A Mathematician–Mathematics Educator Partnership to Teach Teachers

Ruth M. Heaton and W. James Lewis

The mathematical preparation of teachers is both a core problem in and a central solution to improving K–12 mathematics education nationwide ([1], [2], [3], [4]). Currently, there is broad agreement that most teachers, and especially elementary teachers, lack the depth of mathematical and pedagogical knowledge needed to teach mathematics (e.g., [5]). The mathematics knowledge that future teachers gain from their own K–12 education, including competency with basic skills and modest knowledge of algebra and geometry, is insufficient for the work of teaching elementary mathematics. Unfortunately, higher education is not seen as doing its part to "fix" this problem. For example, *Educating Teachers* [6] argues:

> The preparation of beginning teachers by many colleges and universities… does not meet the needs of the modern classroom…. Professional development for continuing teachers…may do little to enhance teachers' content knowledge or the techniques and skills they need to teach science and mathematics effectively. [6, p. 31]

In this article, we emphasize how those with advanced mathematical knowledge can help to resolve the problems of mathematics teacher education. We address two questions:

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• What knowledge, especially mathematical knowledge, do teachers need to have to teach mathematics effectively?

• How can teachers best learn what they need to know?

A Partnership of Expertise

Simply requiring teachers to take more mathematics courses is an inefficient, impractical, and, almost certainly, inadequate response to the problem ([7], [8]). Most courses that teachers might take are not designed to prepare teachers, so the content is far removed from the work of teaching. Furthermore, teachers' collegiate mathematics education has historically been disconnected from their pedagogical preparation. What connections there are remain invisible to students and are not discussed among mathematics and pedagogy instructors. In fact, often there is deep-rooted distrust between the mathematicians and mathematics educators teaching these courses.

At the University of Nebraska-Lincoln (UNL), Lewis, a mathematician, and Heaton, a mathematics educator, have developed a ten-year partnership designed to address problems of elementary mathematics teacher preparation. Following the recommendations for forming interdisciplinary partnerships by the Conference Board of the Mathematical Sciences (CBMS) [9] and the National Research Council (NRC) [6], Lewis[1] and Heaton have integrated the intellectual content of school mathematics and the special blend of mathematical and pedagogical knowledge needed for teaching [10].

[1] *Lewis was chair of the steering committee for [9] and co-chair of the NRC committee that produced [6].*

CBMS [9] proposes what teachers need to know and how best to learn it:

> Prospective teachers need a solid understanding of mathematics so that they can teach it as a coherent, reasoned activity and communicate its elegance and power. Mathematicians are particularly qualified to teach mathematics in the connected, sense-making way that teachers need. For maximum effectiveness, the design of this instruction requires collaboration between mathematicians and mathematics educators and close connections with classroom practice. [9, p. xi]

To our partnership, Lewis brings his expertise as a mathematician and Heaton brings her understanding of learning to teach and research on teaching, coupled with ten years of experience as an elementary classroom teacher. We work closely with teachers in a local public school to connect their courses to real classrooms [6] in an effort to build prospective teachers' deep understanding of mathematics, children, and teaching.

A number of writers have described the intertwined nature of mathematical and pedagogical knowledge that is central to the goals of our program. Ma's [5] description of the profound understanding of fundamental mathematics needed for teaching gives one clear picture. Ball, Thames, and Phelps [10] describe the nature and structure of mathematical knowledge needed for successful teaching as:

> [t]he mathematical knowledge "entailed by teaching"—in other words, mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students. To avoid a strictly reductionist and utilitarian perspective, however, we seek a generous conception of "need" that allows for the perspective, habits of mind, and appreciation that matter for effective teaching of the discipline. [10, p. 399]

In our teaching, Lewis focuses on helping prospective teachers acquire a deep understanding of the content of school mathematics and the attributes of mathematicians seriously engaged in doing mathematics. CBMS refers to these as the "habits of mind of a mathematical thinker" [9, p. 8]. Simultaneously, Heaton works with teachers to use their understanding of mathematics to find the mathematics in the many tasks of teaching mathematics [10]. This helps prospective teachers to develop productive habits of pedagogy [11] and to understand mathematics from the child's point of view. Our public school teacher partners help prospective teachers see the relevance of their coursework in managing the realities of mathematics teaching and learning.

Teaching Mathematical Content for Elementary Teaching

Most mathematicians, including many who take the work of educating teachers seriously, work in isolation from those more directly involved in teacher education. Our view is that this approach is less successful than having mathematicians and educators work in partnership.

