Philosophy, Math Research, Math Ed Research, K–16 Education, and the Civil Rights Movement: A Synthesis

Ed Dubinsky and Robert P. Moses

Robert Moses studied philosophy of mathematics under W. V. O. Quine at Harvard. He has taught high school in New York City; Jackson, Mississippi; and Miami, Florida. A major figure in the civil rights movement of the 1960s, he is currently developing an algebra curriculum for middle and high school mathematics that reaches out effectively to students from underrepresented groups.

Ed Dubinsky spent twenty-five years doing research in functional analysis (including a solution to Grothendieck's bounded approximation problem) and another twenty-five years doing research in undergraduate mathematics education. He has developed a theory of learning topics in undergraduate mathematics and has designed and disseminated innovative curricula in several undergraduate courses. He, too, was active in the civil rights movement of the 1960s.

It is probably our common experiences in struggles for human rights and our commitment to understanding how the mind might work when a student is trying to learn mathematics that has allowed us, in spite of disparate backgrounds and life experiences, to communicate about high school algebra. In any case, the mathematician has been able to contribute to the philosopher-educator's Algebra Project, which has grown and which we both hope will continue to grow in coming years.

It is the purpose of this article to discuss the thinking that has gone into this work and to describe some examples of what has come out of it. First we will give our separate points of view about the epistemology of learning mathematics, then discuss a synthesis of the two approaches, and then describe our high school algebra curriculum as it relates to modular arithmetic. Finally, we will describe how the Algebra Project, founded by Moses, relates to the civil rights movement.

Ed's Story

Throughout the twenty-five years I spent doing research in functional analysis and teaching undergraduate mathematics at six universities in five countries and on three continents, I was always interested in effective teaching. Unfortunately, in spite of trying a myriad of popular methods (modified Socratic, self-paced instruction, mastery learning, etc.), what I produced, more often than not, was *in*effective teaching. I was a good lecturer, enthusiastic about teaching, serious in my attempt to do it well, and I cared about my students. They liked me and my courses, but from everything I could see, they were not learning much more than students of other teachers, and that was woefully inadequate—as many national reports of the 1970s and 1980s concluded.

At one point, I decided in my frustration that if I were to significantly improve my students' learning, I was going to have to figure out something about the process of learning mathematics. That is, I would need to study what might be going on

Ed Dubinsky is a retired professor of mathematics and mathematics education. His email address is `edd@math.kent.edu`.

Robert P. Moses is president of The Algebra Project Inc. He may be contacted at the email address `ben@algebra.org`.

in a student's mind when he or she is trying to understand a mathematical concept. What mental activities need to take place in order for a student to be successful in such learning? I thought that, as I came to know more about the learning process in mathematics, I would be able to figure out pedagogical strategies that would help students engage in appropriate mental activities so as to be more successful. So I began to read. I read a lot over the first two years of my new career (new because, shortly after I started, all of my interest in functional analysis shriveled up). Some of the education literature I read was good; most was not very helpful. It was not until I came across the work of Piaget that I thought I had found an author who understood the mental processes of learning mathematics. I remembered that, as a young student of functional analysis, I had considerable difficulty with the idea of the dual of a locally convex space. I was fine with the notion of a linear functional that acted on elements of a locally convex space to produce numbers—linearly. But the idea of applying actions to these transformations, putting them together in a set, equipping the set with arithmetic and even topologies, was really tough for me. These linear functionals were doing things to elements of a vector space, so how could things be done to them? It was terribly confusing. I struggled for a long time and eventually mastered the mathematics. But I can't say I understood what had gone on in my mind.

