Remembering Paul Malliavin

Daniel W. Stroock and Marc Yor

Paul George Malliavin

On June 3, 2010, Paul Malliavin died at the American Hospital in Paris. At the time of his death, he was four months short of his eighty-fifth birthday.

Malliavin was a major mathematical figure throughout his career. He studied under Szolem Mandelbrojt, who had returned to France after World War II from the United States, where he had been on the faculty of what, at the time, was the Rice Institute. Both Malliavin and Jean-Pierre Kahane received their degrees under Mandelbrojt in 1954, and Yitzhak Katznelson received his from Mandelbrojt a couple of years later. Thus, in less than three years, Mandelbrojt produced three students who would go on to become major figures in mid-twentieth-century harmonic analysis.

Malliavin's own singular contribution to harmonic analysis is described here by Kahane. Like many other definitive solutions to mathematical problems, Malliavin's solution to the spectral synthesis problem killed the field, with the ironic consequence that few young mathematicians even know the statement of the problem, much less the name of the person responsible for its solution.

Not one to rest on his laurels, Malliavin soon turned his attention in new directions. His early work won him an invitation to visit Arne Beurling at the Institute for Advanced Study, where, as Kahane explains here, during a second visit, Malliavin and Beurling completely solved two fundamental problems in classical complex variable theory. After completing his project with Beurling, Malliavin continued to think about complex variable theory and expanded his interests to include analytic functions of more than one variable. This line of research culminated in his joint paper, with his wife Marie-Paule Malliavin. As Gundy explains in his essay here, this paper represents a departure from classical, purely analytic thinking about analytic functions and potential theory. Instead, the ideas in the Malliavin and Malliavin paper can be seen as descendants of stochastic analytic techniques with which Joseph Doob had given a novel derivation of the Fatou theorem for analytic functions on the disk.

Paul Malliavin, surrounded by books, circa 2000.

Photographs, unless otherwise noted, courtesy of Thérèse Malliavin.

It would appear that Malliavin's excursion into probability theory made a lasting impression on him. Ever since Laplace, France has had a proud tradition in probability theory. At the turn of the twentieth century, Emile Borel and Henri Lebesgue were laying the foundations on which Andrey Kolmogorov would build the axiom system which has become the generally accepted one for the mathematical analysis of random phenomena. At the same time, Henri Poincaré's student Louis Bachelier was constructing the model which has recently provided employment for many young mathematicians in the financial industry.

That tradition was continued in France by Paul Lévy, whose uncanny understanding of stochastic processes became, once it was explained to the rest of us by Kyoshi Itô, the basis for much of the work that probabilists have done ever since. Further, under the masterful tutelage of Jacques

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The young boy Malliavin, circa 1940.

Neveu and, on the more analytic side, Gustave Choquet, postwar France was producing a new cohort of mathematicians whose primary interest was probability theory. Until recently, perhaps the most influential of these was Choquet's student Paul-André Meyer, who realized that, in their haste, Lévy, Doob, and others had treated several notions too casually, a situation that Meyer, first by himself and later with his student Claude Dellacherie, remedied.

In spite of the existence of many active French research groups in probability theory, Malliavin charted his own course. He came to the subject as an analyst with wide-ranging interests, and he brought to the subject a vision which only someone with his encyclopedic knowledge of mathematics could provide. Free from prejudice about what topics and methods are or are not "probabilistic", Malliavin trained his formidable technical expertise on aspects of the field that had not been fully considered. His initial project was to understand Brownian motion on a Riemannian manifold from a differential geometric standpoint. Building on a key observation made by David Elworthy and James Eells, Malliavin understood that the Brownian motion on a manifold could be realized by "rolling" a Euclidean Brownian motion in the tangent space onto the manifold. In order to overcome the technical difficulties posed by the nondifferentiability of Brownian paths, he lifted everything to the bundle of orthonormal frames, where he could apply well-established techniques from Itô stochastic differential calculus. As a result, he gave an elegant construction of the Brownian paths on the manifold, one in which they came equipped with an intrinsic notion of parallel transport. Malliavin's ideas were quickly absorbed and exploited by Jean-Michel Bismut, who used them in his proof of the Atiyah–Singer index theorem.

Having thoroughly assimilated Itô calculus, Malliavin began to realize that Itô's stochastic differential equations could be viewed as a prescription for defining nonlinear transformations of Wiener space, transformations that, although they are defined only up to a set of Wiener measure 0, are nonetheless "smooth" and, as such, are susceptible to analysis. This realization was the origin of what Malliavin called the stochastic calculus of variations and what one of the present authors dubbed the "Malliavin calculus", a cursory résumé of whose initial formulation is given below.

Not content to have been its inventor, Malliavin played a leading role in the application of the Malliavin calculus. Over the past twenty-five years, he and his collaborators produced a large body of work in which his calculus played a central role. The essay here by Leonard Gross gives a glimpse into one of the programs in which Malliavin was involved at the time of his death. Missing here are accounts of the many other projects in which Malliavin was engaged. For example, together with his son-in-law Anton Thalmaier, Malliavin wrote a book in 2000 in which various applications of his calculus to mathematical finance are proposed.

Finally, we have included here an homage to Malliavin composed by Michele Vergne. Her essay portrays Malliavin the man, not just Malliavin the mathematician. Most people, even those who have made profound contributions, are unable to sustain their vigor and eventually enter their dotage. Malliavin never did. Indeed, his student and longtime collaborator Hélène Airault was at his bedside as he was dying, and she reports that he was discussing mathematics from behind the oxygen mask covering his face. He was a remarkable individual, and, at least for those of us who were privileged to know him, the world will be a less interesting place now that he is gone.

