The Mis-Education of Mathematics Teachers

H. Wu

If we want to produce good French teachers in schools, should we require them to learn Latin in college but not French? After all, Latin is the mother language of French and is linguistically more complex than French; by mastering a more complex language teachers could enhance their understanding of the French they already know from their school days. To correlate their knowledge of French with their students' achievements, we could look at their grades in Latin!

As ridiculous as this scenario is, its exact analogue in mathematics education turns out to be central to an understanding of the field as of 2011. A natural question is why the mathematics research community should be bothered with a problem in education. The answer is that the freshmen in our calculus classes year after year, and ultimately our math graduate students, are products of this educational philosophy. The purpose of this article is to alert the mathematics community to the urgent need of active participation in the education enterprise. It is a call for action. We will begin by reviewing the state of the mathematical education of teachers in the past four decades, and then give an indication of what needs to be done to improve teachers' content knowledge and why knowledgeable mathematicians' input is essential.

The Early Work of Begle

No one doubts that improvement in school mathematics education depends critically on having effective mathematics teachers in the classroom. The common notion that "you cannot teach what you don't know" underscores our need to produce teachers with a solid knowledge of mathematics. Yet the mathematics education establishment has not maintained a sharp focus on the professional development of both preservice and inservice teachers, in part because what-you-need-to-know turns out to be a contentious issue. It appears that educators[1] are content to let the mathematics community decide what secondary teachers should know and to deal only with the professional development of elementary teachers. In the case of the former, there is too much of the Latin-French syndrome. Mathematicians feed secondary teachers the kind of advanced mathematics that future math researchers should learn and expect the Intellectual Trickle-Down Theory to work overtime to give these teachers the mathematical content knowledge they need in the school classroom. In the case of elementary teachers, too often the quality of the mathematics they are taught leaves much to be desired: the negative evaluation, mostly by mathematicians, of the commonly used textbooks for elementary teachers ([NCTQ][2]) paints a dismal picture of how poorly elementary teachers are served.

A related issue, of course, is whether any correlation exists between mathematics teachers' content knowledge and student achievement. Among the early researchers who tried to establish this correlation was E. G. Begle, the director of SMSG (School Mathematics Study Group), the group that was most identified with the "New Math" of the period 1955–1975. In a 1972 study of 308 teachers of first-year high school algebra ([Begle 1972]), he gave both teachers and students multiple-choice tests to measure teachers' knowledge and student achievement gains.[3] Broadly speaking, he found "little empirical evidence to substantiate any claim that, for example, training in mathematics for

H. Wu is emeritus professor of mathematics at the University of California, Berkeley. His email address is `wu@math.berkeley.edu`.

[1] *I will use "educator" in this article to refer to the mathematics education faculty in universities.*

[2] *See pp. 34–37 and 76–81.*

[3] *Students were given a pretest and a posttest.*

mathematics teachers will have payoff in increased mathematics achievement for their students". Subsequently, he surveyed the empirical literature in mathematics education research and again confirmed that the available evidence did not support the belief that "the more one knows about one's subject, the more effective one can be as a teacher" (p. 51, [Begle1979]).

The 1972 work of Begle is best known for casting doubt on the relevance of mathematical content knowledge to the effectiveness of teaching, but a close examination of this report is extremely instructive. Begle administered two tests to teachers, one on the algebra of the real number system and the other on the level of the abstract algebra of groups, rings, and fields. Analysis of the results indicated to Begle that

> ...teacher understanding of modern algebra (groups, rings, and fields) has no significant correlation with student achievement in algebraic computation or in the understanding of ninth grade algebra. ...However, teacher understanding of the algebra of the real number system does have significant positive correlation with student achievement in the understanding of ninth grade algebra. (Page 8 of original text in [Begle 1972].)

From these findings, Begle arrived at the following two remarkable recommendations:

> The nonsignificant relationship between the teacher modern algebra scores and student achievement would suggest the recommendation that courses not directly relevant to the courses they will teach not be imposed on teachers. The small, but positive, correlation between teacher understanding of the real number system and student achievement in ninth grade algebra would lead to the recommendation that teachers should be provided with a solid understanding of the courses they are expected to teach... (ibid.).

It is to be regretted that Begle did not follow through with his own recommendations. Had that been done, there would have been no need for the present article to be written. Let us put this statement in context. Begle was dealing with high school teachers who are traditionally required to complete the equivalent of a major in mathematics. However, the requirements for math majors are designed mainly to enable them to succeed as mathematics graduate students and, for this very reason, are full of "courses not directly relevant to the courses [teachers] will teach" in the school classroom. Implicitly, Begle recognized back in 1972 a critical flaw in the preservice professional development of high school teachers, namely, they are fed information that doesn't directly help them with their work. In other words, we teach Latin to French teachers and hope that they will become proficient in teaching French. Begle's second recommendation hinted at his awareness of the complementary fact, namely that high school teachers *do* need courses that provide them with a solid understanding of what they teach.

Basic Criteria of Professional Development

Begle's work was carried on by others in the intervening years, notably by [Goldhaber-Brewer] and [Monk]. But the work that is most relevant to the present article is that of Deborah Ball, who some twenty years after Begle considered what teachers need to know about the mathematics of elementary school ([Ball]). Her survey of both elementary and secondary teachers showed that even teachers with a major in mathematics could not explain something as basic as the division of fractions (a basic topic in grades 5 and 6) in a way that is mathematically and pedagogically adequate. Her conclusion is that "the subject matter preparation of teachers is rarely the focus of any phase of teacher education" (p. 465, [Ball]).

A few years later, as a result of my work with the California Mathematics Project (cf. [Wu1999c]), I became alarmed by the deficiency of mathematics teachers' content knowledge and argued on theoretical grounds that improvement must be sought in the way universities teach prospective mathematics teachers ([Wu1999a], [Wu1999b]).[4] The conclusions I arrived at are entirely consistent with those of Begle and Ball, and a slightly sharpened version may be stated as follows. To help teachers teach effectively, we must provide them with a body of mathematical knowledge that satisfies both of the following conditions:

> (A) It is **relevant to teaching**, i.e., does not stray far from the material they teach in school.
> (B) It is consistent with the **fundamental principles of mathematics**.

The rest of this article will amplify on these two statements.

Three Examples

The almost contradictory demands of these two considerations on professional development is illustrated nowhere better than in the teaching of fractions in school mathematics. Although

[4] *I wish I could say I was aware of the work of Begle and Ball at the time that those articles were written, but I can't.*

fractions are sometimes taught as early as second grade nowadays, the most substantial instruction occurs mainly in grades 5–7, and students' difficulty with learning fractions in these grades is part of American folklore.[5] We will henceforth concentrate on fractions in grades 5–7.