It was when I read Piaget's discussions of transformations, the content which they transformed, the fact that these dynamic transformations could be stabilized in one's mind and thereby become contents for higher level transformations, and that this latter step was very *difficult* both historically and for individual students, that I knew I had come home. I began to see that it might be possible to identify mental constructions required to understand a mathematical concept. Working with the ideas of Piaget, I began to express them in an explicit theory called APOS theory. APOS is an acronym for Actions, Processes, Objects, and Schemas. It was developed by a team of mathematicians and mathematics education researchers led by me (see Asiala et al., 1996)

APOS Theory

APOS theory is based on Piaget's principle that an individual learns (e.g., mathematics) by applying certain mental mechanisms to build specific mental structures and uses these structures to deal with mathematical problem situations. According to this principle, for each mathematical concept, there are mental structures one can develop that are appropriate for this concept and that can be used to learn it, understand it, and use it (Asiala et al., 1996). If one has built appropriate structures, very elementary concepts can be grasped easily and early through normal life experiences, trial and error, and discussions with peers. Later, with such structures, more advanced concepts can be learned without undue difficulty via any pedagogical method that relates the concept to the structures. If, however, one does not possess structures appropriate for a concept, it is nearly impossible to learn it.

This aspect of Piaget's theory can explain a phenomenon that seems to be almost universal with respect to learning mathematics. Just about everyone learns the most elementary mathematics: counting, sequential ordering, forming sets, the concept of number. Even as the mathematics becomes less elementary, an individual may feel for a while that the mathematical ideas are almost obvious. One need only have a concept mentioned, perhaps explained, and then it is understood almost immediately and automatically. This period of "automaticity" can last for very different periods of time depending on the individual (from months to decades), but, for everyone, the time comes when the ideas become more difficult. Intervention of others (teachers, colleagues, books) becomes necessary, and learning can be delayed, eventually even stopped. What is happening, according to Piaget's principle—what needs intervention and takes time—is that the individual is building new mental structures to deal with the more complex concepts. At first, with the elementary concepts, the mental structures are built more or less automatically through normal day-to-day experiences. Later, as the mathematics becomes more sophisticated and the requisite structures more complex, intervention, or at least reflection over a period of time, is necessary and, for even the most powerful research mathematician, there are, eventually, mathematical concepts he or she cannot fully understand. The stopping point comes at different places for different people, and one measure of mathematical talent can be the extent of mental structures one is able to build with minimal intervention.

This principle has important consequences for education. Simply put, it says that teaching should consist of helping students use the mental structures they have to develop an understanding of as much mathematics as those available structures can handle. For students to move further, teaching should help them build new, more powerful structures to handle more and more advanced mathematics.

These ideas raise certain questions. Given a mathematical concept, what are the mental structures that can be used to learn it, and, knowing that, how can we help students build them? It is these questions that APOS theory and a pedagogical strategy based on it try to answer.

According to APOS theory, the mental structures are what we call *actions, processes, objects*, and *schemas*. The mental mechanisms used to build these mental structures are called *interiorization* and *encapsulation*. An *action* is a transformation of a physical or mental object that requires specific instruction and must be performed explicitly, one step at a time. A mathematical concept begins to be formed when an action transforms objects to obtain other objects. As an individual repeats and reflects on an action, it may be *interiorized* into a mental process. A *process* is a mental structure that performs the same operation as the action being interiorized, but wholly in the mind of the individual, thus enabling her or him to imagine performing the transformation without having to execute each step explicitly. Given a process structure, one can reverse it to obtain a new process or even coordinate two or more processes to form a new process via composition. If one becomes aware of a process as a totality, realizes that transformations can act on that totality and can actually construct such transformations (explicitly or in one's imagination), then we say the individual has *encapsulated* the process into a mental *object*. In some situations, when working with a mental object, it is necessary to *de-encapsulate* the object back to the process from which it came. While these structures describe how an individual constructs a single transformation, a mathematical topic often involves many actions, processes, and objects that need to be organized and linked into a coherent framework, which is called a *schema*. The mental structures of action, process, object, and schema constitute the acronym APOS.