Jean-Pierre Kahane

Malliavin and Fourier Analysis

It is worth reading Malliavin's articles again and again, and it would be useful to have them collected and published together. He worked in several branches of mathematics, but all his mathematical endeavors have in common a truly exceptional vision which dominates the raw technical accomplishment: he insisted on understanding the problem "from above" before he would delve into the jungle of details involved in its solution. Here I will restrict myself to a description of Malliavin's most important contribution to Fourier analysis. Although it is only one of Malliavin's many achievements, it exemplifies the vision that he brought to all his work.

Rather than presenting the events in chronological order, I will start with Malliavin's *Compte-Rendus* note of April 13, 1959,[1] the one that won him instant recognition. Afterward, I will look back on his 1954 thesis and forward to his collaboration with Beurling in the 1960s.

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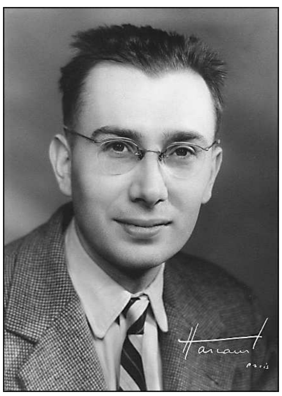
This is a translation (slightly adapted for the Notices*) of an article published in* La Gazette des mathematicians*, no. 126, October 2010.*

[1] *"On the impossibility of spectral synthesis on the line".*

There is much more that I might have, and maybe ought to have, included, but perhaps this brief selection will hasten the day when his collected works are made available.

The Spectral Synthesis Problem

In harmonic analysis, *synthesis* refers to the reconstruction of a sound, signal, function, or some other quantity from its harmonics. For example, periodic functions can be reconstructed via Fourier series. More generally, such reconstruction is possible for functions that are almost periodic in the sense of H. Bohr, functions that are quasi-periodic in the sense of Paley and Wiener, and those that are mean-periodic in the sense of Laurent Schwartz. In all these cases, one associates with an element f of a specified function space the closed subspace $\tau(f)$ generated by the harmonics of f, and the harmonics of f are the generators of the simplest subspaces contained in $\tau(f)$. (With the exception of the mean-periodic case, in the examples cited these subspaces are necessarily one-dimensional.)



Harcourt Studios.

Malliavin as a young man.

In terms of these subspaces, the synthesis problem is that of determining whether the harmonics contained in $\tau(f)$ generate $\tau(f)$. This is a very general question, but suppose that one restricts one's attention to spaces of bounded functions. For example, consider the space $L^\infty(\mathbb{R})$ with the weak topology it has as the dual of $L^1(\mathbb{R})$. (If one uses the strong topology, one recovers Bohr's almost-periodic functions.) The same question can be asked about $L^\infty(G)$ with the weak topology when G is a locally compact Abelian group such as $\mathbb{R}^d$, $\mathbb{Z}$, or $\mathbb{Z}^d$. When G is compact, synthesis always holds, but Laurent Schwartz had shown [22] in 1948 that it fails when $G = \mathbb{R}^3$ or $\mathbb{R}^d$ for any $d \geq 3$. Prior to Malliavin, the answer remained unknown in other cases. In particular, it was unknown for the crucial cases in which G is $\mathbb{R}$ or $\mathbb{Z}$, and finding the answer was a challenge to every analyst of the day.

In his note [15], Malliavin solved the problem when $G = \mathbb{R}$, and he gave the general solution in his *Annales de l'IHES* article [16]. Namely, he proved there that synthesis fails for $L^\infty(G)$ whenever G is a locally compact Abelian group which is not compact.

The problem has many equivalent forms. By duality, it can be seen as a question about the structure of the closed ideals in the convolution algebra $L^1(G)$, in terms of which the question is whether such an ideal is the intersection of the maximal ideals containing it. Alternatively, if Γ is the dual group of G and $A(\Gamma)$ is Wiener's algebra, whose elements are the Fourier transforms of elements of $L^1(G)$, then the synthesis problem is the same as that of determining whether every closed ideal of $A(\Gamma)$ is the ideal of functions in $A(\Gamma)$ that vanish on some closed subset of Γ. If instead of $A(\Gamma)$ one looks at space of continuous functions on Γ that, if Γ is not compact, vanish at infinity, the analogous question has a positive answer. In fact, in that setting, the problem reduces to showing that if $f \in C(\Gamma)$ vanishes on a closed set E and μ is a Radon measure on Γ that is supported on E, then $\langle \mu, f \rangle = \int f \, d\mu = 0$. For $A(\Gamma)$, the problem can be expressed in an analogous way, only the space of Radon measures has to be replaced by the space of "pseudo-measures". That is, one wants to know whether if $f \in A(\Gamma)$ vanishes on a closed $E \subseteq \Gamma$ and if T is a pseudo-measure that is supported on E, then it is necessarily true that $\langle T, f \rangle = 0$.

In the case in which G is a Euclidean space, the space of pseudo-measures can be identified as the space of tempered Schwartz distributions with bounded Fourier transform. In his 1948 counterexample for $\mathbb{R}^3$, Schwartz took E to be the unit sphere $\mathbb{S}^2$ and T to be the derivative in some direction (say, for definiteness, the radial direction) of the surface measure σ for $\mathbb{S}^2$. Because, as $|u| \to \infty$, $\hat{\sigma}(u) = \mathcal{O}\left(\frac{1}{|u|}\right)$, one knows that $\hat{T}(u) = \mathcal{O}(1)$ and therefore that T is a pseudo-measure. Thus Schwartz's counterexample reduces to the trivial task of finding a test function that vanishes on $\mathbb{S}^2$ and has nonvanishing derivative in the direction in which σ was differentiated to get T.

What is the choice of E, T, and f when E is the line $\mathbb{R}$ or circle $\mathbb{T}$? Malliavin's idea was to start with f instead of E and to choose f so that the formal composition $\delta_0' \circ f$ of the derivative of Dirac delta function δ_0 with f can be interpreted as a pseudo-measure whose support is the zero set of f. This idea is beautiful. Wholly aside from the technical challenge posed by its successful implementation, just the realization that it might work is a *tour de force*.

