3N Colored Points in a Plane

Günter M. Ziegler

Why do I want to tell you about the colored Tverberg theorem? Well:

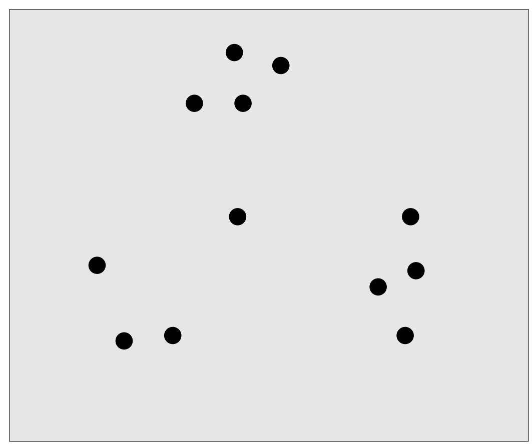
- the setting sounds so harmless, so elementary, as if children were playing with a few colored points in the plane;
- it comes with anecdotes, like the one about a Norwegian mathematician freezing in a Manchester hotel room;
- the latest twist in the story, and the key to a solution, has not been in a very technical proof but rather in the right wording of the problem and of the final outcome;
- an interesting mix of geometric, combinatorial, algebraic, and topological methods have been used;
- after almost two decades, progress is still being made (and I am happy I could contribute to it); and finally
- there is a lot more to do: intriguing conjectures that again sound harmless, elementary, playful.

A Conjecture by Birch

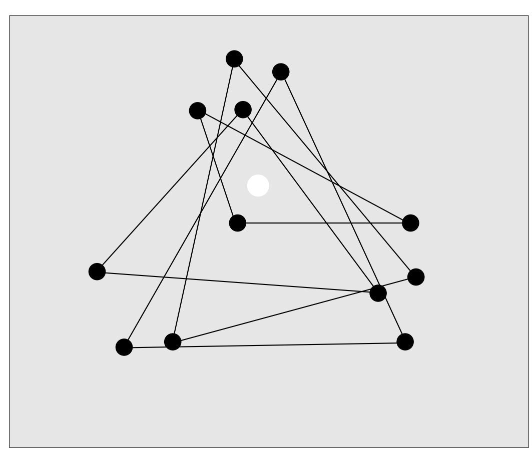
"On 3N points in a plane" is the title of a short paper [6] by Bryan John Birch from 1959. Its main result is right on the first page:

Birch's Theorem 1: *Any* $3N$ *points in the plane determine* N *triangles that have a point in common.*

In order to illustrate this, we may assume that the $3N$ points are in general position. For $N = 4$ we have $3N = 12$ points, and the situation might look like the following diagram:



The claim would be that for *any* such point configuration in the plane there is a partition into triples, such that the corresponding triangles have a point in common:



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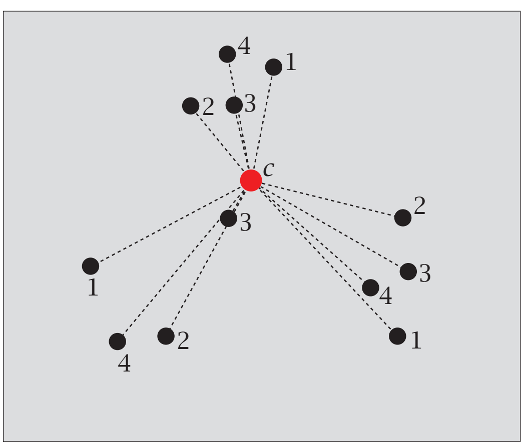
Birch's *proof* for this is remarkable, as it uses a topological fixed point theorem—years before Lovász in 1978 proved the Kneser conjecture using the Borsuk–Ulam theorem, which is commonly seen as the starting point of topological combinatorics [9]. Birch indeed derives his result from the center point theorem, which had been first provided by Bernhard H. Neumann in 1945 [15]:

Center Point Theorem. *For any* $3N$ *points in the plane there is a point* c*, called a "center point", such that any half plane that contains* c *must necessarily also contain at least* N *out of the given* $3N$ *points.*

Birch criticized Neumann's proof for this result: "his proof, though elementary, is long, and does not extend to higher dimensions". He thus proceeded to give his own proof, which also provides an n-dimensional center point theorem. However, apparently Birch did not know that Richard Rado had achieved the result much earlier, published in 1946 [16]. Rado's proof for this is elementary geometry (it uses Helly's theorem—"if any n sets in a finite family of convex sets in $\mathbb{R}^n$ have a point in common, then *all* the sets have a point in common"); see, e.g., [12, Sect. 1.4].