Determining the specific actions, processes, objects, and schemas for a given concept requires research and a specific methodology that I will not discuss in this article. It may be helpful, however, to consider an example from elementary mathematics that will also allow a proposed explanation for a difficulty in arithmetic that is widespread among students and even some teachers. I am talking about the concept of division by a fraction. One understanding of division by a number requires that the number be understood as an object, and the division question is: How many of this object can be found in the dividend? Now think about the notion of a fraction, say 2/3. Initially, one can take a specific object (e.g., a pie or a rectangle), divide it into 3 equal pieces, and pick two of them. If an individual can think of 2/3 only in terms of such an activity, then he or she has an action conception of 2/3. After repeating such an action and reflecting on it, the individual may construct an internal process that allows her or him to imagine dividing an unspecified object into 3 parts and taking 2 of them. This is a process conception of 2/3, and most people, as the result of normal human activity, will come to this point without too much difficulty. It is the next step, necessary for understanding division by 2/3, that is difficult. In order to divide, say, 5 by 2/3, that is, to ask: "How many 2/3s are there in 5?" one must understand that this question requires thinking of 2/3 as an object. Without such an understanding, one can't begin to think about an answer to the division problem. Thus, one must encapsulate the process conception of dividing into 3 parts and selecting 2 into an object which becomes a somewhat abstract entity in the mind of the individual. Most people need help with this process, and it is not immediately obvious how to help students to use the mechanism of encapsulation to come to see the 2/3 process as an object, also called 2/3. In the next section we discuss methods to help students do this.

The above is a very brief description of an analysis that requires considerable research and that must be done for every mathematical concept one wishes one's students to learn. After reading about these ideas applied to very elementary mathematics, we developed APOS theory as a formulation of Piaget's theories that could be applied to more advanced mathematical concepts.

APOS-Based Pedagogy: Writing Computer Code and Programs

I began to look for pedagogical approaches to fit with this theory. I wanted to find ways to induce students to make the mental constructions called for by the theoretical analyses of concepts. I found that one could go a long way in this direction by having the students write certain computer programs or just code. That is, for each mental construction that comes out of an APOS analysis, one can find a computer task of writing a program or code such that, if a student engages in that task, he or she is fairly likely to make the mental construction that leads to learning the mathematics. I am not saying that the computer task is the mental structure but rather that performing the task is an experience that leads to one or more mental constructions.

Here is an example. Consider the concept of function. As with fractions, an APOS analysis says that development of understanding the function concept begins with an action understanding. That is, a function is understood to be an algebraic/trigonometric expression with numbers and a symbol, usually x. The action consists in replacing x with a number, making the calculation specified by the expression, and getting a number as the answer. It is externally directed in the sense that it follows a formula that is external to the individual performing the action. With repetition and reflection, the learner can interiorize this action, which means that he or she builds a mental structure that does the same thing internally that the action does externally. This mental structure is called a process, and it allows an individual to imagine the action as being performed without actually having to perform it. It is then possible to think of the function in terms of "something comes in, something is done to it, something comes out". With a process conception one can coordinate two or more processes to obtain a new process and reverse a process, first in one's mind and then, if needed, with pencil and paper. Finally, if an individual wishes to perform an action on this mental process, he or she first has to see it as a totality and encapsulate it mentally into an object. Then the individual can act on it. (For more details, see Asiala et al., 1996.) Now what kind of pedagogy can be based on such a theoretical analysis?

First, the teacher needs to have an idea of where the students are relative to the construction of requisite mental structures. Is the student restricted to thinking about functions as actions, or is he or she able to understand a function as a process but is still unable to encapsulate these processes as objects? The teacher needs to know this mental activity in order to navigate through the course material. The students may also need to know this in order to have a good idea of their progress. The research provides indicators that can help make reasonable conjectures about where students are relative to an APOS analysis. For example, if a student insists (as many do) that unless there is an explicit formula, there is no function, then such a student is probably at the action level for functions. On the other hand, if he or she is comfortable with forming sets of functions or realizes that the derivative can be interpreted as an operation that transforms a function into another function, then the student may be thinking at the object level for functions.

Working together with several colleagues, we found that a host of mathematical concepts could be analyzed in terms of these actions, processes, and objects. Such analyses could explain student difficulties in terms of mental constructions not made. On the other hand, we found that if we asked students to perform a mathematical action and write a computer program expressing that action, then, in performing this task, the student tended to interiorize the action into a process. Even more exciting was that if the student then wrote another program that accepted the first program as an input, transformed it in some way, and returned a new program, then this student was very likely to encapsulate the process and see it as an object. Going back and forth between object and process conceptualizations of a mathematical idea, so necessary in doing mathematics, resulted from this pedagogy almost effortlessly (Weller et al., 2003).