Malliavin's idea applied equally well to $\mathbb{R}$ and to $\mathbb{Z}$, and its extension to general, noncompact G's (i.e., G's for which Γ is not discrete) is relatively easy. Furthermore, in what is a nice example of the way in which probability theory can simplify otherwise complicated analytic constructions, the use of random trigonometric series can greatly simplify his construction of f (cf. [4] and [10]).

Nowadays there exist many other proofs of Malliavin's theorem. Varopoulos used his theory of tensor algebras to derive the general result from the case, handled by Schwartz, when $G = \mathbb{R}^3$. Returning to the idea of producing E before f, Körner produced a strange set E that is meager in the sense that $C(E) = A(E) \equiv \{\varphi \upharpoonright E : \varphi \in A(\Gamma)\}$

(such a set E is call a "Helson set") and yet is sufficiently robust that it carries a pseudo-measure whose Fourier transform tends to 0 at infinity (such a set is said to be a "set of multiplicity"). Körner's construction is complicated, but it had been known for a long time (cf. [11] and [7]) that the existence of a Helson set of multiplicity would show that spectral synthesis fails.

Identity photo, circa 1975.

Malliavin's theorem has been the subject of many lectures, commentaries, and scholarly articles (cf. [18], [21], and [7]). However, it marked the end of the era in which attention was focused on Wiener's algebra $A(\Gamma)$, which was considered to be an essential object for analysis. Nonetheless, contrary to what one might have supposed, his theorem did not mark the end of commutative harmonic analysis, only the end of a particular period. Today the subject is alive and well, having been rejuvenated by the introduction of new directions in which to go.

The Thesis and the Theorem of Beurling–Malliavin

In 1959 Malliavin was a professor in the faculty of sciences at the university in Caen. He already had a solid reputation in complex analysis. Like me, he had been a student of Szolem Mandelbrojt. Szolem used to tell stories and to ask questions. He had done a joint piece of work with Norbert Wiener, and he posed to Malliavin the following question, which had its origins in that joint work:

What can be said about the set of real zeroes of a holomorphic function f in the right half-space $\{z = x + \sqrt{-1}\,y \in \mathbb{C} : x > 0\}$ *which satisfies an inequality of the form* $|f(z)| \leq M(x)$ *for some* $M : (0, \infty) \longrightarrow (0, \infty)$?

In his doctorat d'etat thesis, Malliavin gave a complete and definitive answer to this question. Simultaneously, his thesis contains several beautiful new results in functional as well as complex analysis. In addition, it won Malliavin an invitation in 1954–1955 to the Princeton Institute for Advanced Study, and it was there that he met Arne Beurling. But it was when Malliavin returned to the I.A.S. in 1960–1961 that he and Beurling launched their extremely fruitful collaboration. During that year, they were able to completely solve two hard and intimately related problems:

(1) the characterization of those entire functions that can written as the quotient of two entire functions, both of which are of exponential type and are bounded on the real line.

(2) the computation of the "totality radius" of a given sequence Λ. That is, find the upper bound of those $a \geq 0$ such that the set $\left\{e^{\sqrt{-1}\lambda x} : \lambda \in \Lambda\right\}$ generates $L^2((-a, a))$.

Although their results did not appear until 1967, the authors knew them as early as 1961, and these results remain jewels in function theory.

The solution to the first problem makes use of the logarithmic integral

$$\int_{-\infty}^{\infty} \log |f(x)| \, \frac{dx}{1 + x^2}$$

and is explained in detail by Paul Koosis in his monographs [12] and [13]. Their answer is that an entire function is of the sort in (1) if and only if its logarithmic integral converges. The most important part of their answer is the statement that if f is an entire function whose logarithmic integral converges, then there is an entire function g of arbitrarily small exponential type that is bounded on $\mathbb{R}$ and for which fg is also of exponential type and bounded on $\mathbb{R}$. Combining this statement with Fourier analysis, one obtains the description of those hyperfunctions that can be regularized by convolution functions having arbitrarily small support.

In order to solve the second problem, Beurling and Malliavin introduced a new notion of density, which they called the "effective density", for a sequence Λ. One way to define this notion involves the notion of a "BM-regular" sequence: a sequence Λ' for which there exists a $D(\Lambda') \in [0, \infty)$, the "density" of Λ', such that

$$\int \left| n(r) - D(\Lambda')r \right| \frac{dr}{1 + r^2} < \infty,$$

where $n(r) \equiv \operatorname{card}(\{\lambda' \in \Lambda' : \lambda' \leq r\})$ is the counting function for Λ'. The effective density of a sequence Λ is the infimum over BM-regular $\Lambda' \subseteq \Lambda$ of $D(\Lambda')$. Mimicking analogous ideas in measure theory, one can associate with each of these densities notions of interior and exterior density, in which case Beurling and Malliavin's effective density becomes the exterior density associated with BM-regular sequences. Beurling and Malliavin's results are easy to describe, but the methods by which they are proved are very elaborate and require intricate refinements of ideas from potential theory (cf. [1] and [2]).

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Richard Gundy

The Contribution of Paul and Marie-Paule Malliavin to the Study of Boundary Values of Harmonic Functions on the Bidisc

In the period 1950–1980, much progress was made understanding the boundary behavior of harmonic functions of several variables in the context of the study of singular integrals. Most of this work was done by mathematicians working in the tradition known as the Calderón-Zygmund school. The major players during this period were Zygmund himself [4] and his students, Alberto Calderón, Elias Stein, Guido Weiss, and later, Stein's student Charles Fefferman. Although Paul Malliavin did not participate in this group, he and his wife did make a significant contribution, described below, to work of their American colleagues [9].