■ Here is how Birch derived his Theorem 1 from the center point theorem: Label the $3N$ points "$1, 2, \ldots, N, 1, 2, \ldots, N, 1, 2, \ldots, N$" in clockwise order around the center point c:



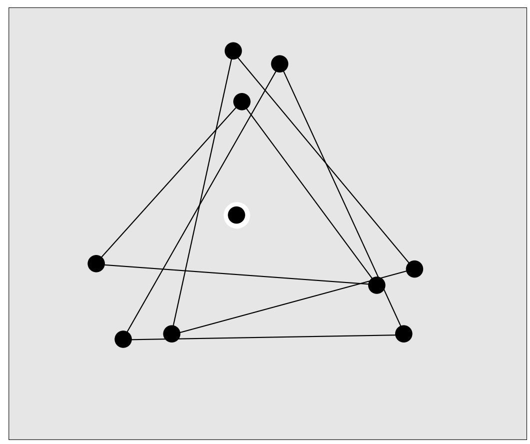
Then the center point c lies in each triangle determined by three points with the same label i. Indeed, each half space with c on the boundary contains at least N contiguous points from the circular sequence, and thus at least one point of each label i.

And here is the idea used by Birch to prove the center point theorem: If a point x is *not* a center point, then some, but not all, of the half spaces that have x on the boundary are "bad", i.e., contain less than N of the $3N$ points. These half planes at x point you into directions to look for improvement. By averaging, Birch uses that to define, on a disk that contains the $3N$ points, a continuous (!) vector field which on the boundary of the disk points inside. By Brouwer's fixed point theorem this vector field must have a zero—which turns out to be a center point. ■

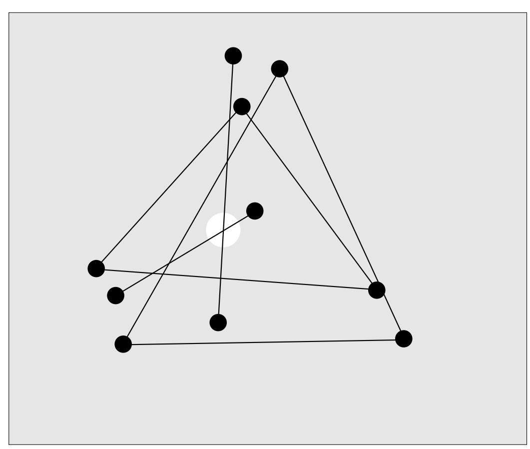
In the same little paper [6] Birch observes that the bound "$3N$" is not really tight, as one can also get a result about only $3N - 2$ points:

Birch's Theorem 1*: *Any* $3N - 2$ *points in the plane can be partitioned into* N *subsets whose convex hulls have a point in common.*

The subsets thus consist of one, two, or three points, and hence their convex hulls are triangles, edges, or single points—and we require that these N convex sets have a point in common. The drawing illustrates this for $N = 4$:



A solution could thus be given by one single point that lies in $N - 1$ triangles, or there could be $N - 2$ triangles that all contain the intersection point of two further edges:

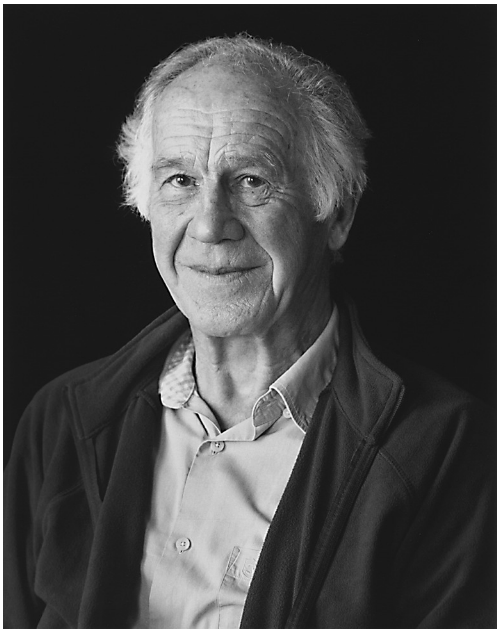


And it is also easy to see that fewer than $3N - 2$ points will in general not be divisible into N subsets whose convex hulls intersect. (Not even the affine hulls will intersect, for codimension reasons.) In that sense, Birch's Theorem 1* is *sharp*.

Who is Bryan John Birch? Where does this problem come from? His paper contains only a small hint at the end:

> In conclusion, I would like to thank Professor Eggleston for his entertaining lectures, which led me to perpetrate this note.

But of course I could try to ask him (he was born in 1931 and is an emeritus professor in Cambridge, England). My email was answered promptly:



Bryan John Birch

> I was an undergraduate at Trinity (College, Cambridge) from 1951-4, and in my third year I took Part III of the Mathematical Tripos; then as now, one attended 6 to 8 "starting graduate student level" courses and took a cross-section exam at the end of the year; nowadays of course it is no longer taken by 3rd year students. One of the courses I took, in the spring of 1954 I would guess, was a very pleasant course on "Convexity" by H. G. Eggleston; this course formed the basis for his Cambridge Tract, published in 1958 according to my references. Eggleston's course contained several proofs of the isoperimetric inequality, as well as Helly's theorem (of course) and I was involved with more complicated plane geometry when I started research in the Geometry of Numbers under Ian Cassels; so it wasn't unnatural for me to be thinking about "3N Points in a Plane"; but one tends not to remember details of what one was thinking about 54 years ago.
>
> I submitted a thesis in competition for a Trinity Junior Research Fellowship in September 1956; this thesis was a compendium of various bits and pieces, including "3N points in a plane". The judges of the competition came from all faculties, so one had to include a summary that laymen could read. I remember that my summary included a picture of the application of the first non-Helly case (7 points in a plane) to the configuration formed by the seven brightest stars in the Pleiades. I don't remember why the paper wasn't submitted till 1959: I was working on several other things—I switched to additive analytic number theory, and then to elliptic curvery—and very probably 1959 was when I gave up trying to prove higher dimensional cases of Tverberg's theorem; another possibility is that I was jogged into writing up by the publication of Eggleston's tract.

Indeed, Birch became famous for his work on elliptic curves—in particular since it led to a one-million-dollar millennium problem that carries his name, the "Birch and Swinnerton-Dyer conjecture".

Another one of his conjectures did not carry such a cash prize but turned out to be important and influential nevertheless: again in his paper "On 3N points in a plane", on the same page as his Theorem 1*, we find a conjectured n-dimensional version of it:

Birch's Conjecture: *Any $(n+1)N-n$ points in $\mathbb{R}^n$ can be partitioned into N subsets, whose convex hulls have a point in common.*

And when Birch now writes that probably 1959 was when he gave up trying to prove "higher dimensional cases of Tverberg's theorem", then he reverses the historical order of things: he gave up trying to prove higher dimensional versions of Birch's theorem, that is, cases of his own conjecture. Tverberg came later.

Tverberg's Theorem

Helge Tverberg, born in 1935, is a Norwegian mathematician. In 1961 he attended a workshop on functional analysis at University College London, where on the sidelines there was also a course on convexity, presumably taught by Rogers [12, p. 16]:

> I found this material fascinating, and read upon it more back in Bergen. Helly's Theorem was especially fascinating, and, in my

> reading, I came upon the following application. Let S be a set of $3N$ points in the plane. Then, there is a point p, not necessarily in S, such that every half-plane containing p contains at least N points from S. It struck me that this would follow simply if it were always possible to split S into N triplets so that the N triangles so formed would have a common point p. For, a half-plane containing p would contain at least 1 vertex from each triangle.