Helping future elementary teachers learn the mathematics they need to know is hard work. Our students choose to become elementary teachers because they love children, not because they love mathematics. Many are weak mathematically. Past experiences have led them to believe they cannot be good in mathematics. They may believe that school textbooks and a teacher's guide are all they need to teach effectively. They may not fully appreciate the "intellectual substance in school mathematics" [9, p. xi]. Few understand the need or expect to be challenged by the need to understand thoroughly the mathematics of the elementary curriculum and can react negatively if their mathematics class proves to be harder than expected. Still, many are quite bright, are driven by a genuine passion to help children learn, and are quite willing to work hard. Thus mathematicians charged with the task of educating future elementary teachers often face a tough audience of learners. Teaching future elementary teachers can be a positive experience for both the students and the mathematician. A partnership between the latter and a mathematics educator and classroom teachers who support and communicate the importance of understanding mathematics helps ease the students' resistance to the mathematician's expectations.

We believe that mathematicians should hold high expectations for what they ask future elementary teachers to learn. Simultaneously, they need to support teachers as learners as they struggle to learn mathematics. Teachers should leave their mathematics courses believing in their ability to do mathematics and to reason about mathematical situations. They need to understand that mathematics is something that can and should make sense.

As Roger Howe wrote to Lewis, "For most future elementary school teachers the level of need is so basic, that what a mathematician might envision as an appropriate course can be hopelessly over the heads of most of the students" [12]. Most mathematicians need mathematics educators to help them to define the core mathematical knowledge of the elementary curriculum. Courses should focus on a thorough development of basic mathematical ideas, and teachers should be encouraged to develop flexibility in their ability to think mathematically, to develop careful reasoning skills, and to acquire mathematical "common sense" [9].

Across the math courses Lewis teaches, his goal is to help teachers become productive mathematical thinkers with a toolbox of skills and knowledge to use to experiment, conjecture, reason, and ultimately solve problems. Developing "mathematical habits of mind" (e.g., [13], [14]) means helping learners to acquire understanding of and experience in using these tools. Although a complete mathematical toolbox includes algorithms, a person with well-developed habits of mind knows why algorithms work and under what circumstances an algorithm will be most effective. Mathematical habits of mind are marked by ease of calculation and estimation as well as persistence in pursuing solutions to problems. A person with well-developed habits of mind will want to analyze all situations, will believe that he or she can make progress toward a solution, and will engage in metacognition: monitoring and reflecting on the processes of reasoning, conjecturing, proving, and problem solving. In the pedagogy courses Heaton teaches, her goal is to help teachers develop pedagogical knowledge and skills that support the development of mathematical habits for elementary students.

The Context of Teacher Education

UNL elementary education majors take twelve hours of mathematics. The first course is typically a general education course, Contemporary Mathematics, which introduces students to many ways in which mathematics is important to our daily lives. It covers topics such as Euler circuits and fair division. Next, they take a number and number sense course with the goal of developing a deep understanding of the arithmetic that is taught in the K–6 curriculum (place value, basic operations, fractions, primes). This is followed by a descriptive geometry course that focuses on understanding the measurement and geometry topics taught in the K–6 curriculum. Lastly, students choose a fourth course from a list that includes several courses designed for future mathematics teachers.

In 2000 we received a Course, Curriculum, and Laboratory Improvement (CCLI) grant from the National Science Foundation (NSF) to rethink elementary mathematics teacher preparation at UNL. The grant helped "purchase" cooperation within Heaton's teacher education department. Lewis was chair of the mathematics department at the time, so cooperation from mathematics was assured. As funding ended, UNL adopted The Mathematics Semester [15], a four-course (ten hours), one-semester integrated immersion program for acquiring and learning how to apply mathematical knowledge to elementary teaching.

In addition to the arithmetic course, The Mathematics Semester includes two pedagogy courses and a two-credit-hour field experience. One goal of the pedagogy courses is to help students understand mathematics from children's perspectives. Another goal is to help them learn how to teach specific mathematical ideas to children. A third is to teach them how to establish and sustain a classroom culture that supports all children in studying mathematics, as well as other subjects. Prior to the development of The Mathematics Semester, the second pedagogy course and the field experience did not include any mathematics. Now they emphasize mathematics, effectively adding two mathematics education courses to the elementary education program. Students participate in The Mathematics Semester as a cohort, taking all four courses with the same group of students. Most of our students end the semester with a positive attitude toward learning and teaching mathematics and a significant increase in their mathematical knowledge for teaching.