Here is a simplified description of the origin of the Malliavin theorem: Let $u(x,y)$ be a harmonic function ($\Delta u = 0$), defined in the unit disc $x^2 + y^2 < 1$. We can always find another harmonic function $\tilde{u}(x,y)$ such that $\partial_x u = \partial_y \tilde{u}$, $\partial_x \tilde{u} = \partial_y u$ (the Cauchy-Riemann equations). If the map $(x,y) \to (u(x,y), \tilde{u}(x,y))$ is one-to-one, its Jacobian $J(x,y) = |\nabla(u(x,y))|^2 = |\nabla(\tilde{u}(x,y))|^2$, by the Cauchy-Riemann equations. Thus, if we wish to calculate the square root of the area of the image of a set Γ in the disc, we must compute an integral. The following functional of u was introduced by Lusin and called the "area integral": $A(u)(\Gamma) := \{\int |\nabla u(x,y)|^2 \chi_\Gamma(x,y) d(x,y)\}^{1/2}$, where the set $\Gamma = \Gamma(\theta)$ is a Stoltz domain, a cone with its axis of symmetry a radius from 0 to a point θ on the boundary of the disc. A remarkable fact is the following: *The set of boundary points $\theta : A(u)(\theta) < \infty$ coincides with the set of boundary points where $N(u)(\theta) := \sup(|u(x,y)| : (x,y) \in \Gamma(\theta)) < \infty$ (up to a set of measure zero); moreover, the harmonic function $u(x,y)$ has a limit along all paths converging to θ within the cone $\Gamma(\theta)$.* Calderón [3] and Stein [9] extended this local equivalence to harmonic functions $u(x,y)$ where $x \in \mathbb{R}^n, y > 0$. Somewhat later Stein and Weiss [10] proved a version of the F. and M. Riesz theorem for harmonic functions defined on the generalized half-plane ($\mathbb{R}^n, y > 0$). In so doing, they defined an H^1 space for functions on $\mathbb{R}^{n+1}_+$. However, it wasn't until 1970, ten years later, that the functionals $A(u), N(u)$ were shown to characterize the classical H^1 space [2]. The extension of this theorem in [2] to the Stein-Weiss H^1-space of several variables is but one of a number of important results in a groundbreaking paper by Fefferman and Stein [5].

Given these developments, other contexts and conjectures spring to mind. A natural context, suggested by Fefferman and Stein, is the Cartesian product of two unit discs, $D_1 \times D_2$ (or two half-planes). In either case, the appropriate Laplacian is $\Delta_{12} := (\Delta_1)(\Delta_2)$. The functionals $A(u), N(u)$ are defined on the product boundary, say $\partial D_1 \times \partial D_2$. In this context, the problem is totally different: for the classical case, the radius variable $0 < r < 1$ is essentially a totally ordered dilation parameter. In the bidisc, the pair of radii (r_1, r_2) are still dilations, but they form a set of parameters that is only *partially ordered*. This produces a major

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In China with his wife Marie-Paule, spring 2010.

obstacle for the following reason: In the disc D (or the ball $B^{n+1} = \{X = (\vec{\theta}, \rho) : 0 \leq \rho \leq 1\, \vec{\theta} \in S^n\}$), the workhorse technique is Green's theorem, applied to a sawtooth region defined by taking the union of all cones $\Gamma(\theta)$ where $A(u)(\theta) \leq \lambda$ (or $N(u)(\theta) \leq \lambda$). Green's theorem provides an estimation of the distribution function of one of these functionals, say $m(\theta : A(u) > \lambda)$, in terms of the distribution function of $N(u)$. However, in the bidisc, Green's theorem is only easily applicable to functions defined on *Cartesian product domains*. Unfortunately, the sawtooth region that arises in the two-parameter setting is *not* necessarily a Cartesian product. Enter the *key idea* provided by the Malliavin collaboration: extend the characteristic function of the sawtooth region to the entire bidisc by an *approximation* that is smooth up to the boundary. In so doing, one obtains a function defined on the entire bidisc, a domain to which Green's theorem is applicable. Now this is easy enough to summarize in a few words. However, the payback comes when one sees the blizzard of error terms that are produced in the subsequent computation. Undaunted, *l'équipe Malliavin* managed to plow through the required estimates with ingenuity and indefatigable courage. To put it mildly!

The breakthrough paper by Marie-Paule and Paul Malliavin contains the proof that $A^2(u)(\theta_1, \theta_2) := \iint(|\nabla u_1|^2 + |\nabla u_2|^2 + |\nabla u_{12}|^2) \cdot \chi_\Gamma(\theta_1)\chi_\Gamma(\theta_2) d(x_1, y_1) d(x_2, y_2)$ is finite for almost every (θ_1, θ_2) where the corresponding nontangential maximal function $N(u)(\theta_1, \theta_2)$ is finite. Subsequently, Jean Brossard [1] obtained the converse result: $N(u)$ is finite almost everywhere on the set where $A(u)$ is finite. Independently, Stein and I [7] obtained the same result, and by refining the Malliavin estimates, we were able to show that the L^p "norms", $0 < p < \infty$, of these functionals were equivalent. In this way, a set of results that can be called H^p theory for the bidisc was created. (A comprehensive survey of these results, including an exposition of the details of the Malliavin contribution with refinements, may be found in the St. Flour Lectures of 1978 [6].) In addition to papers just quoted, several theses and articles followed from these developments.

