Strengthening the Mathematical Content Knowledge of Middle and Secondary Mathematics Teachers

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As a research mathematician and a teacher of mathematics, I have continually enjoyed the thrill of discovering (or re-discovering) mathematics and the excitement of discussing the beauty and utility of mathematics with colleagues and students. Stimulating mathematical conversations have made each day interesting and unique. Although I have taught several different mathematics undergraduate and graduate courses involving a variety of majors, much of my career has been spent working with prospective/practicing middle and secondary mathematics teachers. This teaching trajectory was not accidental, since my original occupational plan was to become a high school mathematics teacher. A desire to continue my studies of advanced mathematics altered this plan, and although my enthusiasm and passion for the subject has not directly impacted middle and secondary students, the mathematical preparation of their teachers has been central to my collegiate life.

Since my primary research area is in commutative algebra, I have taught and developed courses in linear and abstract algebra for prospective/practicing middle and high school teachers. These courses were rich in mathematical ideas, but the connections to important concepts in school mathematics were not always explicitly detailed. Several factors contributed to this shortcoming, but a most significant one was the lack of excellent curricular materials (both at the school and university levels) to help illustrate and demonstrate the critical links. Without such materials, it is challenging and time-consuming for mathematicians, who primarily teach content courses for prospective teachers and who are typically unfamiliar with school mathematics curricula, to make these critical connections. A natural consequence of this predicament was frustrated students (What does this have to do with teaching mathematics in middle or secondary school?) and my own bewilderment (Why can't these students appreciate the beautiful theorems we are proving?).

To help address the need for specialized courses and materials for mathematics teachers, the Conference Board of Mathematical Sciences, in concert with the Mathematical Association of America (with funding provided by the United States Department of Education), developed the *Mathematical Education of Teachers Report* (*MET Report*), 2001. This document carefully articulates a framework for mathematics content courses for prospective teachers that is built upon the premise that "the mathematical knowledge needed for teaching is quite different from that required by college students pursuing other mathematics-related professions."

Mathematics teachers should deeply understand the mathematical ideas (concepts, procedures, reasoning skills) that are central to the grade levels they will be teaching and be able to communicate these ideas in a developmentally appropriate manner. They should know how to represent and connect mathematical ideas so that students may comprehend them and appreciate the power, utility, and diversity of these ideas, and they should be able to

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understand student thinking (questions, solution strategies, misconceptions, etc.) and address it in a manner that supports student learning.

To further clarify the notion of "mathematical knowledge for teaching", consider some authentic mathematics students' questions that school algebra teachers regularly encounter in their teaching practice and should be prepared to address in a mathematically meaningful way.

1. My teacher from last year told me that whatever I do to one side of an equation, I must do the same thing to the other side to keep the equality true. I can't figure out what I'm doing wrong by adding 1 to the numerator of both fractions in the equality $\frac{1}{2} = \frac{2}{4}$ and getting $\frac{2}{2} = \frac{3}{4}$.
2. Why does the book say that a polynomial $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$ if and only if each $a_i = 0$, and then later says that $2x^2 + 5x + 3 = 0$?
3. You always ask us to explain our thinking. I know that two fractions can be equal, but their numerators and denominators don't have to be equal. What about if $\frac{a}{b} = \frac{c}{d}$, and they are both reduced to simplest form. Does $a = c$ and $b = d$, and how should we explain this?
4. I don't understand why $(-3) \times (-5) = 15$. Can you please explain it to me?
5. The homework assignment asked us to find the next term in the list of numbers $3, 5, 7, \ldots$? John said the answer is 9 (he was thinking of odd numbers), I said the answer is 11 (I was thinking odd prime numbers), and Mary said the answer is 3 (she was thinking of a periodic pattern). Who is right?
6. We know how to find 2^2, but how do we find $2^{2.5}$ or $2^{\sqrt{2}}$?
7. My algebra teacher said $\frac{x^2+x-6}{x-2} = \frac{[(x+3)(x-2)]}{(x-2)} = x + 3$, but my sister's boyfriend (who is in college) says that they are not equal, because the original expression is not defined at 2, but the other expression equals 5 when evaluated at 2.
8. My father was helping me with my homework last night and he said the book is wrong. He said that $\sqrt{4} = 2$ and $\sqrt{4} = -2$, because $2^2 = 4$ and $(-2)^2 = 4$, but the book says that $\sqrt{4} \neq -2$. He wants to know why we are using a book that has mistakes.
9. Why should we learn the quadratic formula when our calculators can find the roots to 8 decimal places?
10. The carpenter who is remodeling our kitchen told me that geometry is important. He said he uses his tape measure and the Pythagorean theorem to tell if a corner is square. He marks off 3 inches on one edge of the corner, 4 inches on the other edge, and then connects the marks. If the line connecting them is 5 inches long, he knows by the Pythagorean theorem that the corner is square. This seems different from the way we learned the Pythagorean theorem.