Mathematicians who have a dim memory of their K–12 days may of course wonder why the teachers of these grades must be provided with a knowledge of fractions that is *relevant* to the school classroom. What is so hard about equivalence classes of ordered pairs of integers? Let us recall how fractions are taught in university mathematics courses. As usual, let $\mathbb{Z}$ be the integers, and let S be the subset of ordered pairs of integers $\mathbb{Z} \times \mathbb{Z}$ consisting of all the elements (x, y) so that $y \neq 0$. Introduce an equivalence relation $\sim$ in S by defining $(x, y) \sim (z, w)$ if $xw = yz$. Denoting the equivalence class of (x, y) in S by $\frac{x}{y}$, we call the set of all such $\frac{x}{y}$ the rational numbers $\mathbb{Q}$. Identify $\mathbb{Z}$ with the set of all elements of the form $\frac{x}{1}$, and we have $\mathbb{Z} \subset \mathbb{Q}$. Finally, we convert $\mathbb{Q}$ into a ring by *defining* addition and multiplication in $\mathbb{Q}$ as

$$\frac{x}{y} + \frac{z}{w} = \frac{xw + zy}{yw}, \quad \text{and} \quad \frac{x}{y} \cdot \frac{z}{w} = \frac{xz}{yw}.$$

Of course we routinely check the compatibility of these definitions with the equivalence relation. This is what we normally teach our math majors in two to three lectures; it is without a doubt consistent with the fundamental principles of mathematics. The question is: what could a teacher do with this information in grades 5–7? Probably nothing.

Let us analyze this definition a bit: it requires an understanding of the partition of S into equivalence classes and the ability to consider each equivalence class as one element. Acquiring such an understanding is a major step in the education of beginning math majors. In addition, understanding the identification of $\mathbb{Z}$ with $\{\frac{x}{1} : x \in \mathbb{Z}\}$, or as we say, the injective homomorphism of $\mathbb{Z}$ into $\mathbb{Q}$, requires another level of sophistication.

Surely very little of the preceding discussion is comprehensible to students of ages 10–12, but even more problematic are the definitions of addition and multiplication of rational numbers. For example, consider multiplication once addition has been defined. The definition $\frac{x}{y} \cdot \frac{z}{w} = \frac{xz}{yw}$ makes sense to us because we *want* to introduce a ring structure in $\mathbb{Q}$ and this is the most obvious way to make it work. But can we explain to an average pre-teenager that rings are important and that therefore this definition of multiplication is the *right* definition? If so, what is wrong with defining $\frac{x}{y} + \frac{z}{w}$ as $\frac{x+z}{y+w}$ in accordance with every school student's dream?

[5] *If in doubt, look up* Peanuts *and* FoxTrot *comic strips.*

Because schools were in existence before the introduction of fractions in the 1930s as equivalence classes of ordered pairs of integers, and because fractions have been taught in schools from the beginning, it is a foregone conclusion that some version of fractions has been taught to teachers for a long time. But this version of mathematics makes no pretense of teaching *mathematics*. At least in this case, the relevance to the school classroom has been achieved at an unconscionable cost, namely at the expense of the fundamental principles of mathematics.

Mathematics depends on precise and literal *definitions*, but the way fractions are taught to elementary teachers has almost *no* definitions. The following is a typical example. A fraction is presented as three things all at once: it is a **part of a whole**, it is a **ratio**, and it is a **division**. Thus $\frac{3}{4}$ is 3 parts when the whole is divided into 4 equal parts. Because it is not clear what a "whole" is, the education literature generally resorts to metaphors. Thus a prototypical "whole" is *like* a pizza. Now do we divide a pizza into 4 equal parts according to shape? Weight? Or is it area? The education literature doesn't say. And how to multiply or divide two pieces of pizza? (See [Hart].) As to a fraction being a ratio, $\frac{3}{4}$ can represent a "ratio situation", as 3 boys for every 4 girls. What is the logical connection of boys and girls to pizzas? The education literature is again silent on this point, except to make it clear that every fifth grader had better acquire such a conceptual understanding of a fraction, namely that it can be two things simultaneously. Finally, the fraction $\frac{3}{4}$ is also "3 divided by 4". Now there are *many* things wrong with this statement, foremost being the fact that when students approach fractions, they are either in the process of learning about division of whole numbers or just coming out of it. In the latter situation, they understand $m \div n$ (for whole numbers m and n, $n \neq 0$) to be a partition into equal groups or as a measurement *only when m is a multiple of n*. If m is not a multiple of n, then students learn about division-with-remainder, in which case $m \div n$ yields *two* numbers, namely, the quotient and the remainder. The concept of a *single* number $3 \div 4$ is therefore entirely new to a student trying to learn fractions, and to define $\frac{3}{4}$ in terms of $3 \div 4$ is thus a shocking travesty of mathematics. What *is* true is that, when "part of a whole" is suitably defined and when $m \div n$ is also suitably defined for arbitrary whole numbers m and n ($n \neq 0$), it is a provable **theorem** that, indeed, $\frac{m}{n} = m \div n$. Yet, there is no mention of this fact in the education literature, and such *absence of reasoning* pervades almost all such presentations of fractions.

As a result of this kind of professional development, a typical elementary teacher asks her students to believe that there is a mysterious quantity called *fraction* that possesses three totally unrelated properties and then also asks them to compute with this mysterious quantity in equally mysterious ways. To add two fractions, take their *least common denominator* and then do some unusual things with the numerators to get the sum.[6] Of why, and of how, this concept of addition is related to the concept of adding whole numbers as "combining things", there is no explanation at all.

Recall that we are here discussing the *mathematical* education of elementary teachers. We have to teach them mathematics so that, with proper pedagogical modifications, they can teach it to primary students, and so that, with essentially no modification, they can teach it to students in the upper elementary grades. So how can a teacher teach the addition of fractions in grades 5–7? If a fraction $\frac{a}{b}$ is defined to be a point on the number line, then the sum of two fractions $\frac{a}{b}$ and $\frac{c}{d}$ is, by definition, the total length of the two intervals $[0, \frac{a}{b}]$ and $[0, \frac{c}{d}]$ joined end-to-end—just as is the case of the sum of two whole numbers. In this way, adding fractions is "combining things" again. A simple reasoning then gives $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$. See, for example, pp. 46–49 of [Wu2002].

Next, consider division. The rote teaching of the division of fractions is a good example of the total neglect of the fundamental principles of mathematics, and it has inspired the jingle, "Ours is not to reason why, just invert and multiply." One recent response to such rote teaching is to imitate division between whole numbers by teaching the division of fractions as *repeated subtraction*. Unfortunately, the concept of division in a field cannot be equated with the division algorithm in a Euclidean domain, and the reaction against a defective mathematical practice has resulted in the introduction of another defective mathematical practice. Such a turn of events seems to be typical of the state of school mathematics education in recent times.