Thus Tverberg ran into Birch's problem in an attempt to prove Rado's center point theorem—just the opposite direction from that taken by Birch.

Archives of the Mathematisches Forschungsinstitut Oberwolfach.

Helge Tverberg, 1981.

In 1962 Tverberg attended the ICM in Stockholm, Sweden, and there, after a dinner with Bryan Birch and Hallard Croft from England, before parting at some street corner, he told Croft about his problem about 3N points in the plane. Croft had to disappoint him, the result already being known and having been published by Birch—but he suggested that Tverberg could try his luck on the higher-dimensional case, which Birch had been unable to do.

Then in 1963 Tverberg first solved the three-dimensional case by a complicated proof that consisted of seven separate cases and offered no hope for an extension to higher dimensions.

One year later, in 1964, he then obtained a travel stipend to England, where he wanted to discuss the problem with Birch (then in Manchester) and with Richard Rado (at the University of Reading). Rado had also obtained partial results. What happened then, Tverberg describes as follows [21, p. 16/17]:

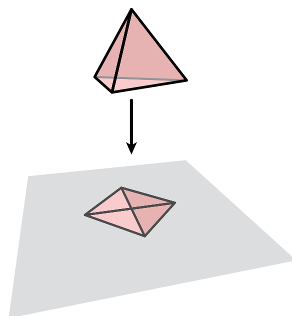
> I recall that the weather was bitterly cold in Manchester. I awoke very early one morning shivering, as the electric heater in the hotel room had gone off, and I did not have an extra shilling to feed the meter. So, instead of falling back to sleep, I reviewed the problem once more, and then the solution dawned on me!
>
> I explained it to Birch, and, after an agreeable day of mathematical conversation with him, returned to Norway to start writing up the result.

Birch disagrees on this: he remembers that Tverberg was not all that interested in explaining his solution, and rather more in seeing a bit of England on his last day. But it's not our job to resolve this apparent contradiction here. In any case, in 1966 (submitted May 8, 1964) Tverberg's paper "A generalization of Radon's theorem" [20] appeared, which he refers to as "T66". The Tverberg theorem is his most famous result, which he came back to again and again. Thus "A generalization of Radon's theorem, II" appeared in 1981 with a new proof, and "On generalizations of Radon's theorem and the ham sandwich theorem" (joint work with Siniša Vrećica) in 1993, which contains a tantalizing conjecture, an extension of Tverberg's theorem to transversals. The original Tverberg theorem now has several different proofs, including those by Tverberg, Roudneff, Sarkaria, and more recently by Zvagel'skii. An especially elegant version is due to Karanbir Sarkaria [17], with further simplifications by Shmuel Onn [4] [11] [12, Sect. 8.3].

Tverberg's theorem, as proved in the "T66" paper, happens to be exactly Birch's conjecture. Nevertheless, we would phrase it differently today. Thus from now on we use d to denote the dimension (that is, $d = 2$ in Birch's classical theorem). We use the letter r for the number of subsets we want to partition into (which was previously denoted by N). And we use the letter N to denote $N := (d+1)(r-1)$ (and thus unfortunately it now means something completely different than before). And instead of discussing $N+1$ points in the plane and the convex hulls of subsets, we now consider an N-dimensional simplex Δ_N (which has $N+1 = (d+1)(r-1)+1$ vertices) and an affine map that in particular positions the $N+1$ vertices of Δ_N in $\mathbb{R}^d$. Thus Tverberg's theorem gets its modern form:

Birch's Conjecture = Tverberg's Theorem: *Let $d \geq 1$, $r \geq 2$, and $N := (d+1)(r-1)$. For every affine map $f : \Delta_N \to \mathbb{R}^d$ there are r disjoint faces of Δ_N whose images under f intersect.*

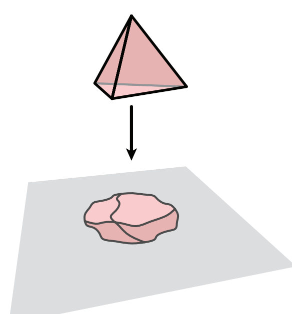
The following diagram illustrates this result for the small parameters $d = 2$ and $r = 2$, where we get $N = 3$, and thus have to consider a map of a tetrahedron (three-dimensional simplex) to the plane.