Based on these ideas, we devised a structured pedagogical approach. It works by a division of the course material into small units, each to last about one week. Each week is a cycle of three kinds of work. First, the students work (usually in cooperative groups) in a computer lab to write programs and code designed to foster mental constructions that can help them build an understanding of the concepts in that unit. They complete this work outside of class. Second, meeting in a classroom, the students work (again in groups) on tasks designed to help them convert the mental structures they have built into understandings of mathematical concepts. Third, based on the assumption that most of the students have at least begun to build understandings that fit with the mathematical ideas held by mathematicians, they are given exercises designed for practice, reinforcing the knowledge they are building, and extending that knowledge (Asiala et al., 1996).

We have designed and implemented undergraduate courses that follow this approach. Textbooks have been written that in their structure and content reflect the three-part cycle. We have conducted empirical studies using both qualitative and quantitative research methodologies of student performance and attitudes. Our results suggest that this approach can be highly effective in helping students learn various advanced mathematical concepts that appear in subjects such as precalculus, calculus, discrete mathematics, abstract algebra, and linear algebra (Weller et al., 2003). It must be acknowledged, however, that this pedagogical strategy requires teachers not only to significantly alter their thinking about learning and teaching but also to exert considerable effort to learn the method. We believe that these requirements are among the things that have limited the widespread adoption of such a strategy in undergraduate mathematics teaching.

Bob's Story

In the 1987–1988 school year, I was a parent volunteer teaching algebra to eighth graders in the open program at the Martin Luther King Jr. school in Cambridge, Massachusetts. My son, Omo, was in the class and wanted very much for some of his friends to be part of the class. He said he felt lonely when he was doing algebra. One of his friends wanted to be part of the group but didn't know his multiplication tables. I agreed to take him in the group and we worked side by side, one on one, every day. When we came to questions about the number line, adding integers on the number line, he always got the same kind of answers. That is, he consistently answered a question different from the one the book was asking. He had only one question about numbers in his mind, namely the "how many" question. My problem was to figure out another question about numbers that he needed to get into his mind.

I finally settled on a "which way" question. This question was a part of his daily routines and vocabulary. He knew how to ask: "Which way to the mall?" or "Which way to a friend's house?" But he didn't have his "how many" questions together with his "which way" questions as part of his concept of number. My problem became how to get his "which way" questions into his number concept on an equal footing with his "how many" questions.

One day, while traveling from Cambridge to Boston, I entered the T-stop on the Red Line at Central Square and noticed that all passengers are called upon to decide whether they are going inbound or outbound—two answers to a "which way" question. At this point, I recalled Quine's ideas about the process of generating elementary mathematics along with the concepts of experiential learning that had been a part of pedagogy at the open program in the Martin Luther King Jr. school. I, along with other teachers, then organized students to take trips on the T and asked them to write, talk, and draw pictures about their trips. We thought of these representations as their commonsense representations, what Quine calls "ordinary discourse". We then asked them to identify important aspects, called *features*, of these representations and discussed with them obvious features that they may not have paid attention to, such as the start and finish of the trip, as well as features that were not so obvious, such as locations and relative positions of stops.

This process, which Quine identifies as a process for mathematizing events, involves moving from ordinary discourse to *regimented language*, that is, the language used in mathematics. Adapting his theories to the classroom, we called the commonsense representations *people-talk* and the regimented or strait-jacketed representations *feature-talk*. We engaged the students in the process of constructing iconic symbols, that is, symbols that are also pictorial representations, as well as abstract symbols for the features that we intended to mathematize, and we developed iconic, as well as abstract, representations for various mathematical features of these trips.

Over time, it became clear that students mathematizing these trips acquire powerful metaphors and concepts for addition and subtraction very different from their arithmetic metaphors for those operations, including the concept of displacement as a mathematical object representing answers to both the "how many" and the "which way" questions. For example, consider the following two questions: "Where is Porter Square in relation to Central Square on the Red Line in Cambridge?" and "Where is Harvard Square in relation to Kendall Square?"