In any intellectual endeavor, a crisis of this nature naturally calls for research and the infusion of new ideas for a resolution. What is at present missing is the kind of education research that addresses students' cognitive development without sacrificing precise definitions, reasoning, and mathematical coherence in the teaching of fractions (see pp. 33–38 in [Wu2008a] for a brief discussion of the research literature).

To improve on fraction instruction in schools, we first need to produce school textbooks that present a mathematically coherent way of approaching the subject, one that proceeds by reason rather than by decree. Several experiments along this line were tried in the past two decades, but let us just say that, from the present perspective, they were not successes. An easier task would be to produce professional development materials for elementary teachers that are sufficiently elementary for students in grades 5–7. This would require a presentation of the mathematics of fractions different from the mathematically incoherent one described above. One way that has been thoroughly worked out is to define a fraction, in an explicit manner, as a point on the number line ([Jensen] and [Wu2002]). It does not matter whether teachers are taught this or possibly other approaches to fractions for school students; the important thing is that teachers *are* taught some version that is valid in the sense of conditions (A) and (B) above so that they can teach it in the school classroom. *It is simply not realistic to expect teachers to develop by themselves the kind of knowledge that satisfies (A) and (B).*

Two additional comments on fractions will further illuminate why we need to specifically address the special knowledge for teaching. At present, a major stumbling block in the learning path of school students is the fact that fractions are taught as *different numbers* from whole numbers. For example, it is believed that "Children must adopt new rules for fractions that often conflict with well-established ideas about whole numbers" ([Bezuk-Cramer], p. 156). The rules here presumably refer to the rules of arithmetic; if so, we can say categorically that there is a complete parallel between these two sets of rules for whole numbers and fractions; the similarity in question is a main point of emphasis in [Wu2002]. If mathematicians who take for granted that $\mathbb{Z}$ is a subring of $\mathbb{Q}$ are surprised by this misconception about fractions and whole numbers, they would do well to ask at which point of teachers' education in K–16 (or, for that matter, a teacher's education, period) they would get an *explicit* understanding of this basic algebraic fact. The unfortunate answer is probably "nowhere", because until the last two years in college, mathematics courses are traditionally more about techniques than ideas, and even for those junior- and senior-level courses, our usual mode of instruction often allows the ideas to be overwhelmed by procedures and formalism (cf. [Wu1999a]). It should be an achievable goal for all teachers to acquire an understanding of the structural similarity between $\mathbb{Z}$ and $\mathbb{Q}$ so that they can teach fractions by emphasizing the similarity rather than the difference between whole numbers and fractions.

A second comment is that school mathematics is built on $\mathbb{Q}$ (the rationals) and not on $\mathbb{R}$ (the reals). $\mathbb{Q}$ is everything in K–12, while $\mathbb{R}$ appears only as a

[6] *This is tantamount to saying that addition cannot be defined in the quotient field of a domain unless the latter is something like a UFD.*

pale shadow.[7] It is this fact that accounts for the need to teach fractions well. We hope all teachers are aware of the dominance of $\mathbb{Q}$ in their day-to-day work,[8] but few are, for the simple reason that we have never brought it to their attention.

In terms of the nitty-gritty of classroom instruction, real numbers are handled in K–12 by what is called the **Fundamental Assumption of School Mathematics** (FASM; see p. 101 of [Wu2002] and p. 62 of [Wu2008b]). It states that any formula or weak inequality that is valid for all rational numbers is also valid for all real numbers.[9] For example, in the seventh grade, let us say, the formula for the addition of fractions,

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd},$$

where a, b, c, d are whole numbers, can be (and should be) proved to be valid when a, b, c, d are rational numbers. By FASM, the formula is also valid for all real numbers a, b, c, d. Thus high school students can write, without blinking an eye, that

$$\frac{1}{\sqrt{2}} + \frac{2}{\sqrt{3}} = \frac{\sqrt{3} + 2\sqrt{2}}{\sqrt{2}\sqrt{3}},$$

even if they know nothing about what $1/\sqrt{2}$ or $\sqrt{2}\sqrt{3}$ means. If this seems a little cut-and-dried and irrelevant, consider the useful identity

$$\frac{1}{1-x} + \frac{1}{1+x} = \frac{2}{1-x^2} \quad \text{for all real numbers } x.$$

If x is rational, this identity is easily verified (see preceding addition formula). But the identity implies also

$$\frac{1}{1-\pi} + \frac{1}{1+\pi} = \frac{2}{1-\pi^2}.$$

Without FASM, there is no way to confirm this equality in K–12, so its validity is entirely an article of faith in school mathematics.

As a final example, let a be any positive number $\neq 1$. Then for all rational numbers $\frac{m}{n}$ and $\frac{p}{q}$, the following law of exponents for rational exponents can be verified (even if the proof is tedious):

$$a^{m/n} \cdot a^{p/q} = a^{m/n + p/q}.$$

Now, FASM implies that we may assume that the following identity holds for all real numbers s and t:

$$a^s \cdot a^t = a^{s+t}.$$

Of course, school mathematics cannot make sense of any of the numbers a^s, a^t, and a^{s+t} when s and t are irrational, much less explain why this equality is valid. Nevertheless, this equality is of more than purely academic interest because it is needed to describe a basic property of the exponential *function* $a^x : \mathbb{R} \to (0, \infty)$.

The preceding discussion brings out the fact that any discussion in high school mathematics is bound to be full of holes, and FASM is needed to fill in those holes. We would like to believe that FASM is a basic part of the professional development of mathematics teachers. Yet, to our knowledge, FASM has never been part of such professional development, with the result that schoolteachers are forced to fake their way through the awkward transition from fractions to real numbers in middle school. It is difficult to believe that, when teachers make a habit of blurring the distinction between what is known and what is not, their teaching can be wholly beneficial to the students. There is definitely room for improvement in our education of mathematics teachers.

Another illustration of the difference between the teaching of mathematics to the average university student and to prospective teachers is the concept of *constant speed*. Consider the following staple problem in fifth or sixth grade:

> If Ina can walk $3\frac{2}{5}$ miles in 90 minutes, how long would it take her to walk half a mile?

A common solution is to *set up a proportion*: Suppose it takes Ina x minutes to walk half a mile; then proportional reasoning shows that "the distances are to each other as the times". Therefore $3\frac{2}{5}$ is to $\frac{1}{2}$ as 90 is to x. So

$$\frac{3\frac{2}{5}}{\frac{1}{2}} = \frac{90}{x}.$$

By the cross-multiplication algorithm:

$$3\frac{2}{5} \cdot x = \frac{1}{2} \cdot 90, \quad \text{so that} \quad x = 13\frac{4}{17} \text{ minutes.}$$

The answer is undoubtedly correct, but what is the reasoning behind the setting up of a proportion? This rote procedure cannot be explained because *the assumption that makes possible the explanation has been suppressed*, the fact that Ina walks at a constant speed. As we know, if there is no assumption, then there is no deduction either. It therefore comes to pass that problem solving in this case is reduced to the rote procedure of setting up a proportion.