In conclusion, I want to remark that the Malliavin paper contains a rather mysterious error term $T^4(u)(\theta_1, \theta_2) := \{\iint(|\nabla u_1|^2 \cdot |\nabla u_2|^2)\chi_\Gamma(\theta_1)\chi_\Gamma(\theta_2)\}$, notable because it is homogeneous of degree *four*. For me, this term has more than a mathematical significance. The following vignette recalls (to me) a scene from Agatha Christie: One evening, probably in the spring of 1978, I was sitting down for dinner at home in New Jersey, at the close of an unremarkable day, when I received a phone call from somewhere on Planet Earth. "Allo (long pause)... Thees eez Malliavin!" (another pause) "What do you know about zee fourth-order term?" Completely flustered, all I could think of was "It's locked in the upstairs bedroom." I wish I could remember what I really said.

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Daniel W. Stroock

Malliavin the Probabilist

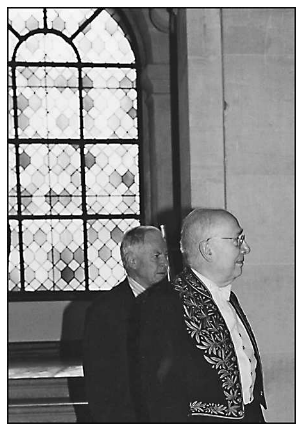
Like Norbert Wiener, Paul Malliavin came to probability theory from harmonic analysis, and, like Wiener, his analytic origins were apparent in everything he did there.

Under the influence of Paul Lévy, most postwar (i.e., Word War II) probabilists have studied stochastic processes as a collection of random paths. For them, the measure determining the distribution of those paths is an *éminence grise* that is best left in the shadows. This perspective gained prominence because of the successes it had in the hands of such masters as K. Itô and J. L. Doob, and no doubt it is responsible for some of the most stunning achievements of probabilists during the last sixty years. However, this was not Wiener's perspective, and it was not Malliavin's either. Instead, for them, the principal object is the measure. Thus, according to Wiener, Brownian motion is a certain Gaussian measure $\mathcal{W}$, now called *Wiener measure*, on the space, *Wiener space*, of continuous paths, and, insofar as possible, he analyzed and exploited it in the same ways that Gauss's and related measures had been in finite-dimensional settings. For example, it is a familiar fact that the Hermite polynomials are a natural, orthogonal basis for the standard Gauss measure on $\mathbb{R}^N$, and Wiener showed that there is an analogous orthogonal basis for his measure on pathspace. More precisely, just as in $\mathbb{R}^N$, it is best to group together all the Hermite polynomials of a fixed degree n and to consider the subspace spanned by them, so Wiener looked at the spaces $Z^{(n)}$ that are obtained by closing in $L^2(\mathbb{W};\mathbb{R})$ the linear span of the nth order Hermite polynomials on Wiener space. His motivation for looking at these subspaces was that he wanted to interpret $Z^{(n)}$ as the subspace of $L^2(\mathbb{W};\mathbb{R})$ consisting of functions that have homogeneous nth order randomness, and, with his usual flair for words, he dubbed them the subspaces of *homogeneous chaos*.[1] Put another way, Wiener was attempting a spectral decomposition of $L^2(\mathbb{W};\mathbb{R})$ in which the spectral parameter is randomness, as opposed to something more conventional, such as frequency.

In a related example, two of Wiener's disciples, R. H. Cameron and W. T. Martin, discovered that Wiener measure is as translation invariant as any measure in infinite dimensions has a right to be. Namely, they showed that if H is the Hilbert subspace of Wiener space whose elements h are absolutely continuous and have square integrable derivative, then translation of $\mathcal{W}$ by an $h \in H$ results in a measure $\mathcal{W}_h$ that is absolutely continuous with respect to $\mathcal{W}$ and has a remarkably simple Radon–Nikodym R_h, all of whose powers are integrable. Using Cameron and Martin's result, one can show that differentiation D_h of a function on Wiener space in the direction of an $h \in H$ admits an adjoint $D_h^\top$, and the existence of this adjoint allows one to make Sobolev-type extensions of D_h and $D_h^\top$ as closed, densely defined operators on $L^2(\mathcal{W};\mathbb{R})$. In addition, one can show that if $\{h_k : k \geq 1\}$ is an orthonormal basis in H, then the operator, known to probabilists as the *Ornstein–Uhlenbeck* operator,

$$\mathcal{N} = \sum_{k=1}^{\infty} D_{h_k}^\top D_{h_k},$$

is self-adjoint and is independent of the particular choice of orthonormal basis. Furthermore, Wiener's spaces of homogeneous chaos are the eigenspaces for $\mathcal{N}$. In fact, $\mathcal{N}\varphi = n\varphi$ for $\varphi \in Z^{(n)}$, a fact that accounts for people doing Euclidean quantum field theory calling $\mathcal{N}$ the *number operator*. All this should come as no surprise to anyone who has dealt with Gaussian measures and Hermite polynomials in finite dimensions, where Cameron and Martin's result is a trivial change of variables and the operator $\mathcal{N}$ is just the ground state representation of the Hermite operator (a.k.a., the harmonic oscillator). However, there are technical difficulties that have to be overcome before one can transfer the finite-dimensional results to infinite dimensions.



Induction in 1979 into the French Académie des Sciences.

The preceding discussion provides a context in which to describe one of Malliavin's most important contributions to probability theory. What he realized is that the Ornstein–Uhlenbeck operator $\mathcal{N}$ can be used as the starting point for a robust integration-by-parts formula for Wiener measure. He was far from the first to attempt integration by parts for functions on Wiener space. Indeed, Cameron and his student M. Donsker had been doing it for years, and integration by parts in Wiener space had been a basic tool of Euclidean quantum field theorists. However, earlier versions had involved functions that are classically (i.e., in the sense of Fréchet) differentiable, whereas Malliavin wanted to apply it to functions that are not even

[1] *It should be pointed out that Wiener's own treatment of this subject was somewhat awkward and that it was Itô who put it on a firm mathematical foundation.*

classically continuous. Specifically, with the goal of proving elliptic regularity results, he wanted to do integration by parts when the functions are solutions to Itô stochastic differential equations. If one thinks of an ordinary differential equation as giving the prescription for turning a straight line into the integral curve of a vector field, then Itô's stochastic differential equations can be thought of as the analogous prescription for converting Brownian (i.e., Wiener) paths into the paths of a more general diffusion. In particular, if $X(\,\cdot\,, x, w)$ is the diffusion path starting at x corresponding to Wiener path w and

$$u(t,x) = \int f(X(t,x,w))\,\mathcal{W}(dw),$$

then u will solve the diffusion equation $\partial_t u = Lu$ with initial data f, where L is the associated *diffusion operator*, the (possibly degenerate) elliptic operator that appears in Kolmogorov's backwards equation.