Underlying both questions is the concept of the relative position of two stops on the Red Line. The answer to both questions is the same: two stops outbound, an answer to both "how many" and "which way".

The geometrical representation of this answer is a displacement two units outbound. Students thought of the movement from Central Square to Porter Square as starting at Central Square and moving two units outbound, and of the movement from Kendall to Harvard as starting at Kendall and moving two units outbound. Thus we have two movements which have the same number of stops and are in the same direction. That is, these two movements represent the same displacement.

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         P           H           C           K
– – – – –•– – – – – –•– – – – – –•– – – – – –•– – – – –
         < – – – – – – – – – –
                     < – – – – – – – – – – –
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We call this diagram an *iconic* representation of the trips. The people-talk representations are the statements:

> Porter Square is two stops outbound from Central Square.
>
> Harvard is two stops outbound from Kendall.

Feature-talk involves explicit reference to location and relative positions of stops. This gives us addition as movement from the location of one stop to the location of another in one of two directions, and subtraction as the comparison of the location of the ending to the location of the starting stop. In other words,

> starting at the location of Kendall and moving two stops outbound one arrives at the location of Harvard

is feature-talk leading to addition, and

> the location of Harvard compared to the location of Kendall is 2 stops outbound

is feature-talk leading to subtraction.

To obtain this mathematization, we select some stop as the benchmark. We then discuss with the students assigning symbols such as 0 for the benchmark, x_1 for the location of Kendall, x_2 for the location of Harvard, and Δx for the displacement. Then the first feature-talk sentence becomes

$$x_1 + \Delta x = x_2,$$

and the second becomes

$$x_2 - x_1 = \Delta x.$$

We can summarize the mathematization of this type of sentence in the following eight steps:

1. Identify the observation sentence.
   Harvard is two stops outbound from Kendall.
2. Identify the name(s) in the sentences.
   Harvard, Kendall.
3. Identify the predicate of the sentences.
   The predicate in this case is the relation of equality ("is") between a name ("Harvard") and the object resulting from applying an operation ("two stops outbound") to a name ("Kendall").
4. Construct an icon for the name(s).
   The students will do this.

5. Construct an icon for the predicate.
   The students will do this.
6. Construct an iconic representation of this sentence.
   This is the Trip Line diagram shown above. The students will do this.
7. Translate the observation sentence into a sentence using regimented language.
   In this case there are two ways of doing so:
   a. Starting at the location of Kendall and moving two stops outbound one arrives at the location of Harvard.
   b. The location of Harvard compared to the location of Kendall is 2 stops outbound.
8. Identify the conventional symbols that are needed to translate the regimented language into conventional mathematical symbols and make that translation.
   We might take $L(H), L(K)$ for the locations of Harvard and Kendall, respectively, and we take $+$ for "move" and $-$ for "compared to". This leads to the following abstract symbolic representation of the two sentences:
   a. $L(K) +^{-} 2 = L(H)$
   b. $L(H) - L(K) =^{-} 2$

This recipe for converting an experience into a mathematical expression can be applied in a wide variety of situations and, together with students actually experiencing the situation, represents our main contribution to the pedagogy referred to as experiential learning.

A Synthesis

The synthesis of the above sets of ideas in our curriculum materials uses the structure described in Bob's story as the basic navigational framework of the material while paying attention to possible actions, processes, and objects that students might be constructing in their minds, as described in Ed's story. Thus writing computer programs has been replaced by playing certain games, discussing them, and writing about them. On the other hand, many of the specifics of the games are driven by the need to make certain mental constructions suggested by APOS theory.