How did school mathematics get to the point that "constant speed" is not even mentioned or, if mentioned, is not explained in the school classroom? It comes back to the issue of how we educate our teachers. The only time university mathematics deals with constant speed is in calculus, where a motion along a line $f(t)$ describing the distance from a fixed point as a function of time t is said to have *constant speed* if its derivative $f'(t)$ is a constant. There are teachers who don't take calculus, of course, but even those who do

[7] *This fundamental fact seems to have escaped Begle, as evidenced by his tests for teachers ([Begle1972]).*

[8] *Better yet, one hopes that all state and national standards reflect an awareness of this fact as well, but that is just a forlorn hope.*

[9] *A trivial consequence of continuity and the density of* $\mathbb{Q}$ *in* $\mathbb{R}$.

will see constant speed as a calculus concept and nothing else. Because we do not see fit to help prospective teachers relate university mathematics to school mathematics, such a misconception about constant speed will remain with them. In the school classroom, they realize that there is no place for the derivative and therefore conclude that it is impossible to discuss constant speed. Once this realization sets in, they fall back on what they learned as students in K–12, which is not to talk about constant speed at all. So the tradition continues, not just in the classroom instruction but also in textbooks.

Having taken calculus is usually considered a badge of honor among middle and elementary school teachers, and some professional development programs go out of their way to include calculus exactly for this reason. The example of constant speed is but one of the innumerable reasons why having taken a standard calculus course does not ensure a teacher's effectiveness in the school classroom.[10]

Professional development of teachers ideally should include the instruction that in the school curriculum the concept of *speed* is too subtle to be made precise, but that one should use instead the concept of **average speed in a time interval** $[t, t']$, which is the quotient

$$\frac{\text{(the distance traveled from time } t \text{ to time } t')}{(t' - t)}.$$

A motion is said to have **constant speed** K if, for *every* time interval $[t, t']$, the average speed is always equal to K, i.e.,

$$\frac{\text{(the distance traveled from time } t \text{ to time } t')}{(t' - t)} = K.$$

Once this concept is introduced, the setting up of a proportion in the preceding example can be explained *provided Ina is assumed to walk at a constant speed.* For then her average speeds in the two time intervals $[0, 90]$ and $[0, x]$ are the same, and therefore

$$\frac{3\frac{2}{5}}{90} = \frac{\frac{1}{2}}{x},$$

and this equality is equivalent to the proportion above.

Of course school students would find it difficult to grasp the idea that the average speed in *every* time interval is a fixed number, and education researchers should consider how to lighten the attendant cognitive load. But that is a different story. Our concern here is whether prospective teachers are taught what they need to know in order to carry out their duties, and once again we see the gulf that separates what is mathematically correct in a university setting from what is pedagogically feasible in a school classroom. What is needed to bridge this gulf is the concept of *customizing abstract mathematics for use in the school classroom.* This is the essence of mathematics education (see [Wu2006] for a full discussion). In this case, it is a matter of taking apart the concept of the constancy of the derivative of a function and reconstructing it so that it makes sense to school students.

As a final example to illustrate the chasm between what we teach teachers and what they need to know, consider the fundamental concepts of congruence and similarity in geometry. The gaps in our teachers' knowledge of these two concepts are reflected in the existing school geometry curricula. For example:

> (i) In middle school, two figures (not necessarily polygons) are defined to be *congruent* if they have the same size and same shape and to be *similar* if they have the same shape but not necessarily the same size. In high school, congruence and similarity are defined in terms of angles and sides, but only for polygons. There is no attempt to reconcile the more precise definitions in high school with the general ones in middle school.
>
> (ii) In middle school, the purpose of learning about congruence is to perceive the inherent symmetries in nature as well as in artistic designs such as Escher's prints, tessellations, and mosaic art. Likewise, the purpose of learning about similarity is to engage in fun activities about enlarging pictures. In high school, students prove theorems about congruent and similar triangles in a geometry course but otherwise never again encounter these concepts in another course in school mathematics.
>
> (iii) Because similarity is more general than congruence and because two figures are more likely to be similar than congruent, some curricula ask teachers to teach similarity before congruence in middle school.[11]

As a result of the neglect by universities, our teachers' conception of congruence and similarity is largely as fragmented and incoherent as the practices described in (i)–(iii) above. Not every

[10] *Calculus is by definition, as well as by design, a technique-oriented subject.*

[11] *It is possible to define similarity as a bijection of the plane that changes distance of any two points by a fixed scale factor k and to define a congruence as the case of $k = 1$. This approach is, however, basically impossible to bring off in a school classroom.*

school geometry curriculum is guilty of all three, but most are guilty of the first two. So long as university mathematics courses do not address issues arising from school mathematics, teachers will not be sufficiently well informed to reject such mathematical illiteracy, and publishers will continue to get away with the promotion of this kind of illiteracy. We must create a university mathematics curriculum for prospective teachers to help them *look back* at such school concerns as the meaning of congruence and similarity and why these concepts are important in mathematics. By contrast, preservice teachers are given at least some access to such topics as the curvature of curves, Gaussian curvature of surfaces, finite geometry, projective geometry, non-Euclidean geometries, and the foundations of geometry. They are *not*, however, taught plane Euclidean geometry. This last is exactly what teachers need because it is usually taught poorly in schools. They desperately need solid information about school geometry in order to better teach their own geometry classes.

Thus we see in this case the same scenario that we saw with fractions played all over again: mangled definitions, critical gaps in mathematical reasoning, and insufficient attention to mathematical coherence; above all, students are given no purpose for learning these concepts except for fun, for art appreciation, or for the learning of boring geometric proofs.

