have big gaps in their mathematical knowledge. I have found this not only with college students, but also with schoolchildren. A few years ago I taught sixth-grade math, every morning for a whole year, to a group of students who were acknowledged by other teachers at the school to be functioning below grade level in math. Many of the students were still struggling with basic arithmetic facts. Near the beginning of the year, when I asked my students to write a word problem for whole number division, most of the students couldn't write any problems at all. But, despite the deficits, students still came up with valuable comments and insights throughout the year, and their interest in abstract mathematical ideas surprised me at times. When we discussed the circumference and area of a circle, I showed the students a printout of several thousand digits of pi. I told them that the digits go on forever without stopping and without a repeating pattern. Their eyes grew big. "For real!?" they said. When I asked the students where pi would be on a number line, Santiago described how he thought about the location of pi, explaining that we'd have to keep zooming in forever on the number line to see exactly where pi is located.

In all my teaching, whether sixth grade or at the college and graduate levels, I've found that gaps and difficulties can coexist with insightful thoughts and interest in mathematical ideas and with enthusiasm for math. It's easy to get frustrated with our students' knowledge gaps and misconceptions, but by recognizing that all of our students have mathematical potential and by seeking out our students' ideas, we can make our teaching more satisfying and more interesting.

What Can We Contribute to Pre-K–12 Education and What Can We Learn?

It's not surprising when I say that mathematicians have much to offer teachers and mathematics educators because of their broader, deeper view of mathematics. Mathematicians can help teachers and mathematics educators learn more math and learn connections between school math and more advanced math. But, perhaps surprisingly, there is plenty of *mathematics* that teachers and mathematics educators know but that mathematicians may not know explicitly or may not know in a way that applies to school mathematics.

For example, imagine that you are teaching third graders about division and that you want them to solve a variety of division word problems. What kinds of problems will you give them for 15 ÷ 3? You will surely have the students solve problems about dividing 15 objects equally among 3 groups, such as dividing 15 cookies equally among 3 people, or dividing 15 blocks equally among 3 containers. But you might not think to have students solve problems that involve dividing

15 objects into groups of 3 each, such as dividing 15 cookies into packages of 3 each, or dividing 15 blocks into containers that each hold 3 blocks. These two different perspectives on what division means correspond to two different equations, which are related by commutativity.

$$\begin{aligned} 3 \times ? &= 15 \\ ? \times 3 &= 15 \end{aligned}$$

We know not to take commutativity for granted because of the existence of nonabelian groups and noncommutative rings. The commutative property of multiplication is important to third graders too because it helps them lighten the load of learning the single-digit multiplication facts. But will third graders understand that whole number multiplication is commutative? Is it obvious? In fact, no; even from a third-grade perspective, the commutativity of multiplication of whole numbers is not obvious, as shown in Figure 4.

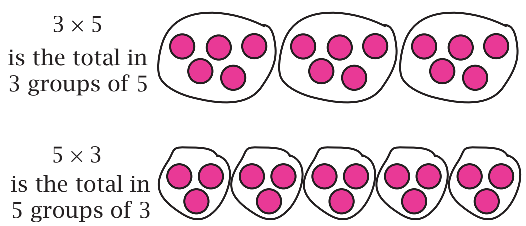


Figure 4. A third-grade perspective on why commutativity of multiplication is not obvious.

After seeing many examples, third graders may come to expect that multiplication really is commutative, but what is a third-grade way to see why whole number multiplication is commutative? (Note: no Peano axioms!) The existence of two-dimensional arrays, which can be decomposed either into equal rows or into equal columns, as in Figure 4, shows why whole number multiplication is commutative. As simple as arrays are, the existence of these structures now strikes me as saying something much deeper and more surprising about two-dimensional Euclidean space than I had previously appreciated.

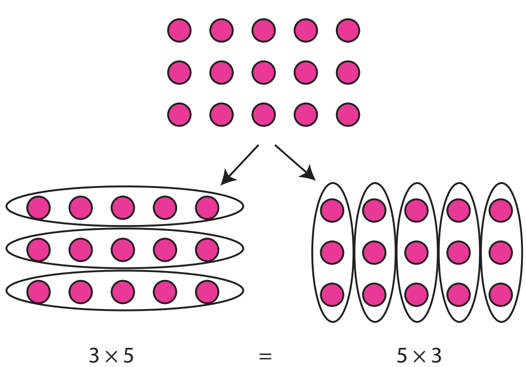


Figure 5. A third-grade perspective on the commutativity of multiplication.

My examples so far have concerned only whole number multiplication and division. But examples of surprisingly intricate details that are involved in understanding elementary math are everywhere. Did you know, for example, that there are many ways to explain why it makes sense to divide fractions according to the "invert and multiply" rule, including ways that involve analyzing word problems and drawing simple pictures? Who knew!