The Topological Tverberg Theorem

The modern version of the Tverberg theorem is not only more succinct (and a bit more abstract), but it also has the advantage of suggesting a generalization, known as the "topological Tverberg theorem", which the Hungarian mathematicians Imre Bárány and András Szűcs, together with the Russian Senya B. Shlosman, presented in 1981.

Topological Tverberg Theorem: [5] *Let* $d \geq 1$, $r \geq 2$, *and* $N := (d+1)(r-1)$. *For every* continuous *map* $f : \Delta_N \to \mathbb{R}^d$ *there are* r *disjoint faces of* Δ_N *whose images under* f *intersect.*



The name "topological Tverberg theorem" happens to be pretty bad terminology, not only since neither Tverberg nor the theorem is "topological" but—much more seriously—since in the version I have just stated this is not a theorem, but only a *conjecture*. Indeed, Bárány et al. claimed and proved this only for the case where $r \geq 2$ is a prime. Only later was this extended to the case of prime powers r, first by Murad Özaydin in an unpublished preprint from 1987. We refer to the wonderful textbook *Using the Borsuk–Ulam Theorem* by Jiří Matoušek [13] for details and references. In any case the conjecture remains open and a challenge up to now for $d \geq 2$ and nonprime powers r.

"… We Need a Colored Version …"

A team of three Hungarians, Imre Bárány, Zoltan Füredi, and László Lovász, in an influential computational geometry paper "On the number of halving planes" (conference proceedings version 1988, journal publication 1990 [3]) stumbled upon a situation with three disjoint sets A, B, C of points in the plane and observed:

> For this we need a colored version of Tverberg's theorem.

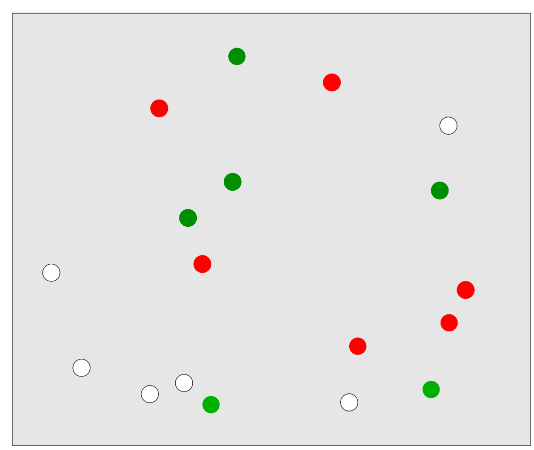
In their paper they needed only a very simple small special case:

Let A, B, C *be sets of* t *red, green, resp. blue points in the plane, then one can find* $r = 3$ *disjoint triples consisting of one of point of each color such that the convex hulls of the triples have a point in common.*

They gave a proof for $t = 7$, asserted they also had a proof for $t = 4$, but also noted that they had no counterexample even for $t = 3$.

The call for a colored version of Tverberg's theorem was seen as a challenge and attacked immediately. The first answer, by Imre Bárány and David Larman in 1991, treated the case of $3r$ points in the plane, with three different colors:

> "… given r red, r white, r green points in the plane …".

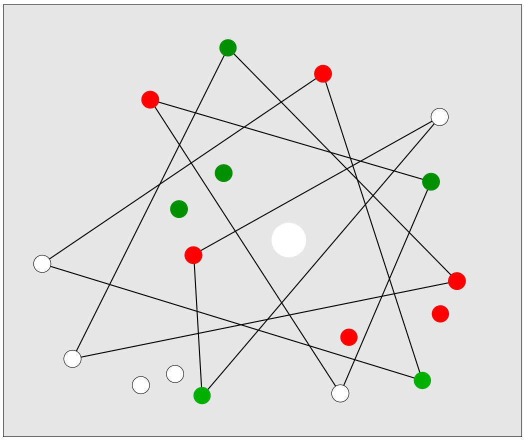


(Question: Why these particular three colors? I have recently asked David Larman; he didn't know. Perhaps his coauthor managed to slip in the colors of the Hungarian flag?)