However, we should not accept these results of years of neglect as immutable, because there are ways to make mathematical sense of school geometry and, in particular, of congruence and similarity. We can begin with the instructions on the *basic rigid motions* of the plane (translations, rotations, and reflections) more or less informally by the use of hands-on activities; after all, one has to accept the fact that the concept of a transformation is difficult for students, and it won't do to insist on too much formalism at the outset. We can do the same with the concept of a *dilation* from a point (i.e., central projection of a fixed scale factor from that point). Then we can define **congruence** as a finite composition of basic rigid motions and **similarity** as the composition of a dilation and a congruence. But, as in all things mathematical, precision is not pursued for its own sake. In the present situation, students can now make direct use of translations, rotations, and reflections to prove the congruence of segments and angles; such proofs are far more intuitive than those using the traditional criteria of ASA, SAS, and SSS. In addition, it is a rather simple exercise to *assume* the abundant existence of basic rigid motions in the plane in order to prove all the usual theorems in Euclidean geometry, including those on similar triangles (cf. [CCSS] and Chapter 11 in Volume II of [Wu2011b]). The requirement of "invariance under congruence" in such a mathematical development further highlights the fundamental role of congruence in the definitions of length, area, and volume (cf. Chapter 7 of [Wu2010] and Chapter 18 in Volume III of [Wu2011b]). This is one way to make teachers aware of what congruence and similarity are and why they are part of the basic fabric of mathematics itself.

In advanced mathematics, the basic rigid motions of Euclidean n-space $\mathbb{R}^n$ are defined in terms of orthogonal transformations and coordinates, and a dilation is also defined in terms of coordinates. Here we use rigid motions and dilations instead as the basic building blocks of geometry in order to define coordinate systems in the plane in a way that is usable in middle and high schools. This is another example of the *customization* of abstract mathematics for use in schools.

Such content knowledge for mathematics teachers is not yet standard fare in preservice professional development, but it should be.

The Role of Mathematicians

The three mathematical examples above indicate what needs to be done to customize abstract mathematics for use in the K–12 classroom, but they are only the tip of the iceberg. Almost the entire K–12 curriculum needs careful revamping in order to meet the *minimum* standards of mathematics, and this kind of work calls for input by mathematicians. The mathematical defects of the present curriculum are, in my opinion, too pronounced to be undone by people outside of mathematics. Research mathematicians have their work cut out for them: consult with education colleagues, help design new mathematics courses for teachers, teach those courses, and offer constructive criticisms in every phase of this reorientation in preservice professional development. My own systematic attempt to address the problem is given in [Wu2011a] (for elementary school teachers) and in [Wu2011b] (for high school teachers); a third volume for middle school teachers will include [Wu2010]).[12]

For those who don't care about the details, an outline of what is possible for the K–8 curriculum can be found in [Wu2008b]. Such an outline also appears in [MET], which was written in 2001 to give guidance on the mathematical education of math teachers to university math departments. Its main point was to bring research mathematicians into the discussion of mathematics education. Although others may disagree with me, my own opinion is that its language is not one that speaks persuasively to mathematicians and that the mathematics therein

[12] *[Wu2011b] is the text for the sequence of three-semester courses, Mathematics of the Secondary School Curriculum, which is required of all math majors at UC Berkeley with a teaching concentration.*

fails to respect the fundamental principles of mathematics more often than it should.

To (research) mathematicians, the mathematics of K–12 obviously holds no mystery. If they have to develop the whole body of knowledge *ab initio*, strictly as mathematics, they can do it with ease. But if they hope to make the exposition speak to the teachers, then they will have to spend time to learn about the school classroom. If one mathematician's experience is to be trusted, the pedagogical pitfalls of such an undertaking can be avoided only if mathematicians *can get substantive input about the K–12 classroom.* For starters, one can go to the local school district office to look at the textbooks being used; reading them should be an eye-opener. One should also try to talk to inservice teachers about their experiences and their students' learning difficulties; make an effort to visit a school classroom if possible. But the ultimate test is, of course, to get to teach (inservice or preservice) teachers *the mathematics of K–12* and solicit honest feedback about their reactions. If the mathematics department and the school of education on campus are on good terms, then the whole process of getting in touch with teachers can be expedited with the help of one's education colleagues.

There is another crucial contribution that research mathematicians can make, one that seems to be insufficiently emphasized in education discussions up to this point. In their routine grappling with new ideas, mathematicians need to know, for survival if nothing else, the intuitive meaning of a concept perhaps not yet precisely formulated and the motivation behind the creation of a particular skill and to have a vague understanding of the direction they have to pursue. These needs completely parallel those of students in their initial attempt to learn something new. This part of a research mathematician's knowledge would surely shed light on students' learning processes. Here, then, is another important resource that should not go to waste in our attempt to help teachers and educators better understand teaching.

The Fundamental Principles of Mathematics

Having invoked the "fundamental principles of mathematics" several times throughout this article, I will now summarize and make explicit what they are and why they are important. I believe there are at least five of them. They are interrelated and, to the extent that they are routinely violated in school textbooks and in the school education literature (to be explained below), teachers have to be aware of them if they hope to teach well.

(1) *Every concept is precisely defined, and **definitions** furnish the basis for logical deductions.* At the moment, the neglect of definitions in school mathematics has reached the point at which many teachers no longer know the difference between a definition and a theorem. The general perception among teachers is that a definition is "one more thing to memorize". We have already pointed out that the concepts of a fraction, constant speed, congruence, and similarity are in general not defined in the school mathematics education literature. It is sobering to point out that many more bread-and-butter concepts of K–12 mathematics are also not correctly defined or, if defined, are not put to use as an integral part of reasoning. These include: number, rational number (in middle school), decimal (as a fraction in upper elementary school), ordering of fractions, length-area-volume (for different grade levels), slope of a line, half-plane of a line, equation, graph of an equation, inequality between functions, rational exponents of a positive number, and polynomial.

(2) *Mathematical statements are **precise**. At any moment, it is clear what is known and what is not known.* Yet there are too many places in school mathematics in which textbooks and education materials fudge the boundary between what is true and what is not. Often a heuristic argument is conflated with correct logical reasoning. For example, the identity $\sqrt{a}\sqrt{b} = \sqrt{ab}$ for positive numbers a and b is often explained by assigning a few specific values to a and b and then checking for these values by a calculator. (For other examples, see pp. 3–5 of [Wu1998].) Sometimes the lack of precision comes from an abuse of notation or terminology, such as using $25 \div 6 = 4\ R\ 1$ to express "25 divided by 6 has quotient equal to 4 and remainder 1" (this is an equality of neither two whole numbers nor two fractions). At other times an implicit assumption is made but is not brought to the fore; perhaps the absence of any explicit statement about FASM is the most obvious example of this kind of transgression.

(3) *Every assertion can be backed by logical **reasoning**.* Reasoning is the lifeblood of mathematics and the platform that launches problem solving. Given the too frequent absence of reasoning in school mathematics (cf. the discussion of fractions and constant speed above), how can we ask students to solve problems if teachers do not have the ability to engage students in logical reasoning on a consistent basis?