Even if you aren't interested in learning cool ways of explaining why "invert and multiply" is valid, what can mathematicians learn from the work of mathematics educators and teachers? I can summarize the most important thing I have learned: to improve teaching and learning in mathematics, we must take into account not only the mathematics itself—how to organize it, how to explain lines of reasoning clearly and logically, how the mathematical ideas are connected to other ideas both in and outside of math—but also *what students think*—what paths they tend to take as they develop understanding of mathematical ideas, where the difficulties lie, what errors and misconceptions tend to occur, what captures students' interest. We must attend to where our students are in their understanding of the material we are trying to teach them, not just by marking their answers right or wrong (which of course is important), but also by looking into the source of our students' errors. What ideas have our students not yet grasped and how can we help them learn those ideas? What misconceptions do they have and how can we help them see why these are misconceptions? What gets students excited about math and interested in learning it?

We might think that studying student thinking is only the job of mathematics education researchers and that the rest of us who teach math could safely dispense with it. Top-notch teaching might seem to be just a matter of having a well-structured course and a good book and then presenting the material clearly and enthusiastically in class, assigning good homework, and holding students accountable by giving tests. All these things are components of good teaching and can contribute to student learning, but they are not enough for excellent teaching. Most of us who teach have had the experience of delivering some beautifully polished lessons and carefully designed homework sets only to find out from students' performance on the test that they didn't actually grasp the ideas. What was missing? Most likely, our lectures didn't connect with students' existing knowledge and didn't help students engage with the material at a level where they could make sense of it. In our enthusiasm to share exciting mathematical ideas, we might have failed to see that our students weren't ready to appreciate the ideas. We probably gave answers to questions before students even grasped what the questions were and why the questions were significant. We showed students mathematical tools for solving problems before the students saw the need for those tools. We didn't learn how our students were thinking and therefore we weren't able to help them build the ideas up in their own minds.

So I have learned from mathematics educators that there is no "royal road" to mathematics teaching and learning. It will never be just a matter of getting students who are adequately prepared when they enter our classes, and it will never be just a matter of delivering polished material. Teaching is a deeply human activity because, like conversation, it requires a give and take between the teacher and the students. Good teaching will always be hard work because it requires a teacher to know the mathematics *and* to take his or her students' thinking into account when making instructional decisions. Good teaching requires knowing the mathematical ideas and how to connect and scaffold them to make them accessible to students, *and* it requires finding out how students are thinking and then using this information in lectures, problems, and activities. Good learning will always be hard work for students because it requires them to engage actively with the material, to think about what they do and don't understand, and to persevere in making sense of the ideas.

Even if you are not interested in learning more about pre-K–12 math or in learning about the work of mathematics educators and about results from mathematics education research, why should mathematicians, mathematics educators, and teachers work together?

We Are All in This Together: Collective Responsibility for Improving Pre-K–College Math Education

If we care about the pipeline of students going into math and about the strength of our profession in the future, then we simply must take the whole system of mathematics education into account. Students arrive at college with a long history of learning math, and that history affects their initial choices of math in college and their attitudes toward math as they enter their initial college math classes. These initial classes, together with a student's mathematical background, affect a student's decision to take further math classes or not, and they affect whether the student decides to become a math teacher. This means that all of us who teach math, pre-K teachers, elementary school teachers, middle school teachers, high school teachers, college teachers—all of us—must think collectively and systemically about improving our system.

Think about this: if you teach a calculus course, some of your students may go on to become teachers who will teach high school, middle school, or elementary school students. These students' experiences in your math class inform them about what math is and how it's done. Do your students view explaining ideas and making sense of lines of reasoning as an important part of math? Or do they see math as plowing through a large volume of stuff that doesn't make sense? Students' experiences and views—not just your intentions—will inform their future teaching if they become teachers. So whether you want to be involved in pre-K–12 mathematics education or not, if you teach math to college students, you *are* involved in pre-K–12 mathematics education because some of your students might someday become teachers.

If we—mathematicians, mathematics educators, and teachers—are the community that is responsible for improving the mathematics education of all students, then we all bear *collective* as well as *individual* responsibility for improvement of the mathematics education system as a whole. Individually, we are responsible for constantly seeking to improve our own teaching. Collectively, enough of us must work together to cause the community as a whole to move along a path of constant improvement.

But here is something puzzling: why is it that our system of doing research promotes vigorous activity and striving for excellence, whereas at no level of teaching, from pre-K through the graduate level, do we have such a system? In research, we have a system of publication, presentation, and peer review in which we build on each other's ideas and constantly strive to move the field forward. The acts of publishing and presenting research findings are public activities, and because these activities are filtered by a peer review system, they allow us to compete for each other's admiration, and thus they provide us with an incentive to think hard about our work and to keep trying to improve it.

Wouldn't it be wonderful if teaching were a public activity, in the way that research is, in which we build on other people's good ideas and compete for each other's admiration? Wouldn't it be great if all of us who teach math were to take pride in the things we know well and yet at the same time be humble, expect to learn more, and recognize that in each one of us, knowledge, skill, and insightfulness coexist with gaps and areas that need improvement? I think it would be truly exciting to have a vibrant community of math teachers at all levels—the community of math teachers from prekindergarten through graduate school—thinking together about mathematics teaching and spurring each other on to do better and better work for the sake of all of our students.

Acknowledgments

I would like to thank Michael Ching, Pete Clark, and Mark Saul for commenting on earlier drafts.