Solutions to Itô's equations have much to recommend them, but from a classical analytic perspective they are dreadful when viewed as functions of w. Indeed, they are defined only up to a set of $\mathcal{W}$-measure 0, and so they are not amenable to any classical notion of differentiation. Undaunted by this formidable technicality, Malliavin realized that $w \rightsquigarrow X(t,x,w)$ nonetheless ought to be differentiable in the sense of Sobolev. After all, in infinite dimensions, there is no Sobolev embedding theorem to prevent there from being functions that are classically discontinuous and yet infinitely differentiable in the sense of Sobolev. To wit, any element of $Z^{(n)}$ will be smooth in the sense of Sobolev, but few elements of even $Z^{(1)}$ will be classically continuous. With this in mind, Malliavin constructed a Schwartz-type space of functions on Wiener space.[2]

Rather than use powers of the operators D_h to measure regularity, he used powers of $\mathcal{N}$, because the spectral and stability properties of $\mathcal{N}$ make it easier to understand questions about its domain than the corresponding questions about the domain of D_h. Because everything had to be based on properties of $\mathcal{N}$, his integration-by-parts formula had to be an application of the self-adjointness of $\mathcal{N}$, and for this purpose he took advantage of the Leibniz rule satisfied by second-order elliptic operators. That is, for $\mathbb{R}$-valued F and G on Wiener space,

$$\text{(1)} \qquad \mathcal{N}(FG) = F\mathcal{N}G + G\mathcal{N}F + \langle F, G\rangle,$$

where the symmetric, bilinear operation $\langle F, G\rangle$ has the crucial properties that it is nonnegative and satisfies

$$\text{(2)} \quad \langle \varphi\circ F, G\rangle = \varphi'\circ F\langle F, G\rangle \quad \text{for } \varphi \in C^1_{\mathrm{b}}(\mathbb{R};\mathbb{R}).$$

Starting from (2), he argued that, because

$$\varphi'\circ F\langle F,F\rangle = \mathcal{N}(F\varphi\circ F) - F\mathcal{N}(\varphi\circ F) - \varphi\circ F\mathcal{N}F,$$

then if $\langle F,F\rangle > 0$, one can write

$$\varphi'\circ F = \frac{\mathcal{N}(F\varphi\circ F)}{\langle F,F\rangle} - \frac{F\mathcal{N}(\varphi\circ F)}{\langle F,F\rangle} - \frac{\varphi\circ F}{\langle F,F\rangle};$$

and therefore, because $\mathcal{N}$ is self-adjoint,

$$\begin{aligned}&\int \varphi'\circ F\,d\mathcal{W}\\ &\quad = \int \varphi\circ F\left[F\mathcal{N}\left(\frac{1}{\langle F,F\rangle}\right) - \mathcal{N}\left(\frac{F}{\langle F,F\rangle}\right)\right.\\ &\qquad\qquad\qquad\qquad \left. - \left(\frac{\mathcal{N}F}{\langle F,F\rangle}\right)\right] d\mathcal{W},\end{aligned}$$

which, by taking advantage of (1) and (2), can be rewritten as

(3)

$$\int \varphi'\circ F\,d\mathcal{W} = -\int \varphi\circ F\left[\frac{2\mathcal{N}F}{\langle F,F\rangle} + \frac{\langle F,\langle F,F\rangle\rangle}{\langle F,F\rangle}\right] d\mathcal{W}.$$

The virtue of (3) is that it allows one to draw conclusions about the distribution of F. Indeed, if μ is the distribution of F, then (3) says that, in the language of Schwartz distributions, $\partial\mu = -\psi\mu$, where $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ is the function such that $\psi\circ F$ is the conditional expectation value $\mathbb{E}^{\mathcal{W}}[\Psi \mid \sigma(F)]$ of

$$\Psi \equiv \frac{2\mathcal{N}F}{\langle F,F\rangle} + \frac{\langle F,\langle F,F\rangle\rangle}{\langle F,F\rangle}.$$

Elementary analysis shows that if $\psi \in L^p(\mu;\mathbb{R})$ for some $p > 1$, then μ admits a density f that is uniformly Hölder continuous with a Hölder exponent depending only on p. Furthermore, since $\|\psi\|_{L^p(\mu;\mathbb{R})} \le \|\Psi\|_{L^p(\mathcal{W};\mathbb{R})}$, one can estimate $\|\psi\|_{L^p(\mu;\mathbb{R})}$ entirely in terms of Wiener integrals.

Of course, there are at least two technical details that have to be confronted in order to carry out Malliavin's program. For one thing, one has to check that F is in the domain of the operations that one wants to perform on it. When F is the solution to an Itô equation in which the coefficients are smooth, Malliavin's description of his Schwartz space in terms of $\mathcal{N}$ makes this a difficult problem. The Japanese school, especially S. Watanabe and his student H. Sugita, vastly simplified matters by describing the same Schwartz space in terms of the operators D_h. A key ingredient in their approach was provided by P. A. Meyer, who showed that E. M. Stein's Littlewood–Paley theory for symmetric semigroups can be

[2] *This had been done before by P. Kree, but Kree's construction had, at least for the applications that Malliavin had in mind, a fatal flaw: the space he built was not an algebra. In the absence of a Sobolev embedding theorem, the only way to achieve an algebra is to abandon L^2 and deal with all L^p spaces simultaneously.*

used to prove that for any orthonormal basis $\{h_k : k \geq 1\}$ and any $p \in (1, \infty)$,

$$\|\mathcal{N}^{\frac{1}{2}}F\|_{L^p(\mathcal{W};\mathbb{R})} \sim \left\| \left(\sum_{k=1}^{\infty} |D_{h_k}F|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathcal{W};\mathbb{R})},$$

where $\sim$ means here that one side is dominated by a constant times the other.