(4) *Mathematics is **coherent**; it is a tapestry in which all the concepts and skills are logically interwoven to form a single piece.* The professional development of math teachers usually emphasizes either procedures (in days of yore) or intuition (in modern times) but not the coherence (structure) of mathematics. The last may be the one aspect of mathematics that most teachers (and dare I say also educators) find most elusive. The lack of awareness

of the coherence of the number systems in K–12[13] may account for teaching fractions as "different from" whole numbers (so that the learning of whole numbers becomes almost divorced from the learning of fractions). We mentioned earlier an example of curricular incoherence when similarity is discussed before congruence. A more common example is the almost universal "proof" of the theorem on *equivalent fractions*, which states: *For all fractions* $\frac{m}{n}$ *and for any nonzero whole number* c,

$$\frac{m}{n} = \frac{cm}{cn}.$$

The "proof" in question goes as follows:

$$\frac{m}{n} = \frac{m}{n} \times 1 = \frac{m}{n} \times \frac{c}{c} = \frac{cm}{cn}.$$

The problem with this argument is that this theorem must be proved essentially as soon as a fraction is defined, but multiplication of fractions, the most sophisticated of the four arithmetic operations on fractions,[14] comes much later in the usual development of fractions.

The coherence of mathematics includes (but of course is not limited to) the sequential development of concepts and theorems; the progression from the logically simple to the logically complex cannot be subverted at will. However, for people who have not been immersed in mathematics systematically and for a long time, it is almost impossible to resist the temptation to subvert this sequential development. The two preceding examples testify eloquently to this fact.

(5) *Mathematics is goal-oriented, and every concept or skill in the standard curriculum is there for a* ***purpose***. Teachers who recognize the purposefulness of mathematics gain an extra tool to make their lessons more compelling. When congruence and similarity are taught with no mathematical purpose except to do "fun activities", students lose sight of the mathematics and wonder why they were made to learn it.[15] When students see the technique of completing the square merely as a trick to get the quadratic formula rather than as the central idea underlying the study of quadratic functions, their understanding of the technique is superficial. But perhaps the most telling example of teaching mathematics without a purpose is teaching students by rote to round off whole numbers, to the nearest hundreds or to the nearest thousands, *without* telling them why it is useful (cf. section 10.3 of [Wu2011a]). Most elementary students consider rounding a completely useless skill that is needed only for exams. If teachers can put rounding off in the context of the how and the why of estimations, they are likely to achieve better results.

The Mathematics Teachers Need to Know

I hope that this discussion of the fundamental principles of mathematics convinces the reader that there is substantive mathematics about the K–12 curriculum that a teacher must learn. This body of knowledge may be elementary, but it is by no means trivial, in the same sense that the theory behind the laptop computer may be elementary (just nineteenth-century electromagnetic theory as of 2000) but decidedly not trivial. This discussion in fact strongly bears on the central question of the moment in mathematics education: *exactly* what kind of content knowledge for teachers would lead to improved student achievement? (Cf. Begle's work, mentioned at the beginning of this article.) Although research evidence on this issue is lacking, it is not needed as a first step toward a better mathematics education for teachers. For whatever this knowledge may be, it must *include* the mathematics of the school curriculum presented in a way that is consistent with the fundamental principles of mathematics. Let me be as explicit as I can: I am not making any extravagant claims about the advanced mathematics teachers need to know or even whether they need to know advanced mathematics, only that *they must* ***know*** *the content of what they teach to their students.* Here I am using the word "know" in the unambiguous sense that mathematicians understand this term:[16] *knowing* a concept means knowing its precise definition, its intuitive content, why it is needed, and in what contexts it plays a role, and *knowing* a technique[17] means knowing its precise statement, when it is appropriate to apply it, how to prove that it is correct, the motivation for its creation, and, of course, the ability to use it correctly in diverse situations. In this unambiguous sense, teachers cannot claim to know the mathematics of a particular grade without also knowing a substantial amount of the mathematics of three or four grades *before* and *after* the grade in question (see Recommendation 19 of [NMP1]). This necessity that math teachers actually *know* the mathematics

[13] *Whole numbers, integers, fractions, rational numbers, real numbers, and complex numbers.*

[14] *The sophistication comes from the fact that at least three things must be explained about* $m/n \times k/\ell$ *before it can be effectively used by students: (1) it is the area of a rectangle of sides* m/n *and* k/ℓ*, (2) it is the number that is the totality of* m *parts when* k/ℓ *is partitioned into* n *equal parts, and (3) it is equal to* $(mk)/(n\ell)$*. Either (1) or (2) can be used as the definition of* $m/n \times k/\ell$ *and the other will have to be proved, and then the seductive formula (3) must also be proved. Too often, the deceptive simplicity of (3) is the siren song that causes many shipwrecks in the teaching of fraction multiplication.*

[15] *At least according to math majors I have taught at Berkeley.*

[16] *Educators usually use the word "know" in its literal sense: being able to memorize a fact, a definition, or a procedure.*

[17] *Usually referred to as "skill" in the education literature.*

they teach sheds light, in particular, on why we want all high school teachers to know some abstract algebra: this knowledge allows them to really understand why there are only two arithmetic operations (+ and ×) instead of four, in what way the rational functions are similar to rational numbers, and that the axiomatic system they encounter in geometry[18] is part of a universal practice in mathematics. The necessity that teachers know the mathematics they teach also explains why we want all teachers of high school calculus to know some analysis rather than just lower-division calculus.

At the moment, most of our teachers do not know the materials of the three grades above and below what they teach, because our education system has not seen to it that they do. We have the obligation to correct this oversight.

Content Knowledge and Pedagogical Knowledge

The title of this article is about the education of mathematics teachers, but we have talked thus far only about learning mathematics, not about the methodology of teaching it. While knowing mathematics is undoubtedly necessary for a teacher to be effective, it is clearly not sufficient. For example, while we want all teachers to know precise definitions and their role in the development of mathematical skills and ideas, we do not wish to suggest that they teach school mathematics in the definition-theorem-proof style of graduate mathematics courses. The fact remains, however, that the more teachers know about a definition (the historical need it fulfills, why a particular formulation is favored, what ramifications it has, etc.), the more likely it is that they can make it accessible to their students. The same comment applies to every one of the fundamental principles of mathematics.

This then brings up the tension that exists at present between some mathematicians' perception of the most urgent task in a mathematics teacher's education and some educators' perception of the same. Mathematicians tend to believe that, because the most difficult step in mathematics teachers' education is to learn the necessary mathematics, giving them this knowledge is the number one priority in professional development. Quite understandably, some educators believe that the really hard work lies in the pedagogical part of the education that channels the teacher's content knowledge into the school classroom. As this theory goes, teachers learn the mathematics better if it is taught hand in hand with pedagogy. The main point of these conflicting perceptions—whether learning the pedagogy or learning the mathematics is more difficult to achieve—can at some point be resolved by a large-scale study to see whether it is a lack of genuine understanding of content knowledge or weak pedagogical skills that contribute more to student nonlearning in the classroom.[19] In the meantime, some small-scale studies, e.g., [Ball] and [Ma], indicate that teachers' lack of content knowledge is the more severe problem. The available anecdotal evidence points in the same direction.