Remark. The Situations Project, a collaborative project of the Mid-Atlantic Center for Mathematics Teaching and Learning and the Center for Proficiency in Teaching Mathematics, is developing a practice-based framework for mathematical knowledge for teaching at the secondary school level. The framework creates a structure for identifying and describing important mathematics that underlies authentic classroom questions ("situations") arising in teaching practice (e.g., identifying and describing various mathematical ideas connected to questions such as those previously listed).

In addition to knowing and communicating mathematics, teachers of mathematics must be prepared to:

- Assess student learning through a variety of methods.
- Make mathematical curricular decisions (choosing and implementing curriculum), understand the mathematical content of state standards and grade-level expectations, communicate mathematics learning goals to parents, principals, etc.

This kind of mathematical knowledge is beyond what most teachers experience in standard mathematics courses in the United States (*Principles and Standards for School Mathematics*, NCTM, 2000), but there are a growing number of institutions, mathematicians, and mathematics educators who are determined to improve their teacher education programs along the lines recommended in the *MET Report*.

Two Collaborative Projects: Mathematicians and Mathematics Educators Working Together to Improve Mathematics Teacher Education

Collaborative efforts between mathematicians, mathematics teacher educators, classroom teachers, statisticians, and cognitive scientists have yielded (and continue to yield) innovative foundational mathematics and mathematics education courses and materials for prospective and practicing teachers that fundamentally address the need to improve the mathematical and pedagogical content knowledge of teachers. These collaborations have provided a greater understanding of the varying perspectives on important issues regarding the teaching and learning of mathematics and have significantly contributed (and continue to do so) to the improvement of mathematics teacher education in the United States. What follows are two examples of such fruitful collaboration.

I. Connecting Middle School and College Mathematics

Using the *MET Report* as a basic framework, a group of research mathematicians and mathematics educators at the University of Missouri-Columbia, in combination with a group of classroom teachers from Missouri, jointly developed four foundational college-level mathematics courses for prospective and practicing middle-grade teachers and accompanying textbooks as part of the NSF-funded project, *Connecting Middle School and College Mathematics* [$(CM)^2$], ESI 0101822, 2001–2006. These courses and materials were designed to provide middle-grade mathematics teachers with a strong mathematical foundation and to connect the mathematics they are learning with the mathematics they will be teaching. The four mathematics courses focus on algebra and number theory, geometric structures, data analysis and probability, and mathematics of change and serve as the core of the twenty-nine-credit-hour mathematics content area of the College of Education's middle-school mathematics certificate program at the University of Missouri-Columbia. For those practicing elementary or middle-grade teachers seeking graduate mathematics experiences to improve their mathematics content knowledge, cross listed, extended versions of the four core courses developed under the $(CM)^2$ Project are offered for graduate credit.

In an effort to help students explore and learn mathematics in greater depth, the four companion textbooks that were developed as part of the $(CM)^2$ Project (*Algebra Connections, Geometry Connections, Calculus Connections, Data and Probability Connections*, Prentice Hall, 2005, 2006), employ a unique design feature that utilizes current middle-grade mathematics curricular materials in the following multiple ways:

- As a springboard to college-level mathematics.
- To expose future (or present) teachers to current middle-grade curricular materials.
- To provide strong motivation to learn more and deeper mathematics.
- To support curriculum dissection—critically analyzing middle-school curriculum content—developing improved middle-grade lessons through lesson study approach.
- To use college content to gain new perspectives on middle-grade content and vice versa.
- To apply middle-grade instructional strategies and multiple forms of assessment to the college classroom.

Throughout each book, the reader finds a number of classroom connections, classroom discussions, and classroom problems. These instructional components are designed to deepen the connections between the mathematics that students are studying and the mathematics that they will be teaching. The classroom connections are middle-grade investigations that serve as launching pads to the college-level classroom discussions, classroom problems, and other related collegiate mathematics. The classroom discussions are intended to be detailed mathematical conversations between college teachers and preservice middle-grade teachers and are used to introduce and explore a variety of important concepts during class periods. The classroom problems are a collection of problems with complete or partially complete solutions and are meant to illustrate and engage preservice teachers in various problem-solving techniques and strategies. The continual process of connecting what they are learning in the college classroom to what they will be teaching in their own classrooms provides teachers with real motivation to strengthen their mathematical content knowledge.

