On the Work of Louis Nirenberg

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Louis Nirenberg received the first Chern Medal Award at the International Congress of Mathematicians (ICM) in August 2010. Sponsored by the International Mathematical Union and the Chern Medal Foundation, the award will be given every four years at the ICM to an individual whose lifelong outstanding achievements in the field of mathematics warrant the highest level of recognition. The award consists of a medal and a monetary award of US$500,000. There is a requirement that half of the award will be donated to organizations of the recipient's choice to support research, education, outreach or other activities to promote mathematics.
The *Notices* solicited the following article about the work of Louis Nirenberg. The International Mathematical Union also issued news releases about the Chern Medal Award, which appeared in the December 2010 *Notices*.

It is an honour to be asked to write this celebration of the award of the first Chern Prize of the International Mathematical Union to Louis Nirenberg, but the author's enthusiasm for the task is matched by considerable doubts about his adequacy for it. Nirenberg is a giant in the subject of partial differential equations and has made fundamental contributions to that area over more than half a century. The author is far from being a PDE specialist, and this account is of necessity highly selective, emphasising those topics (mainly connected with geometry) that the author knows a little better. I will say nothing about huge swathes of Nirenberg's work, which might well be the focus of other essays—for example, his work with Agmon and Douglis on very general elliptic boundary value problems, or his place as one of the fathers of pseudodifferential operators. Our small selection perhaps gives a glimpse of the range of his work, melding PDE theory with classical differential geometry, with the theory of complex manifolds, with harmonic analysis, even though this leaves out much else, such as his work on fluid mechanics.

Isometric Embedding

Nirenberg's Ph.D. thesis [8] completed the solution of a famous problem in differential geometry, developing 1916 work of Weyl. The statement is very simple: an abstract Riemannian metric on the 2-sphere with positive curvature can be realised by an isometric embedding in $\mathbf{R}^3$. (Another proof was given by Pogerolov at about the same time.) This is a wonderful result for several reasons. The statement is easy to grasp and decisive. The strategy of proof, by the "continuity method", is a model for many other PDE problems in differential geometry and elsewhere. For example, the overall strategy in Yau's proof of the Calabi conjecture has just the same shape. Finally, while it might appear an elementary problem, some serious difficulties arise. Nearly one hundred years after Weyl's paper and nearly sixty years after Nirenberg's and having in mind the huge development in "geometric analysis" over this period, one might think that supplying a proof would nowadays be a straightforward exercise, but that is very far from the case.

Let g_1 be a Riemannian metric of positive curvature on S^2. In the continuity method one first shows that g_1 can be joined to a standard "round" metric g_0 by a path g_t, $t \in [0,1]$ of metrics of positive curvature. This is quite easy, using the fact that any metric is conformal to a round metric. Now one considers the set $T \subset [0,1]$ of parameter values t such that g_t can be realised by an isometric embedding. The task is to show that T is both open and closed, and hence must be the whole interval (since the round metric is certainly realised by an embedding and so T is not empty).

A first difficulty is that, if set up in the obvious way as a PDE for a map $\iota : S^2 \to \mathbf{R}^3$, the isometric embedding problem is degenerate. Infinitesimal

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variations of the map in the normal direction change the induced metric by an algebraic, rather than a differential, operator. Thus the usual approach to prove openness, through the inverse function theorem in Banach spaces, does not immediately apply. A more sophisticated machine, the Nash-Moser theory, can be applied [3], but this came later and Nirenberg used an intricate and ingenious argument, partly following Weyl, to get around the difficulty.