My personal experience, from having taught elementary and middle school teachers for eleven summers in four states (sometimes more than once in a given year) and having taught prospective high school teachers for four years at Berkeley, is that, in an overwhelming majority of cases, their mathematical preparation leaves a lot to be desired.[20] It is also the case that even when I inject pedagogical issues into my teaching from time to time, the teachers are usually so preoccupied with learning the mathematics that the pedagogical discussion hardly ever takes place. Some standard statistics, such as those in *A Nation at Risk* (see "Findings Regarding Teaching" in [NAR]), are consistent with this overall picture. It is for this reason that I have focused exclusively in this article on teachers' content knowledge.

This discussion of content knowledge should be put in the context of Lee Shulman's 1985 address ([Shulman]) on pedagogical content knowledge, i.e., the kind of pedagogical knowledge *specific to the teaching of mathematics* that a math teacher needs in order to be effective. There are two things that need clarification in such a discussion: what this *mathematical content knowledge* is and what the *associated pedagogical knowledge* is. Deborah Ball and her colleagues have recently begun to codify both kinds of knowledge in their attempt to reform math teachers' education (cf. [Ball-TP2008]). What must not be left unsaid is the obvious fact that, without a solid mathematical knowledge base, it is futile to talk about pedagogical content knowledge.

The Need for Inservice Professional Development

At the beginning of this article, I mentioned the disheartening results of Deborah Ball's survey of teachers on their understanding of fraction division ([Ball]). I would venture a guess that, had her teachers been taught *the mathematics of K–12* in a way that respects the five fundamental principles of mathematics, the results of the survey

[18] *This is not to be interpreted as an advocacy of teaching high school geometry by the use of axioms.*

[19] *For teachers in first and third grades, the large-scale study of [Hill-RB] found positive correlation between teachers' content knowledge and student achievement. So content knowledge is likely a major factor even at such an early stage of student learning.*

[20] *Again, see [Hill-RB].*

would have been far more satisfactory.[21] Until we improve on how we teach mathematics to teachers in the universities, defective mathematics will continue to be the rule of the day in our schools. It is time for us to break out of the vicious cycle by exposing teachers to a mathematically principled version of the mathematics taught in K–12.

Unfortunately, such short-term exposure in the university may not be enough to undo thirteen years of mis-education of prospective teachers in K–12. Uniform achievement in the content knowledge of all math teachers will thus require heavy investments by the state and federal governments in sustained inservice professional development. To this end we need inservice professional development that directly addresses content knowledge. Funding for such professional development, however, may be hard to get, for content knowledge does not seem to be a high-priority consideration among government agencies. For example, in a recent survey by Loveless, Henriques, and Kelly of winning proposals among the state-administered Mathematics Science Partnership (MSP) grants from forty-one states ([Loveless-HK]), it was found that: "Some of the MSPs appear to be offering sound professional development. Many, however, are vague in describing what teachers will learn." Typically, these "MSPs' professional development activities tip decisively towards pedagogy". For example, although the professional workshops described in [TAMS] were not part of the review in [Loveless-HK], they nevertheless fit the description of this review. The [TAMS] document begins with the promising statement that the "TAMS-style teacher training increases teachers' content knowledge". But other than mentioning "teacher workshops focused on data analysis and measurement.... Early grade teachers also studied length, area, and volume", the rest of the discussion of mathematics professional development focuses on persuading teachers to adopt "constructivist, inquiry-based instruction". The lack of awareness in [TAMS] about what content knowledge elementary teachers need in their classrooms is far from uncommon. It is time to face the fact that the need for change in the funding of inservice professional development is every bit as urgent as the need for more focus on content knowledge in the preservice arena.

Concluding Remarks

To conclude, let me add two observations. The mathematics taught in K–12 is the main source of the mathematical information of not only our schoolteachers but also of the mathematics education faculties and school administrators.[22] Mathematics education cannot improve so long as educators and administrators remain mathematically ill-informed as a result of the negligence of the mathematics community. It is doubtful, for example, that the research literature on fractions would slight logical reasoning (cf. pp. 33–38 in [Wu2008a]) had the researchers been exposed to a presentation of K–12 mathematics consistent with the five principles above. Many mathematics educators have likewise been denied this exposure and, as a result, have developed a distorted view of what mathematics is about. As this article tries to show, the cumulative gap between what (research) mathematicians take for granted as mathematics and what teachers and educators *perceive* to be mathematics has caused enormous damage in mathematics education. It is imperative that we minimize this damage by straightening out at least the *mathematics of K–12*, and we cannot possibly do that without first creating a corps of mathematically informed teachers. The latter has to be the mathematics community's immediate goal.

To lend some perspective on the communication gap between mathematicians and educators, it must be said that such miscommunication is by no means unusual in any interdisciplinary undertaking. In his celebrated account of the discovery of the double-helix model of DNA ([Watson]), James Watson recalled that at one point of his and Francis Crick's model building,[23] they followed the standard reference on organic chemistry[24] to pair the bases like-with-like. By luck, the American crystallographer Jerry Donohue happened to be visiting and was sharing an office with them, and Donohue told Watson not only that his (Watson's) scheme of pairing was wrong but also that such information given in most textbooks of chemistry was incorrect (p. 190, ibid.). In Watson's own words:

> If he [Donohue] had not been with us in Cambridge, I might still have been pumping for a like-with-like structure. (p. 209)

In other words, but for the fortuitous presence of someone truly knowledgeable about physical chemistry, Crick and Watson might not have been able to guess the double helix model, or at least the discovery would have been much delayed.

The moral one can draw from this story is that, if such misinformation could exist in high-level science, one should expect the same in mathematics education, which is much more freewheeling. This suggests that real progress in teacher education will

[21]*Note that the work of Hill, Rowan, and Ball ([Hill-RB]), while not directly verifying this hypothesis, is nevertheless fully consistent with it.*

[22]*If anyone wonders where administrators come in, let me say that the number of horrendous decisions in school districts on mathematics textbooks and professional development would easily fill a volume.*

[23]*In Cambridge, England.*

[24]The Biochemistry of Nucleic Acids *by J. N. Davidson.*

require both the education and the mathematics communities to collaborate very closely and to be vigilant in separating the wheat from the chaff. In particular, given the long years during which incorrect information about mathematics has been accumulating in the education literature and school textbooks, there should be strong incentive for educators to seek information about the K–12 mathematics curriculum anew and to begin some critical rethinking.