We can make other uses of a synthesis of the two "stories". Consider, for example, the relation that appears in every Algebra 1 high school textbook:

$$a - b = a +^{-} b.$$

Here, a, b are any two integers. As we saw in the discussion of trips in Bob's story, an integer can be interpreted as a movement of a certain number of steps in a certain direction or as a location on a line. So is an integer a movement or a location? The APOS theory in Ed's story resolves this seeming ambiguity. If an integer is interpreted as a movement, then this is a process in the sense of APOS theory. The encapsulation of that process is an object that, in the case of an integer, is a location on the number line. With the mechanism of encapsulation and its opposite, de-encapsulation, we may go back and forth between interpreting an integer as either a movement or a location.

Now, suppose we start at a location b and make the movement

$$a +^{-} b.$$

This movement is constructed by moving from the benchmark to the location a, making the movement ^{-}b to arrive at the location $a +^{-} b$, which is then de-encapsulated to a movement that we also call $a +^{-} b$.

Now we can start at the location b and make the movement $a +^{-} b$, which, by our interpretation of addition, brings us to the location

$$b + (a +^{-} b),$$

which, using standard properties of integers,[1] is equal to the location a. To summarize, we have said that if we start at b and make the movement $(a +^{-} b)$, then we arrive at a. According to our interpretation of subtraction, this movement is just $a - b$. So we have:

$$a - b = a +^{-} b.$$

Now this relation may seem too obvious to mention to experienced mathematicians, but it appears explicitly in almost every high school algebra text and is one of the more difficult parts of beginning algebra.

To develop this material for the classroom, we divide the content into segments. Each segment begins with an experience, such as a game. The students play the game and record salient information. Each student then writes a description of what happened in the plays of the game. They are encouraged to write in complete sentences, organized in paragraphs (people-talk). Then, in a classroom discussion, the teacher helps them identify the features of the game (feature-talk), the operations that were performed with these features, and the predicate that describes the goal of the game (process of mathematization). The students are then asked to work in teams to answer certain questions designed to move them further toward mathematization of the situation. This is completed with the teacher describing the mathematics in language and symbols that are used by mathematicians.

We can also use this approach to interpret two equations that are so important in the mathematics that comes after algebra:

$$x_2 - x_1 = \Delta x,$$
$$x_1 + \Delta x = x_2.$$

The first relation says, according to our interpretation of subtraction, that the comparison of x_2 with x_1 is Δx. That is, it is the movement that takes

[1] *These properties are developed in our curriculum before the treatment being described.*

us from x_1 to x_2. In other words, if we begin at x_1 and make the movement Δx, we arrive at x_2, which, according to our interpretation of addition, is precisely the second equation.

Of course these two equations involve no more than very simple arithmetic, but, in order to do that arithmetic with any kind of understanding, students need to have useful interpretations—metaphors if you like—for the equations. We believe that the metaphors we have presented for addition and subtraction of integers can provide the necessary interpretations.

An Example

As a final example, here is a brief outline of curriculum material based on certain games for the topic of modular arithmetic. In discussing these games and what happens in the classroom, we will explain how this pedagogy relates to the ideas in Ed's story and in Bob's.

The first goal of this unit is for students to understand the mathematical operation of division-with-remainder of a positive integer a by a positive integer b in terms of the classic equation,

$$a = qb + r, \quad r = 0, 1, 2, \ldots, b - 1. \tag{1}$$

The curriculum begins with a game called Winding Around Positions. There are twelve stations that could represent hours on a clock or the Chinese years zodiac. A reference station is selected (in general, selections are made by the class with some input from the teacher), and one student sits at that station throughout the game. The class selects an integer, and a second student goes to the starting position and then moves through the stations, counting until the selected number is reached. While the student is moving, note is taken of the number of times the second student passes by the first and of the final position reached by the second student.

The features of this game are: the starting position, number of positions to be moved, number of winds, and final position. The operation is to count the positions, and the predicate asks how many winds there are. The purpose of this game is for the students to construct a mental process of moving through the stations and winding around the circle. We do this by first getting the students to perform the action of multiplying explicit numbers b by numbers q and adding quantities r that are less than b and second by interiorizing this action into a process that does the same. The reason for doing this is that an APOS analysis expresses the mental process underlying (1) as the reversal of the process of multiplying b by q and adding r.