Anyway, here is the answer suggested by Bárány and Larman:

Colored Tverberg Theorem: *Let* $d \geq 1$ *and* $r \geq 2$, *and* $f : \Delta_N \longrightarrow \mathbb{R}^d$ *affine (order at least continuous), where the* $N+1$ *vertices of* Δ_N *carry* $d+1$ *different colors, and every color class* C_i *has size* $|C_i| \geq t$ *for a sufficiently large* t. *Then* Δ_N *has* r *disjoint* rainbow faces, *whose images under* f *intersect.*

Of course again here neither Tverberg nor the theorem is colored. And the statement just proposed is not a useful theorem, as it does not specify the "sufficiently large t". A *rainbow face* refers to a d-dimensional face of the simplex whose $d+1$ vertices carry the $d+1$ different colors. In the case $d = 2$ thus we have at least $3t$ points in the plane, which carry the three different colors. The claim is that then there are r rainbow triangles that have a point in common:



For $d = 2$ Bárány and Larman proved this, and indeed they obtained a "sharp" colored version of Birch's theorem: for $d = 2$ it suffices to require that $t \geq r$. For $d = 1$ the analogous colored version of Tverberg's theorem is a nice exercise. For $d > 2$ Bárány and Larman presented it as a conjecture.

This was answered by a breakthrough paper by Rade Živaljević and Siniša Vrećica from Belgrade, Yugoslavia, published in 1992 [24]. They introduced new concepts and methods to topological combinatorics (in particular, "chessboard complexes") and thus could show that the colored Tverberg, in the version just given, holds for $t \geq 2r-1$, if r is a prime, and thus also for $t \geq 4r-3$ and all $r \geq 2$, due to Bertrand's postulate that there is always a prime between n and $2n$ [1, Chap. 2].

Živaljević and Vrećica's breakthrough got a lot of attention. In particular, Jiří Matoušek in Prague was so excited that he gave a course that eventually led to the textbook [13] mentioned before, which develops all the mathematics from scratch that is needed to arrive eventually, in the last section of the book, at the Živaljević–Vrećica proof of the colored Tverberg theorem.

Nevertheless, the gap between $t = r$ and $t \geq 4r-3$ remained: the colored Tverberg theorem of Živaljević and Vrećica is not sharp. And also it is not a generalization or strengthening of the classical, color-free Tverberg theorem.

A sharp version was obtained only very recently, again by a team of three. Pavle V. M. Blagojević from Belgrade, my Berlin Ph.D. student Benjamin Matschke, and I were ready in October 2009 to present a proof that $t = r$ suffices when $r+1$ is a prime (and thus $t = r + o(r)$ is good enough for large r).

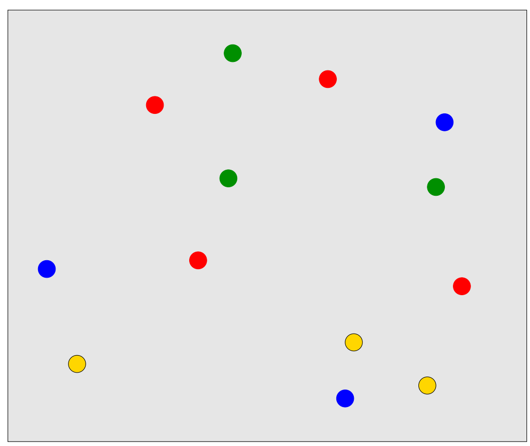
This result was a surprise to us (and even more to others, perhaps), since we arrived at it via considerable detours and it needed a substantial change of perspective. Indeed, we arrived at a *new* colored Tverberg theorem that uses more colors, requires different assumptions about the color classes, and contains the classical Tverberg theorem as a special case—and turns out to be much easier to prove.