The closedness comes down to establishing a priori estimates. One estimate goes back to Weyl. The Gauss curvature K of an embedded surface is given by a quadratic expression in the second fundamental form B, and one finds that

$$\Delta K = b\Delta b - \langle B, \nabla\nabla b\rangle + |\nabla b|^2 - |\nabla B|^2 + K(b^2 - 4K),$$

where Δ is the intrinsic Laplacian on the surface and $b = \mathrm{Tr}B$ is the mean curvature. Then an application of the maximum principle, considering the point where b attains its maximum, gives an a priori bound on b that translates into a C^2 bound on the isometric embedding $\iota : S^2 \to \mathbf{R}^3$. The other major component in Nirenberg's proof is to promote this first to a $C^{2,\alpha}$ bound and then to all higher derivatives. This follows from general theorems about elliptic equations in two variables that he had developed at about the same time [9], together with another ingenious differential geometric device, considering the equation satisfied by the distance on the surface to a fixed origin in $\mathbf{R}^3$.

Nirenberg has many other papers related to classical differential geometry and isometric embedding. There is a particular difficulty at points at which the sign of the Gauss curvature changes and the PDE changes from elliptic to hyperbolic type.

Complex Geometry

Another of Nirenberg's renowned achievements in geometry is the Newlander-Nirenberg theorem on the integrability of almost-complex structures [10], which is a foundation stone for the Kodaira-Spencer-Nirenberg treatment of deformations of complex manifolds [5]. (In the context of this essay, note that the authors of [10] thank Chern for bringing the problem to their attention.) Recall that an almost-complex structure on a $2n$-dimensional manifold M is a bundle map $I : TM \to TM$ with $I^2 = -1$. The question is: when does such an almost-complex structure come from a complex structure; that is, when can one find a diffeomorphism between the neighbourhood of any point in M and a polydisc in $\mathbf{C}^n$ that takes I to the standard almost-complex structure on $\mathbf{C}^n$? The eigenspaces of I yield a decomposition of the complexified tangent bundle $TM \otimes \mathbf{C} = T' \oplus T''$, and a necessary condition is that sections of T' are closed under the Lie bracket. The Newlander-Nirenberg theorem asserts that this condition is also sufficient. This is formally analogous to the Frobenius integrability condition, for a real sub-bundle of the tangent bundle to define a foliation, and when the data is real-analytic one can derive the result from the Frobenius theorem by a complexification argument that goes back, in the case $n = 1$, to Gauss. But if the data is C^∞ or worse, different methods are required.

There are now many different approaches to the proof of this integrability theorem. We have to construct a diffeomorphism between some neighbourhood $N \subset M$ and a polydisc $B \subset \mathbf{C}^n$. The character of the problem appears rather different depending on whether one seeks to construct a map $f : N \to B$ or a map $g : B \to N$. Of course, at the end of the day, once one has shown that the constructed maps are diffeomorphisms, one follows from the other by inversion, but at the outset the problems look different. If we seek a map f, then the problem is *linear*, the condition is just that f be a vector of holomorphic functions and the PDE to be satisfied is $\overline{\partial}_M f = 0$ where $\overline{\partial}_M$ is the natural Cauchy-Riemann operator defined on M. In the classical case when $n = 1$ the problem can easily be solved this way. One takes the neighbourhood N very small so that, in suitable coordinates $\overline{\partial}_M$ can be written as a small perturbation of the standard Cauchy-Riemann operator which is essentially just $\frac{\partial}{\partial \overline{z}}$. Then a local holomorphic function can be constructed by perturbation methods, using the explicit integral operator inverting $\frac{\partial}{\partial \overline{z}}$. The essential difficulty that appears when $n > 1$ is that the equation $\overline{\partial}_M f = 0$ is *overdetermined*. Indeed, we have to use the integrability condition in some way, and in fact if this condition does not hold there will typically be *no* nonconstant local solutions of the equation $\overline{\partial}_M f = 0$.