The next game is played with the same setup but, instead of beginning with a single number, the students select a number of winds and an increment (which must be between 0 and 11). Here the features are essentially the same, but the operation is to multiply the number of winds by 12 and add the increment to respond to the predicate, which is: how many positions have been traversed? The mathematization to which the students are led is the basic division-with-remainder formula (1). This expresses a mental process in which a single traversal of all twelve stations has been encapsulated into a "wind".

The next game is designed to help the students reverse the mental process of multiplying the number of winds by 12 and adding the increment. It is also played with twelve stations representing the hours on a clock. A number of hours is given to represent time elapsed. Working in teams, the students begin at 12 and count around the clock to determine the number of winds and the increment that gives the final time on the clock. In this game, the features are: the time elapsed, the number of winds, and the remainder or end time. The operation consists of dividing the time elapsed by 12 to find the number of winds (quotient) and the end time (remainder). The mathematization of this game is division-with-remainder. It is symbolized by the same formula (1), which now is seen as expressing the reversal of a process. That process consists in multiplying a number of winds by 12 and adding an increment to obtain a total. The reversal consists in starting with the total, determining the number of winds, and determining the remainder.

All of the games are now repeated, with the twelve hours on a clock replaced by the seven days of the week. Then there is a summary discussion in which the ideas are mathematized to obtain the notion of an integer mod n where n is $12, 7$, or any positive integer. This permits a discussion of *equivalence* mod n, partitions of a set of integers, and the relationship between equivalence and partition.

One can then return to the clock and days-of-the-week games to do arithmetic, using the same epistemological perspective and the same pedagogy. For addition, one simply plays the winding game with two numbers. With the first number, one begins at the starting point (12 o'clock or Sunday) and then, with the second number, one begins at the ending point reached by the first number. A deep mathematical idea that can be represented in the game (and hence is likely to be accessible to the students) is that one can add two numbers a and b mod n by either adding first and then finding the equivalent mod n or finding the equivalents first and then adding mod n. Of course the standard group properties of $\mathbb{Z}_n$ with addition mod n can be discussed entirely in terms of trips around the clock or in the calendar.

For multiplication, we play the addition game several times using the same number. This leads to multiplication as repeated addition through the use of all of the same pedagogy, including people-talk, feature-talk, mathematization through operations on the features and evaluating a predicate, and assigning symbols. The result is the concept of multiplication mod n. Since the bases 12 and 7 are used, the students can experience directly the mathematical phenomena of the axioms for a field being satisfied in the mod 7 system but not in the mod 12 system. Some of the brighter students may even be interested in thinking about the properties of 12 and 7 that lead to this difference.

The Algebra Project and the Civil Rights Movement

The United States, to lay down the economic foundations for the caste system established after the Civil War, built a steel industry on the backs of the indentured slavery of young black men in Alabama (Blackmon, *Slavery by Another Name*) and established its textile industry on the pittance doled out to sharecroppers picking cotton in Mississippi (Barry, *Rising Tide*; Lemann, *Redemption* and *The Promised Land*). The civil rights movement dismantled the manifestations of the caste system in public accommodations, voting, and the National Democratic Party; however, the clearest manifestation of this caste system remains in its public schools (*U.S. v. State of Mississippi*, Civil Action 3312). The Algebra Project, a direct descendent of the 1961 to 1965 Mississippi Theater of the civil rights movement, tackles head-on this dimension of the nation's unfinished work (Moses, testimony to the U.S. Senate Judiciary Committee).

It is our contention that, with the ascendance of information technology and the increasing complexity of our society, mathematics joins reading and writing as a literacy needed for full citizenship. Like it or not, history has thrust mathematicians and specialists in mathematics education into the middle of a central American dilemma: the reconciliation of the ideals in the Declaration of Independence and the United States Constitution with the structures of race and caste and the legacies of slavery and Jim Crow.

Briefly, in 1875, Congress refused to consider President Grant's appeal for a constitutional amendment to guarantee at the level of the federal government the right to an education for all children, including those of the freed slaves. It did pass a civil rights bill, but the Supreme Court of 1883 declared that Congress had no right to do this, thus setting the stage for eighty-one years of rigid race and class divisions (Civil Rights Cases, 1883; see also Justice Harlan's dissent).