A New Colored Tverberg Theorem

Here it is:

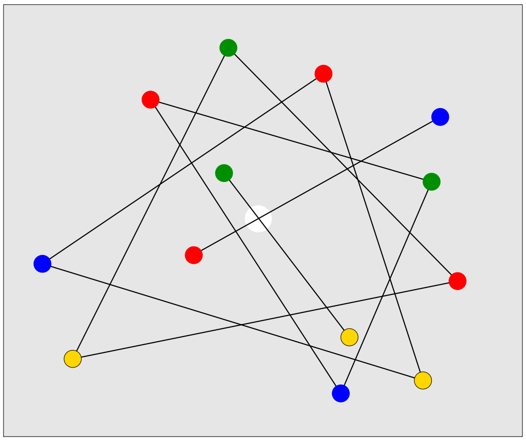
New Colored Tverberg Theorem [7]: *Let $d \geq 1$, $r \geq 2$ prime, $N := (d+1)(r-1)$ and $f : \Delta_N \longrightarrow \mathbb{R}^d$ affine (or at least continuous), where the $N+1$ vertices of Δ_N have at least $d+2$ different colors, and each color class C_i has size $|C_i| \leq r-1$. The Δ_N has r disjoint rainbow faces whose images under f intersect.*

Thus, for example, we consider the following situation in the plane ($d = 2$) for $r = 5$, in which the $N+1 = 3\cdot 4+1 = 13$ points have four different colors and no color is used more than $r-1 = 4$ times (see the following figure).



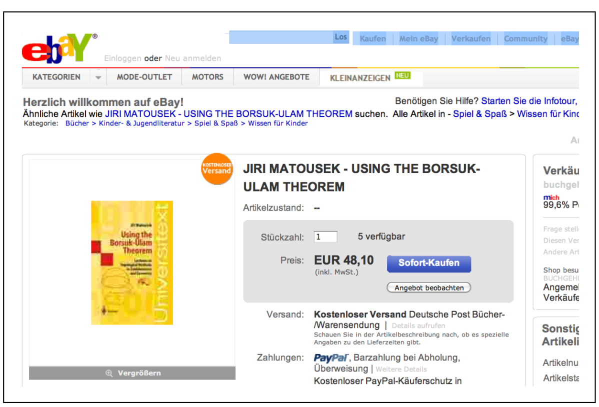
In comparison to the original colored Tverberg theorem the number of colors has changed (not $d+1$ any more, but at least $d+2$), the sizes of the color classes have changed (not "large enough" any more, and at least r, but instead less than r), and the definition of a *rainbow face* has changed (they do not have to carry all the colors any more, and indeed they can't, but now they are defined as faces on which no vertex color appears more than once). One of the solutions looks like the figure at the top of the next page. Try to find one yourself, before you turn the page!

Our theorem admits the special case that all color classes have size 1; thus all the vertices of the simplex Δ_N have distinct colors, and thus all faces have the rainbow property, and thus we obtain the original topological Tverberg theorem by Bárány–Shlosman–Szűcz as a special case.



In a second special case we could have $d+1$ color classes of size $r-1$, and one final color class would consist of a single additional vertex with a separate color. This special case turns out to be important since we can derive the sharp classical Tverberg theorem from it (for primes r), but also since this special case indeed also implies the theorem in full generality. Both these reductions are elementary geometric (and the ideas needed have been used before in similar contexts).

Thus we must prove the new colored Tverberg theorem for the case of color classes $|C_0| = |C_1| = \cdots = |C_d| = r-1$ and $|C_{d+1}| = 1$. For this we use the by-now classical configuration space/test map scheme, which can be learned from the textbook [13]—which recently on German eBay appeared in the category "Books > Children's & Youth Literature > Fun & Games > Knowledge for Children". So this is certainly a book you and your family cannot do without:



According to this scheme we have to show that a certain equivariant map does not exist. More precisely, we want to show that there is no continuous map

$$F : (\Delta_{r-1,r})^{*(d+1)} * [r] \longrightarrow_G S^{N-1}$$

that is compatible with the action of a finite group G (here a cyclic or a symmetric group). Here the topological space on the left-hand side is a simplicial complex that encodes the sets of r points on r disjoint rainbow faces of Δ_N; the right-hand side encodes the r-tuples of points (not all equal) they would map to.