The second detail is a more interesting one. In fact, it is less a detail than the heart of the whole program: namely, in deriving (3), it was assumed that $\langle F, F \rangle$ is strictly positive, and in applying (3) it is necessary to have sufficient control on its positivity to estimate the $L^p(\mathcal{W};\mathbb{R})$-norm of Ψ. Gaining such control can be a very challenging problem. In applications to solutions to Itô's equations, $\langle F, F \rangle$ can be recognized as a pathwise measure of ellipticity. In particular, when the diffusion operator L is uniformly elliptic, it is relatively easy to check that the corresponding $\langle F, F \rangle$ will have reciprocal moments of all orders. However, Malliavin was not ready to settle for a rederivation of classical elliptic regularity results. He wanted to show that his method could also be applied to derive regularity results of the sort that Hörmander had proved for subelliptic operators. Although Malliavin pointed the way, it required considerable effort by several authors to achieve his goal, and it must be admitted that in the end their effort was not rewarded by the discovery of many facts that more traditional analytic methods had not already revealed.

In conversation, circa 2000.

Expanding on the remark at the end of the preceding paragraph, one should recognize why it is that Malliavin's ideas do not give an efficient way of looking at questions such as elliptic regularity. Indeed, his approach takes an inherently finite-dimensional problem, lifts it to an infinite-dimensional setting, performs the analysis in infinite dimensions, and then projects that analysis back down to finite dimensions. This is a little too much like going to a neighbor's house by way of the moon: it works, but it is not efficient. Thus it was not until Malliavin's ideas were applied to intrinsically infinite-dimensional problems that they came into their own. There is now a major industry, populated (somewhat worryingly) in part by financial engineers, who are making such applications. All this is still in the early stages of development, but there can be no doubt that the program that Malliavin initiated is one of the great contributions to modern probability theory.

Leonard Gross

Among the many areas of mathematics to which Paul Malliavin contributed in the past twenty years, the general problem of constructing an interesting theory of infinite-dimensional manifolds, on which there is a useful measure for doing harmonic analysis, was the focus of a large part of his work. Such a manifold must be able to support a notion of differentiation that relates well to the measure. For example, one should be able to carry out integration by parts at the very least. Preferably, there should also be some kind of natural second-order, infinite-dimensional "elliptic" differential operator Δ, which plays the role of a Laplacian. After Cameron and Martin's work in the 1940s and 1950s, showing that "advanced calculus" on Wiener space, using Wiener measure, could be developed in an interesting way, it became reasonable to seek interesting examples of not necessarily linear infinite-dimensional manifolds on which to develop some analog of finite-dimensional harmonic analysis. Early work in this direction began in the 1960s (H. H. Kuo, D. Elworthy, J. Eells). Malliavin initially sought examples in the form of spaces of continuous functions from an interval or circle into a finite-dimensional Riemannian manifold M. The measure induced by Brownian motion on M (or by pinned Brownian motion) would provide, in this case, a natural measure of interest on these spaces of continuous paths into M. The required integration by parts theorems were established in the late 1980s and early 1990s by J. M. Bismut, B. Driver, E. Hsu, and Malliavin and his wife.

But in the last ten years Malliavin aimed at producing a natural Brownian motion on the diffeomorphism group of the circle, thereby replacing the finite-dimensional manifold M by an infinite-dimensional one. As is often the case, construction of a Brownian motion in a manifold is more or less equivalent to the construction of a heat semigroup $e^{t\Delta}$ acting on functions over the manifold. In case the manifold is the diffeomorphism group of the circle, there is a natural infinite-dimensional Laplacian. It is determined by regarding the tangent space at the identity to be vector fields on S^1 of Sobolev class $H_{3/2}$. The index 3/2 arises for several seemingly very different reasons. It is the natural Sobolev class for action on the argument of loops into a compact Lie group. It is

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also the correct Sobolev index for capturing the Weil-Petersson metric on this Lie algebra.

It is well understood, at least for linear spaces, that if the symbol of an operator Δ is the norm on some infinite-dimensional Hilbert space H, then the associated heat semigroup $e^{t\Delta}$ cannot act reasonably on functions over H but will act well on functions over a suitable enlargement of H. Otherwise said, the associated Brownian motion will have continuous sample paths into the larger space but will jump out of H immediately. The analog for nonlinear manifolds, such as the diffeomorphism group, is much more difficult.

Malliavin and his coauthors showed, over the last ten years, that, for the natural Laplacian (whose symbol is the Weil-Petersson metric), it suffices to enlarge the group of $H_{3/2}$ homeomorphisms of S^1 to the group of Hölder continuous homeomorphisms. This enlargement, along with the obvious weakening of the topology, supports the associated Brownian motion with continuous sample paths.

The evolution of this work can be traced through the papers [10], [1], [9], [7], [4], [2], [5], [11], [3], [12], [13], [14], [8], and [6], many of which were written jointly with one or more of Malliavin's coworkers, Hélène Airault, Ana-Bela Cruzeiro, Jiagang Ren, and Anton Thalmaier. The program was completed only very recently, in the paper [6].