The Mathematical Content

Although future elementary teachers need to understand the mathematics they will teach, it is important that a college-level math class for teachers not completely mirror the elementary mathematics curriculum. Mathematicians will be pulled by their students to teach useful pedagogical strategies, such as providing math activities that could be done with fourth graders or a prescribed method for teaching fractions. A mathematician working with a mathematics educator can—and should—resist this pull. In the context of a partnership, the former can concentrate on mathematics while the latter attends to pedagogy.

Lewis, for example, teaches core mathematical content to teachers [16, 17], emphasizing problem solving, communication, and reasoning and proof [18]. In doing so, he models for future teachers a way of moving away from thinking about mathematics only as mastery of basic skills and computational fluency toward a definition of mathematics that recognizes and values mathematical proficiency as defined in *Adding It Up* [19]—including strategic competence, adaptive reasoning, and productive disposition.

Procedural fluency and conceptual understanding are important to both teachers and their students. It is a mistake to dismiss one in favor of the other [20]. Either the NCTM process standards [18] or the components of mathematical proficiency outlined in *Adding It Up* [19] can be used as organizational structures for mathematicians' pedagogy. The mathematics educator can then, with students in the pedagogy course, analyze and reflect on the mathematician's teaching, considering specific pedagogical strategies used by the mathematician and the varied outcomes.

A Mathematical Example from Practice

In developing the habits of mathematical thinkers, instructors must identify interesting problems that are accessible but for which solutions or means of finding solutions are not immediately obvious. Such problems make an important contribution to the mathematical education of teachers, even though they may not connect directly to the particular content being studied in class. The problems should be challenging enough that students will want to seek out other members of the class with whom to work.

Lewis assigns these problems as weekly homework and expects the work to be accomplished outside of class time. The problems and their solutions are rarely discussed in class. If the problems are particularly challenging, Lewis might hand out a solution he himself has created after students have worked on them and turned in their own solutions. The solution then serves as a model for how they might communicate their solutions. By analyzing someone else's mathematical arguments and explanations, students can learn how to construct their own arguments.

Students just learning the careful reasoning necessary in mathematics have trouble if their first experience is with a mathematical proof. Even if they are asked for a straightforward proof (such as the proof that an odd number plus an odd number is even), they are not likely to engage. Students have much more success if they are given a problem to be solved, an answer to be found, or a solution to be justified. Such contexts offer students opportunities to ask and answer questions to help them move through the construction of a mathematical argument.

As a semester begins and students are adjusting to new courses and a new schedule, Lewis's first goal is to have students understand that there is important mathematics they need to understand but do not yet understand well. He believes that it is important that the first homework assignment establish his expectation that students will put time into their mathematics course. One such problem was based on a Puzzler used on the National Public Radio show *Car Talk* [21].

The Chicken Nugget Conundrum involved both intentional and unintentional complexity. The mathematical complexity was planned. Not planned, and a surprise to Lewis, was the unintentional linguistic complexity in the problem. As a result, some students found themselves off track, unable to solve the problem because they did not understand the question being asked. For example, some students interpreted the sentence, "You can only buy them in a box of six, a box of nine, or a box of twenty" to mean they could only consider multiples of six, nine, or twenty, not different combinations, despite the example that clarifies this point. Others interpreted "Explain why it is possible to have a combination that equals any number larger than *N*" to mean that they needed to show that some number larger than, say, forty-three was possible. These examples point out not only linguistic but also cultural differences. Whereas precision in language is highly valued by mathematicians, teachers (and many others) tend to be linguistically imprecise. Teachers need to acquire the habit of mind that precise language is important in mathematics.

The Chicken Nugget Conundrum

There's a famous fast-food restaurant you can go to where you can order chicken nuggets. They come in boxes of various sizes. You can only buy them in a box of six, a box of nine, or a box of twenty. Using these order sizes, you can order, for example, thirty-two pieces of chicken if you want. You'd order a box of twenty and two boxes of six. Here's the question: What is the largest number of chicken pieces that you cannot order? For example, if you wanted, say, thirty-one of them, could you get thirty-one? No. Is there a larger number of chicken nuggets that you cannot get? And if there is, what number is it? How do you know your answer is correct?
A complete answer will:
i) Choose a whole number "N" that is your answer to the question.
ii) Explain why it is not possible to have a combination of "boxes of six" and "boxes of nine" and "boxes of twenty" chicken nuggets that add to exactly N pieces of chicken.
iii) Explain why it is possible to have a combination that equals any number larger than N.