Subsequent advances in several complex variables, due to Kohn and Hörmander, do lead to proofs of the integrability theorem via linear theory along the above lines, but the original Newlander-Nirenberg approach takes the other point of view, with the construction of a map $g : B \to N \subset M$. Taking $n = 2$ for simplicity, we will have $g(z, w) \in N$ for z, w in the standard disc $D \subset \mathbf{C}$. The condition we want to achieve requires that, for each fixed w the map $\gamma_w : D \to M$ defined by $\gamma_w(z) = g(z, w)$ is holomorphic, so we seek a family of *holomorphic curves* γ_w in M, parametrised by w. From the point of view of the local theory of holomorphic curves, almost-complex manifolds behave much like complex ones—in contrast to the situation with holomorphic functions above. Thus one can produce families of curves γ_w and the integrability condition can be brought in to show that such a family can be constructed so that $\gamma_w(z)$ is also holomorphic in the w variable. Related ideas have become important in the context of Gromov's theory applying holomorphic curves in almost-complex manifolds to symplectic topology. One area in which the Newlander-Nirenberg theorem has been crucial is "twistor theory", which encodes equations satisfied by a metric on space-time in the integrability condition for an almost-complex structure on twistor space.

Now we turn to the work of Kodaira, Nirenberg, and Spencer [5]. Many readers will be familiar with the simplest example of variation of complex structure—the classification of Riemann surfaces of genus 1 by a discrete quotient of the upper half plane. The Kodaira-Nirenberg-Spencer theory gives a vast extension of this idea to a general compact complex manifold M. Deformations of the almost-complex structure can be parametrised by certain tensors $\mu \in \Omega^{0,1}(TM)$—the differential forms of type $(0,1)$ with values in the tangent bundle of M—and the integrability condition takes the form of a first-order nonlinear PDE

$$\overline{\partial}\mu + [\mu, \mu] = 0.$$

The problem of deformation theory is, roughly speaking, to describe the small solutions μ of this equation, modulo the action of the diffeomorphism group of the manifold. If we naively linearise the equation about $\mu = 0$ by dropping the quadratic term, we simply have the equation $\overline{\partial}\mu = 0$, which states that μ defines a class in $H^1(TM)$ viewed as the Dolbeault cohomology defined by the complex

$$\Omega^0(TM) \xrightarrow{\overline{\partial}} \Omega^1(TM) \xrightarrow{\overline{\partial}} \Omega^{0,2}(TM) \xrightarrow{\overline{\partial}} \dots.$$

Kodaira, Nirenberg, and Spencer considered the case when $H^2(TM) = 0$, which is just the statement that for any $\rho \in \Omega^{0,2}(TM)$ with $\overline{\partial}\rho = 0$ one can solve the equation $\overline{\partial}\nu = \rho$. They construct deformations parameterised by a neighbourhood of 0 in $H^1(TM)$ with a universal property that implies that *any* sufficiently small deformation of M is isomorphic to one in their family. Their method is to construct μ as a power series $\mu = \mu_0 + t\mu_1 + t^2\mu_2 + \dots$ where μ_0 satisfies $\overline{\partial}\mu_0 = 0$ and the subsequent terms are found by solving equations of the form $\overline{\partial}\mu_i = \rho_i$ with ρ_i determined by $\mu_0, \dots, \mu_{i-1}$. The proof was an early application of Hodge theory, which gives a preferred solution to an equation $\overline{\partial}\nu = \rho$, admitting Hölder estimates. The theory was later extended by Kuranishi [6] to cases in which $H^2(TM)$ does not vanish.

For a very simple example of these ideas, go back to a Riemann surface of genus 1 (a torus). Then TM is trivial, so the cohomology group $H^1(TM)$ is the same as $H^1(\mathcal{O})$, which is isomorphic to $\mathbf{C}$. The theory gives a family of deformations of the complex structure parameterised by a small disc in $\mathbf{C}$, and of course this is just what we see in the familiar explicit representation using the upper half plane. A striking example of the power of the theory comes in the case of K3 surfaces. Here one gets certain "visible" deformations arising from algebraic geometry. For example, if we consider the family of K3 surfaces defined by smooth surfaces of degree 4 in $\mathbf{CP}^3$, then one finds a nineteen-dimensional family of deformations (the quartic polynomial has thirty-five coefficients and the linear group has dimension sixteen). On the other hand, calculation shows that $\dim H^1(TM) = 20$ and the theory tells us that there is actually a twenty-dimensional family of deformations. This is an analytical statement—the generic member is not an algebraic surface at all, and the picture is inaccessible from algebraic geometry.