The Court decided that, for the purpose of access to public accommodations, the nation's constitutional people were decisively citizens of states rather than citizens of the nation, a constitutional status applicable to the vote and membership in the national political party structures as well as to public school education.

The Supreme Court's landmark 1954 decision did not challenge, with respect to their education, this constitutional status of the nation's children. Rather it affirmed the "equal protection" clause of the Fourteenth Amendment: states, rather than the federal government, have a constitutional obligation to provide their citizens equal access to public school education. As James Bryant Conant reminded us in 1961, the nation's caste system thus found its clearest manifestation in its education system (Conant, 1961).

Such inequality was confirmed in 1968, when four hundred Mexican American high school students left school to march on their school board to demand better physical facilities and better teachers. Their mothers sued, and their case, "San Antonio Independent School District v. Rodriguez" was decided March 21, 1973:

Justice Lewis Powell's majority opinion in Rodriguez held that education was not a fundamental right, since it was guaranteed neither explicitly nor implicitly in the Constitution.

Powell's decision, in effect, guaranteed that public school education remained the clearest manifestation of the nation's caste system, which now extended over class as well as race. This situation still holds today.

When, in 1960, Kennedy stepped into the presidency, black students at historically black universities and colleges stepped into history: "On February 1, 1960, four African American college students sat down at a lunch counter at Woolworth's in Greensboro, North Carolina, and politely asked for service. Their request was refused. When asked to leave, they remained in their seats. Their passive resistance and peaceful sit-down demand helped ignite a youth-led movement to challenge racial inequality throughout the South" (C. Vann Woodward, 2001).

The sit-in students demanded, in effect, a change in their constitutional status: for purposes of access to public accommodations, they demanded status as citizens of the nation rather than citizens of a state. This demand was made crystal clear a year later, with the Freedom Rides.

Thanks largely to Ella Baker, the sit-in movement was transformed into a network of sit-in leaders called the Student Nonviolent Coordinating Committee, or SNCC. Then, thanks largely to Amzie Moore, SNCC transported the sit-in energy into Mississippi to focus on the constitutional status of sharecroppers in the Mississippi Delta, especially with respect to the right to vote. SNCC organized sharecroppers not only to demand constitutional status as citizens of the nation with respect to voting rights but also to demand an equivalent status with respect to participation in the National Democratic Party structure, making it possible for a Democratic Party Convention to consider an African American as its presidential nominee.

Robert (Bob) Moses, coauthor of this article and president and founder of the Algebra Project, was the director of SNCC's Mississippi operations. He left Mississippi in 1965, left the country in 1966, and made his way to Tanzania with his wife Janet, where they started their family. They returned to the United States in 1976 with their four children: Maisha, Omowale (Omo), Tabasuri (Taba), and Malika. Bob's job in the family was to make sure the kids did their math, a job he enlarged as a parent volunteer in the Open Program of the Martin Luther King School in Cambridge, Massachusetts, to teach Maisha and three of her classmates algebra when she hit the eighth grade in 1982. Bob got a MacArthur fellowship in 1982 and settled into the issue of algebra for all the eighth graders in the Open Program, thereby launching the "Algebra Project", which inevitably found its way into Mississippi and the issue left hanging from the Mississippi civil rights movement of 1961–1965: the constitutional status of children in the nation with respect to their public school education. It seems clear that, unless children become decisively citizens of the nation for the purposes of their public school education, public school education will remain the clearest manifestation of the nation's caste system.

Conclusion

Today, the Algebra Project, working together with sister organizations such as the Young People's Project, with support from the National Science Foundation and other public as well as private agencies, is a national movement that is trying to transform the educational experiences of children from the underserved lowest quartile of our population. It is a prime example of how people from the academic fields of philosophy, mathematics research, mathematics education research as well as teachers and administrators from the field of K–16 education and also those of us who struggle for social and economic justice in the United States can find common ground, work together, and contribute to solving some of the major problems facing our country in the twenty-first century.

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