Below are several solutions Lewis received:
Parts i and ii:

> *1) I've found that the largest number of chicken pieces that I cannot order is forty-three. I know that forty-three is correct because it is not divisible by any number except one and forty-three. This makes a prime number. I also know that forty-three is the correct number because it cannot be broken down into any combination of the numbers twenty, nine, and six. With the number forty-three, it is not possible to have a combination of multiple "boxes of six", "boxes of nine", or "boxes of twenty" because you cannot use these numbers to reach forty-three.*
>
> *2) You cannot have any combination that adds to forty-three because it can't evenly divide by six, nine, or twenty. It is not a multiple of fifteen and it can't be evenly divided in half.*

> *3) There is no combination for forty-three, but there is for some of the numbers below, and all the numbers above. I had noticed in the early numbers that it was skipping by ten. Five did not work, fifteen did, twenty-five did not. Then once it got higher they doubled or added.*
>
> *4) It is not possible because the numbers don't have common divisors. Although forty-three is a prime number. if [sic] we had a three-pack we could fulfill the order with one three-pack and two orders of twenty.*

Part iii:

> *5) It's possible to have a combination greater then forty-three. This is because you can buy all the multiples of the numbers. For example, if you buy eighteen, you can buy thirty-six and seventy. Or if you by [sic] twenty you can buy forty, sixty, eighty, one hundred, etc.*
>
> *6) After forty-three, I went up to sixty-five and everything between forty-three and sixty-five could be done. Beyond forty-three, each number would be able to work because they are multiples of six, nine, or twenty. This could go on forever, but I figured after forty-three they will somehow work!*
>
> *7) Every number after forty-three is possible whether you add six, nine or twenty multiple combinations will give you any number will give you more than forty-three.[sic]*

These answers include poor grammar, imprecise language, information irrelevant to the solution, "trust me" arguments, and evidence that some of the students did not understand the question they were trying to answer. For example, in argument #3, it is not useful to note that one cannot order five nuggets. Nor is it useful to talk about what would be possible under different conditions from those given in the problem (see argument #4). Argument #6 indicates the student believes a proof is found in the existence of a large number of examples. We leave it to readers to complete an assessment of these answers.

Lewis was surprised by the level of difficulty the problem posed for students and the many weaknesses in the justifications of their solutions. Lewis assigned low marks to our students' work but then took time in class to discuss the weaknesses in students' solutions by sharing some of the false explanations and asking students to discuss them.

Lewis changed what he first feared was a disaster into a lesson on communication, using the false explanations to help students learn about the nature of mathematical argument, the importance of using precise language, and different reasonable methods of justification. After the discussion, students were offered the opportunity to redo their explanations. Most took advantage of the opportunity, meeting with Lewis and putting considerable energy into communicating their understanding of a solution. It was common to receive two-page typed solutions for what was only a ten-point assignment. Susan's solution, here considerably shortened, was typical:

> *It is easy to see that no more than two boxes of twenty could be used. If you subtracted the two boxes of twenty from forty-three, only three would remain… (and) there are no boxes that offer only three chicken nuggets…. If you were to use only one box of twenty, you would subtract twenty from forty-three resulting in twenty-three. These twenty-three would have to come from a combination of boxes of nine or six. Both nine and six are multiples of three, so any combination resulting from boxes of nine, boxes of six, or both would have to be divisible by three also. Twenty-three is not divisible by three so any combination… would not work. …the only other option is to use no boxes of twenty…. Just like twenty-three, forty-three is not divisible by three either. …With all the possible options eliminated it is clear that forty-three cannot be created using a combination of twenty, nine, or six.*

Despite the stress of this process (for everyone concerned), our students made significant progress quickly, responding to the challenge to reason about mathematics, to use careful language, and to communicate their understanding. By the third homework assignment, most students were meeting regularly in groups to work on the homework and to produce careful explanations. They regularly offered a two- to three-page solution to a ten-point homework problem. The high quality of their work made the homework easier to grade. The homework problems thus established a culture of mathematical explanation that carried over to class and our study of the mathematics taught in the elementary classroom.