The classical "there is no such map" result of this type is the Borsuk–Ulam theorem, which says that there is no continuous map

$$F : S^n \longrightarrow_{\mathbb{Z}_2} S^{n-1}$$

that would be compatible with the antipodal action of the group $\mathbb{Z}_2$.

For our more complicated setup we have even provided three different proofs. The easiest one may be found in [8]; it uses the mapping degree. One interesting aspect of this proof is that it asks us to *count* the number of Tverberg partitions for a certain point configuration, where we get the result $(r-1)!^d$. We eventually conclude that $(r-1)!^d$ must be divisible by r, which is not true if r is a prime. (See also [23] for a sketch of this proof, and [14] for an elementary "topology-free" version.)

The second proof (which we found first) is technically more demanding; it uses equivariant obstruction theory, which one can learn from Tammo tom Dieck's book on transformation groups [19, Sect. II.3] and then apply to our concrete situation. Here one has to act with care, as the action of the symmetric group is not free. This proof is not only more difficult than the first one, but it also yields more: we get that the configuration space/test map proof scheme works, even if we use the full symmetric group, *if and only if* $(r-1)!^d$ is not divisible by r, that is, if r is a prime and in the uninteresting case $r=4$, $d=1$. In all other cases, the equivariant map F in question does exist, and we cannot conclude anything.

The third proof, also from [8], is the most complicated one: it computes the Fadell–Husseini index, an ideal of the cohomology ring of the group that we have acting. However, it also yields even more: We get the full theorem directly, without previous reduction to the special case of color classes of sizes $d-1$ resp. 1, and thus it can be extended to a proof of the transversal generalization of the new colored Tverberg theorem.

> *"Proofs should be communicated only by consenting adults in private"*
> — Victor Klee (U. Washington)

Questions, Problems, Challenges

1. As mentioned above, for the classical Tverberg theorem we have "elementary" linear algebra proofs that would work for *all* $r \geq 2$. Is there a similar proof also for the affine case of the new colored Tverberg theorem?

2. The Tverberg theorems, whether colored or not, promise to us the *existence* of a specified type of partition of a point configuration. How would one *find* one? Is it easy to find such a partition, can one *compute* one in polynomial time? This is not clear at all—not even for the probably much simpler colored Carathéodory theorem of Bárány [2] [12, Sect. 8.2].

3. Whoever plays around with instances of the Tverberg theorem will notice that typically there is not only one Tverberg partition, but indeed *many* of them. For the configuration of $(d+1)(r-1)+1$ points in $\mathbb{R}^d$ that is suggested by our third figure, there are exactly $(r-1)!^d$ distinct partitions. Gerard Sierksma from Groningen, Netherlands, has conjectured that there are always (that is, for all point configurations) at least $(r-1)!^d$ distinct Tverberg partitions. Indeed, he has offered one whole Gouda cheese for a proof of his conjecture, which is why it is known as "Sierksma's Dutch Cheese Problem". This problem is open even for the case of $d = 2$. Lower bounds on the number of solutions have been obtained by Aleksandar Vučić and Rade Živaljević [22] in the case of primes and by Stephan Hell [10] for the case of prime powers. It may well be that good lower bounds are easier to prove for the Tverberg theorem with colors and that those could eventually be put together to yield tight bounds for the case without colors.

4. The case when r is not a prime power continues to be the greatest challenge. For $d = 2$ Torsten Schöneborn and I have recast this into a graph drawing problem [18]. Thus, for the smallest case of $r = 6$, we would ask whether every drawing of the complete graph K_{16} in the plane either has a vertex that is surrounded by five triangles (with winding number not equal to zero) or whether some crossing of two disjoint edges is surrounded by four triangles. Counterexample, anyone?

> *"I was of course flabbergasted by the variety of generalisations that have blossomed in that particular garden!"*
>
> — Bryan Birch (2010)

Acknowledgments

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