Last but not least, all through this article I have put great emphasis on getting teachers and (consequently) educators to *know* the mathematics of K–12. This should in no way be interpreted as saying that the mathematics of K–12 is all a teacher needs to know. Contrary to Begle's belief, there is no such thing as knowing (in the sense described above) *too much* mathematics in mathematics education. Every bit of mathematical knowledge will help in the long run. However, faced with the almost intractable problem of improving the education of *all* math teachers, it is only proper that we focus on a modest and doable first step: make sure that mathematics teachers all know the mathematics of K–12. Let us get this done.

Acknowledgments

I am extremely grateful for the two referees' extensive and constructive comments, and to Ralph Raimi for many factual and linguistic corrections. They have made this article substantially better. I am also indebted to Richard Askey, Deborah Ball, Raven McCrory, Xiaoxia Newton, and Rebecca Poon for their invaluable contributions in the preparation of this article.

References

[Ball] D. L. Ball, The mathematical understandings that prospective teachers bring to teacher education, *Elementary School Journal* **90** (1990), 449–466.

[Ball-TP2008] D. L. Ball, M. H. Thames, and G. Phelps, *Content Knowledge for Teaching: What Makes It Special?* http://jte.sagepub.com/cgi/content/abstract/59/5/389.

[Begle1972] E. G. Begle, *Teacher Knowledge and Student Achievement in Algebra*, SMSG Reports, No. 9, 1972. http://www.eric.ed.gov/ERICWebPortal/custom/portlets/recordDetails/detailmini.jsp?_nfpb=true&_&ERICExtSearch_SearchValue_0=ED064175&ERICExtSearch_SearchType_0=no&accno=ED064175.

[Begle1979] *Critical Variables in Mathematics Education: Findings from a Survey of the Empirical Literature*, Mathematical Association of America, 1979.

[Bezuk-Cramer] N. Bezuk and K. Cramer, Teaching about fractions: What, when and how? In: *New Directions for Elementary School Mathematics*, P. R. Trafton, & A. P. Schulte (editors), National Council of Teachers of Mathematics, Reston, VA, 1989.

[CCSS] *Common Core State Standards for Mathematics*, June 2, 2010. http://www.corestandards.org/.

[Goldhaber-Brewer] D. D. Goldhaber and D. J. Brewer, Does teacher certification matter? High school certification status and student achievement, *Educational Evaluation and Policy Analysis* **22** (2000), 129–146.

[Hart] Kathleen Hart, Mathematics content and learning issues in the middle grades. In: *Mathematics Education in the Middle Grades*, National Academy Press, 2000, 50–57.

[Hill-RB] H. C. Hill, B. Rowan, and D. Ball, Effects of teachers' mathematical knowledge for teaching on student achievement, *American Educational Research Journal* **42** (2005), 371–406. http://www-personal.umich.edu/~dball/articles/index.html.

[Jensen] G. Jensen, *Arithmetic for Teachers*, American Mathematical Society, Providence, RI, 2003.

[Loveless-HK] T. Loveless, A. Henriques, and A. Kelly, *Mathematics and Science Partnership (MSP) Program Descriptive Analysis of Winning Proposals* (May 10, 2005). http://www.ed.gov/searchResults.jhtml?oq=.

[Ma] L. Ma, *Knowing and Teaching Elementary Mathematics,* Lawrence Erlbaum Associates, Mahwah, NJ, 1999.

[MET] *The Mathematical Education of Teachers*, Conference Board of the Mathematical Sciences, 2001. http://www.cbmsweb.org/MET_Document/index.htm.

[Monk] D. H. Monk, Subject area preparation of secondary mathematics and science teachers, *Economics of Education Review* **13** (1994), 125–145.

[NAR] *A Nation at Risk*, U.S. Department of Education, Washington, DC, 1983.

[NCTQ] *No Common Denominator*, Full Report, National Council on Teacher Quality, June 2008. http://www.nctq.org/p/publications/reports.jsp.

[NMP1] *Foundations for Success: Final Report*, The Mathematics Advisory Panel, U.S. Department of Education, Washington DC, 2008. http://www.ed.gov/about/bdscomm/list/mathpanel/report/final-report.pdf.

[NMP2] *Foundations for Success: Reports of the Task Groups and Sub-Committees*, The Mathematics Advisory Panel, U.S. Department of Education, Washington DC, 2008. http://www.ed.gov/about/bdscomm/list/mathpanel/reports.html.

[Shulman] L. Shulman, Those who understand: Knowledge growth in teaching, *Educational Researcher* **15** (1986), 4–14.

[TAMS] *Never Too Late to Learn: Lessons from the Teachers Academy for Math and Science.* http://www.projectexploration.org/web/pdf/tams2009/.

[Watson] J. D. Watson, *The Double Helix*, Atheneum, New York, 1968.

[Wu1998] H. Wu, *The Mathematics Education Reform: What Is It and Why Should You Care?*, 1998. http://math.berkeley.edu/~wu/reform3.pdf.

[Wu1999a] ———, On the education of mathematics majors, in *Contemporary Issues in Mathematics Education*, edited by E. Gavosto, S. G. Krantz, and W. G. McCallum, MSRI Publications, Volume 36, Cambridge University Press, 1999, 9–23. http://math.berkeley.edu/~wu/math-majors.pdf.

[Wu1999b] ———, *Pre-service Professional Development of Mathematics Teachers*, April, 1999. http://math.berkeley.edu/~wu/pspd2.pdf.

[Wu1999c] ———, Professional development of mathematics teachers, *Notices Amer. Math. Soc.* **46** (1999), 535–542. http://www.ams.org/notices/199905/fea-wu.pdf.

[Wu2002] ———, Chapter 2: Fractions [Draft], 2002. http://math.berkeley.edu/~wu/EMI2a.pdf.

[Wu2006] ———, How mathematicians can contribute to K–12 mathematics education, *Proceedings of International Congress of Mathematicians, 2006*, Volume III, European Mathematical Society, Madrid, 2006. Zürich, 2006, 1676–1688. http://math.berkeley.edu/wu/ICMtalk.pdf.

[Wu2008a] ———, *Fractions, Decimals, and Rational Numbers*, February 29, 2008. http://math.berkeley.edu/~wu/NMPfractions4.pdf.

[Wu2008b] ———, *The Mathematics K–12 Teachers Need to Know*, December 17, 2008. http://math.berkeley.edu/~wu/Schoolmathematics1.pdf

[Wu2010] ———, *Pre-Algebra*, March, 2010. http://math.berkeley.edu/~wu/Pre-Algebra.pdf.

[Wu2011a] ———, *Understanding Numbers in Elementary School Mathematics*, Amer. Math. Society, to appear.

[Wu2011b] ———, *Mathematics of the Secondary School Curriculum I, II, III*, in preparation.