Translating Mathematical Knowledge into Classroom Practice

Over time, we have integrated the goals and practices of our teaching mathematics and pedagogy. We began simply by scheduling the mathematics and pedagogy classes back-to-back in the same university classroom and by requiring students to take all four classes in the same semester. We used separate syllabi but experimented with assignments that were given in multiple courses. Currently, we have one integrated syllabus for the entire Mathematics Semester, including several assignments that require students to apply what they learn from one course setting to another. For these assignments, we evaluate students together, and they receive single grades that count in more than one course.

These integrated assignments give students practice in some of the "mathematical tasks of teaching" [10]. Assignments include such things as modifying tasks to be easier or harder. We also ask students to plan, teach, and reflect on math lessons, in elementary school settings that are one-on-one, in small groups, or with an entire class of students. Students are also asked to appraise mathematical topics within reform curricula and to identify within these topics intellectually rich problems for children. The students are required as well to recognize the mathematical knowledge that teachers need to teach the topics well.

One major assignment is the Learning and Teaching Project. Students use a challenging homework problem from Lewis's class to plan a lesson in which they work with a K–5 child in the field. The goal is to adapt the problem so as to offer the child a successful learning experience. Students need to consider vocabulary, instructional representational tools, a sequence of tasks, and possible questions to move the child along in his or her thinking about the problem.

One such assignment begins with Crossing the River [22], a problem that Lewis first saw at a conference that Ira Papick organized in Missouri. An edited version of the problem is shown in the box.

When the problem is assigned in the second week of the semester, students often struggle to explain their reasoning and to state and solve the problem for a adults and s students. When they receive the Learning and Teaching assignment later in the semester, they may think it is an unreasonable problem for a second-grade or even fifth-grade student and look to their cooperating teacher at the elementary school to validate their belief. Fortunately, the teachers always support us because Heaton has developed a strong partnership with them.

Our students videotape themselves teaching and write a paper about their experience. The assignment thus proves valuable in many ways. Students come to realize that they cannot teach mathematics successfully unless they understand it themselves. Students also must use their pedagogical knowledge to prepare appropriate manipulatives, to plan how they will present information and ask questions, and to anticipate the difficulties the children will have. Often, in their papers, students write about their surprise that young children can be creative and successful with challenging mathematics assignments. This integrated learning experience would not be possible without our mathematician/educator/teacher partnership.

Crossing the River

A group of adults go on a camping trip with a group of fourth-grade students. They come to a river that is too deep to wade across. They find a boat, but it isn't very big. The adults are rather big, and only one adult can fit in the boat at one time, but the boat can hold any two fourth-grade students. The students have experience boating, and each can safely row across the river by themselves.

If there are four adults and two students on the trip, is it possible to get all of them across the river? If yes, how many one-way trips across the river will it take? What if there were five adults and only one student? What if there were five adults and two students or four adults and six students? How can the problem be generalized? Solve the general problem or at least several more cases.

Expanding the Partnership

The key to improving K–12 mathematics education is to build teachers' mathematical and pedagogical knowledge, and the need is not limited to the context of preparing future elementary teachers. Many current K–12 teachers have similar needs. The separate expertise of a mathematician and a mathematics educator, joined in a successful partnership, is the right foundation to support this kind of work. At Nebraska, our partnership has resulted in two large NSF grants for Math Science Partnerships (MSP), Math in the Middle Institute Partnership and NebraskaMATH. Information about these grants is available on our website (`http://scimath.unl.edu/`).

Conclusion

As we look back on ten years of working together, we are convinced that our partnership has been the key to our success. Heaton's courses are more mathematical than they were a decade ago. Lewis's courses have a much stronger connection to the work of teaching elementary mathematics. By supporting each other, we are able to hold our students to high standards and to help them learn both mathematics and how to teach mathematics so that they end the semester with a positive attitude toward mathematics.

The partnership that began with a CCLI grant has given us the opportunity to work with hundreds of mathematics teachers who are eager to learn more mathematics in a context that enables them to be more successful teachers. Many of our colleagues are now involved in teaching teachers, either as part of The Mathematics Semester or through one of our MSP grants. Over the past five years, over thirty mathematics graduate students have benefited from working in our projects as part of instructional teams, thus enhancing their ability to teach and their knowledge of teachers. This experience has proven quite valuable as they earn Ph.D.s and apply for jobs. The partnership is also supporting a substantial research program in mathematics education.

We encourage others to join us in working across department lines to benefit both their departments and the future teachers they educate.

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