Analysis

One of Nirenberg's most famous achievements is his introduction, with F. John, of the function space BMO [4]. The L^p norms of functions are a familiar tool in PDE theory and analysis generally, but often the information one has is limited to some fixed value of p. For example, the L^2 norm of the derivative of a function appears as the Dirichlet integral in the theory of harmonic functions and harmonic maps. Rather than varying the exponent, one can vary the domain of the integrals considered. The Morrey spaces M_p on $\mathbf{R}^n$ are defined by the finiteness of the norm

$$\|\phi\|_{M_p} = \sup_B \frac{1}{|B|^{1-1/p}} \int_B |\phi|.$$

Here B runs over the set of balls in $\mathbf{R}^n$, and we write $|B|$ for the volume. Hölder's inequality implies that $L^p \subset M_p$, and in fact many results about L^p extend to M_p. Thus one can get good information about a function f from consideration of the M_p norm of $|\nabla f|^2$—which is defined by the Dirichlet integral over balls—as opposed to the less accessible L^{2p} norm of the derivative.

The Morrey norm when $p = \infty$ reduces to the L^∞ norm. John and Nirenberg's BMO (bounded mean oscillation) norm is somewhat different:

$$\|\phi\|_{BMO} = \sup_B \frac{1}{|B|} \int_B |\phi - \underline{\phi}_B|,$$

where $\underline{\phi}_B$ is the mean value of ϕ over B. A basic example of an unbounded function in BMO is $\log|x|$ on $\mathbf{R}$. The famous John-Nirenberg inequality fills out the idea that BMO is "slightly larger" than L^∞ but smaller than any L^p. For a function ϕ supported on the unit ball $B \subset \mathbf{R}^n$, with integral zero and with BMO norm 1 the inequality gives a fixed bound on the integral of $e^{a\phi}$ for a certain definite value of a, depending on n. The proof uses the Calderón-Zygmund cube decomposition. It is related to the fact, discovered subsequently by Fefferman, that BMO is the dual of the Hardy space H^1. Indeed, a function in H^1 has an "atomic decomposition" [11] $f = \sum \lambda_i f_i$ (a.e.) where $\lambda_i \in \mathbf{R}$, $\sum |\lambda_i| < \infty$, each function f_i is supported on a ball B_i has integral zero and is bounded in modulus by $|B_i|^{-1}$. Given this representation (which is related to wavelet expansions) it is clear, at least, that if ϕ is in BMO, then the integral $\int \phi f$ is well defined, and this leads to the duality between the two spaces. Many fundamental results of harmonic analysis involving L^p spaces fail for the extremes $p = 1, \infty$, and the pair BMO, H^1 provide the correct substitutes. For example, if a harmonic function on a half-plane has normal derivative on the boundary in L^∞, then the tangential derivative need not be in L^∞ but it must be in BMO. In PDE theory the John-Nirenberg inequality often appears through a Sobolev-type estimate for functions at the critical exponent $n/(n-1)$. A compactly supported function f with derivative in $L^{n/n-1}$ is not continuous, but it does lie in BMO and hence satisfies the corresponding exponential integral estimate.

General PDE Theory

It is ridiculous to attempt to describe Nirenberg's massive contribution to PDE theory in the few lines here. We are looking at a period of more than a half century in which the literature on even, say, elliptic second-order equations runs to many thousands of pages and whose intricate developments are covered in texts such as [2]. The awesome number of citations to Nirenberg's papers is one measure of the central nature of his contributions to this huge field. Nevertheless, let us try to pick out some strands.