II. Nebraska Algebra

(Part of the NSF project, NebraskaMATH, DUE-0831835, 2009–2014). For several decades, school algebra has occupied a unique position in the middle and secondary curricula and even more so in recent times with the expectations of "algebra for all" (Kilpatrick et al., 2001). Not only is algebra a critical prerequisite for higher-level mathematics and science courses, but also it is essential for success in the work force (ACT, 2005). Most recently, several national reports have called for an intensified focus on the learning and teaching of school algebra (National Mathematics Advisory Panel; NCTM Focal Points; MET; MAA report, *Algebra: Gateway to a Technological Future*). Although the specific recommendations of these reports have some differences, all of them agree that "strategies for improving the algebra achievement of middle and high school students depend in fundamental ways on improving the content and pedagogical knowledge of their teachers" (Katz, 2007).

Employing the teacher-education recommendations of the aforementioned reports, with the ultimate goal of extending success in algebra to all students in Nebraska, a collaborative group of mathematicians, mathematics educators, classroom teachers, statisticians, and cognitive psychologists recently developed an integrated nine-graduate-credit-hour sequence designed to help practicing Nebraska Algebra I teachers to become master Algebra I teachers with special strengths in algebraic thinking and knowledge for teaching algebra to middle and high school students. Two of the courses in the program (Algebra for Algebra Teachers and Seminar in Educational Psychology: Cognition, Motivation, and Instruction for Algebra Teachers) are taught in a two-week summer institute (the first two cohorts completed these courses in the summers of 2009 and 2010) and, during the academic year following their participation in the Nebraska algebra summer institute, teachers return to the classroom and work with

an instructional coach or teaching mentor as they strive to transfer knowledge gained in the summer institute into improved classroom practice. In addition, teachers take a three-graduate-credit-hour yearlong pedagogy class focused on enhancing their ability to teach algebra to all students and to become reflective practitioners.

The Algebra for Algebra Teachers course was designed to help teachers better understand the conceptual underpinnings of school algebra and how to leverage that understanding into improved classroom practice. Course content and pedagogy development was strongly influenced by national reports and research findings, as well as by the collaborative expertise of mathematicians, mathematics educators, and classroom teachers.

The course content begins with a review of key facts about the integers, including the Euclidean algorithm and the fundamental theorem of arithmetic. The integers modulo n are studied as a tool to broaden and deepen students' knowledge of the integers, since questions concerning integers can often be settled by translating and analyzing them within the framework of this allied system. From this foundation, the course of study involves polynomials, roots, polynomial functions, polynomial interpolation, and polynomial rings $k[x]$, where k is the field of rationals, reals, or complex numbers. Special attention is paid to linear and quadratic polynomials/functions in connection to their importance in school algebra. Fundamental theorems established in the context of the ring of integers are studied in the context of $k[x]$, e.g., the division algorithm, Euclidean algorithm, irreducibility, and unique factorization. Additionally, applications (such as the remainder theorem, factor theorem, etc.) are considered, and other results in polynomial algebra (such as the rational root test, multiple roots and formal derivatives, Newton's method, etc.) are studied in depth.

The course pedagogy combines collaborative learning with direct instruction and was designed to provide teachers with dynamic learning and teaching models that can be employed in the school classroom. Course assessments include individual and collective presentations, written assignments, historical assignments, mathematical analyses of school curricula, extended mathematics projects, and a final course assessment.

Conclusion

A fundamental tenet of our courses and material development is that mathematics teachers should not only learn important mathematics, but they should also explicitly see the fundamental connections between what they are learning and what they teach (or will teach) in their own classrooms. Moreover, while learning this mathematics, they should directly experience exemplary classroom practice, creative applications to a wide variety of state-of-the-art technology, and multiple forms of authentic assessment. The work we have accomplished, and that which we hope to accomplish, has occurred through the collaboration of mathematicians, mathematics educators, classroom teachers, statisticians, and cognitive psychologists. The combined expertise and perspective of these professionals have significantly strengthened our efforts, and we are hopeful that these and future collaborations will help contribute to the improvement of mathematics teacher education in the United States.

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