As a model for a nonlinear PDE consider the Monge-Ampère equation

$$\det\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right) = e^\rho,$$

where u is to be convex. Linear theory enters when one considers the equation satisfied by a derivative $h = \frac{\partial u}{\partial x_k}$ which is $L(h) = \rho_k$, where L is the linear operator

$$L(h) = \sum a^{ij} \frac{\partial^2 h}{\partial x_i \partial x_j},$$

(a^{ij}) being the inverse matrix of the Hessian $\frac{\partial^2 u}{\partial x_i \partial x_j}$. The idea is that a good theory of linear operators with suitably general coefficients a^{ij} will allow bootstrapping to derive improved information about the derivatives. One important issue is the *uniform ellipticity* of the linear equation: a bound on the ratio of the maximum and minimum eigenvalues of (a^{ij}). Another is the *continuity* of the coefficients a^{ij}. If these satisfy a fixed modulus of continuity, then the linear operator L can be approximated by a constant coefficient operator on balls of a fixed size, and the situation is rather well controlled.

The two-dimensional case is special because an operator L can be related to a generalised Cauchy-Riemann operator and thus to the theory of quasiconformal maps. We have mentioned in the first section above Nirenberg's early work [9] deriving regularity theorems, using this approach, for a very general class of elliptic PDE in two dimensions (related to work of Morrey). Here the crucial point is to show that (a^{ij}) uniformly elliptic implies a $C^{1,\alpha}$ bound on $\frac{\partial u}{\partial x_k}$ and hence a Hölder ($C^{,\alpha}$) modulus of continuity of a^{ij}. A renowned development in the years around 1960, due to de Giorgi, Nash, and Moser, was a general regularity theory for "quasilinear" elliptic equations in higher dimensions. The John-Nirenberg inequality was applied by Moser [7] to (quoting from [2]) "bridge a vital gap" in the proof of the fundamental Harnack inequality in the relevant linear theory. A general theory for fully nonlinear equations in higher dimensions, such as the Monge-Ampère equation, came later with the work of Caffarelli, Nirenberg, and Spruck [1]. This concerns the Dirichlet problem, where u is defined on a convex set $\Omega \subset \mathbf{R}^n$ with prescribed value on the boundary. The crucial issue turns out to be deriving a modulus of continuity for the second derivatives of u along the boundary. Caffarelli, Nirenberg, and Spruck obtained a logarithmic modulus of continuity by an extremely delicate application of the maximum principle (construction of a barrier function). This crucial logarithmic bound, finer than any Hölder estimate, has some relation, in spirit and content (involving the relation between tangential and normal derivatives), with the ideas surrounding BMO.

References

[1] L. Caffarelli, L. Nirenberg, and D. Spruck, The Dirichlet problem for nonlinear second order differential equations, I, *Comm. Pure Appl. Math.* **37**, no. 3 (1984), 369–402.

[2] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, 1983.

[3] R. Hamilton, The inverse function theorem of Nash and Moser, *Bull. Amer. Math. Soc.* **7**, no. 1 (1982), 65–222.

[4] F. John and L. Nirenberg, On functions of bounded mean oscillation, *Comm. Pure. Applied Math.* **14** (1961), 415–426.

[5] K. Kodaira, L. Nirenberg, and D. C. Spencer, On the existence of deformations of complex analytic structures, *Ann. of Math. (2)* **68**, no. 3 (1958), 450–459.

[6] M. Kuranishi, *New proof for the existence of locally complete families of complex structures*, in Proc. Conference on Complex Analysis, Minneapolis, Springer, 1965.

[7] J. Moser, On Harnack's theorem for elliptic differential equations, *Comm. Pure Appl. Math.* **14** (1961), 577–591.

[8] L. Nirenberg, The Weyl and Minkowski problems in differential geometry in the large, *Comm. Pure. Appl. Math.* **6** (1953), 337–394.

[9] ______ On nonlinear elliptic partial differential equations and Hölder continuity, *Comm. Pure. Appl. Math.* **6** (1953), 103–156.

[10] A. Newlander and L. Nirenberg, Complex analytic coordinates in almost-complex manifolds, *Ann. of Math. (2)* **65**, no. 3 (1957), 391–404.

[11] E. Stein, *Harmonic Analysis*, Princeton Univ. Press, 1993.