The titles of some of these papers seem to have more to do with analytic function theory than diffeomorphisms of the circle. It happens that a Brownian motion on the diffeomorphism group of S^1 induces, at least informally, a Brownian motion on the space of closed Jordan curves in the plane, as well as on certain spaces of univalent holomorphic functions on the unit disc. At an informal level the link is easy to understand: Suppose that J is a Jordan curve and that f is a holomorphic map from the open unit disk D onto the interior of the region bounded by J. It is known that f extends to a continuous function $\hat{f}$ on the closure of D and that $\hat{f}$ is a bijection of S^1 onto J. Moreover, there is a holomorphic function h from the exterior of the unit disk onto the unbounded component of the complement of J. This also extends to the closure of the exterior of the unit disk, producing another continuous bijection $\hat{h}$ from S^1 onto J. Insisting that $\lim_{z\to\infty} h(z)/z > 0$ makes the choice of h unique. On the other hand, f fails to be unique, but only to the extent that its argument can be changed by a D-preserving Möbius transformation. For any such function f the map $(\hat{f})^{-1} \circ \hat{h} : S^1 \to S^1$ is a homeomorphism ϕ of S^1. What homeomorphisms can arise in this way? It is known that if ϕ is C^∞, then there exists a Jordan curve and corresponding functions f and h whose ratio, as above, is ϕ. Ignoring the nonuniqueness of f, which only requires replacing Jordan curves by the space of shapes, the procedure of mapping a given homeomorphism ϕ to the pair f, h, the so-called "welding" procedure, would give a Brownian motion on the space of shapes if one could extend the welding procedure to those homeomorphisms of S^1 that are needed to support a Brownian motion as above, namely a large subset of the Hölderian homeomorphisms. This is accomplished in the paper [6], which welds deep stochastic analysis techniques with analytic function theory.

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Michèle Vergne

Malliavin and I

Pisa, June 2010: I am in Pisa, I learn via email about the death of Paul Malliavin. If anyone seemed to me to be immortal among immortals, it was he. I think of him introducing me to the Académie and guiding me on a visit to the Institut Library and the Bibliothèque Mazarine. We could see the Seine through the windows. He told me about Cardinal Richelieu, to whom I think he attributed the sentence: "I shall regret the beauty of this place when in the other world".

I.H.P., May 1968: I am twenty-five-years old, long hair, in blue jeans. My heroine is Louise Michel. I can well see myself sent to hard labor for my ideas.

Photo by D. Stroock.

Visit to Versailles in 2008.

Slogans, demonstrations: the system must be changed. The dusty "Institut Henri Poincaré" must be destroyed. General meetings, declarations. When Malliavin intervenes with his soft voice, I shout: "Malliavin is a bourgeois, Malliavin is a fascist". If the red guard were gathering their battalions, I would be with them and would send Malliavin out to serve the people. Malliavin continues to smile serenely. Ever since then, when I meet him on the streets of Paris, he tips his hat and addresses me with a ceremonious: "Chère Madame".

At the Theater, January 2009: Malliavin is presenting a candidate for membership in the Académie des Sciences. It won't be an easy win. I sit beside him at the green, oval desk. He draws a few crumpled sheets of paper out of his pocket. "Madame, I would like to have your opinion of my speech," and, in a low voice, he starts to whisper: "Already a hundred years ago, Elie Cartan...." Then comes his turn to speak, and in a loud voice, he declaims: "Already a hundred years ago...." Bravo, clap, clap, and his candidate is elected.

The Poisson Summation Formula: Malliavin and I are ecstatic about the Poisson summation formula. No doubt, he knows all its finest and deepest aspects. I don't, but I nonetheless think it is the most beautiful formula in mathematics.

Two Things That Malliavin Loves, Mathematics and Influencing People's Destiny: These are not unrelated. Speaking about a colleague, he often lauds the beauty of his work: "Demailly's annulation theorem is extraordinary; Madame, consider that it does not require pointwise estimates, but only in the mean...." Villani's work overjoys him.

A Reception at the Malliavins' Home: I am invited to a reception at the Malliavins' home. I go with my daughter Marianne. She was eight years old at the time. We enter a paved courtyard, we go up some stairs, we ring a bell, and we enter an apartment, immersed in semi-darkness. Overflowing bookshelves cover the walls; the seats are antique, I fear that the furniture would disintegrate into dust were the curtains to be drawn. Malliavin talks to my daughter, he finds an old, illustrated edition of *The Children of Captain Grant* for her. She sits on a window sill and reads passionately, while the other guests, mainly mathematicians from all over the world, tell each other about their lives. Marie-Paule becomes nervous: the petits fours must be eaten, the Bertillon sorbets must be tasted....

Temptations: Malliavin phones me, he wants to propose me as a candidate for the Académie. I object: "My father and mother are dead, it is too late to please them, it would give me no personal pleasure." Malliavin responds: "Madame, we are not Académicians for our pleasure, but to serve our country". He then invites me to come to the Académie and leads me to the *salle des séances*. In the dim light, the white marble heads observe the scene. That night, I have a dream: there is a lit niche, and inside the niche, a bottle of whiskey sitting on a pedestal (I had just seen again *Rio Bravo*). I realize that, more than anything else, I wish to have a draught of whiskey and that I would also like to enter the Académie. I phone Malliavin: "Yes, I agree to be nominated." Anyway, I am totally incapable of saying *no* to Malliavin.

No: I am elected. My sister Gilberte does not survive for my election, and my sister Martine is about to die. Now other plans are afoot behind the scene: a representative of France for the executive committee of the International Mathematical Union is needed. Malliavin and Jacques-Louis Lions contact me, they have decided it should be me.

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This is a translation (slightly adapted for the Notices*) of an article published in* La Gazette des mathematicians, *no. 126, October 2010.*

Malliavin calls me daily. "No, I do not want to, I cannot." After each phone call, a wave of anxiety suffocates me. I feel like a fraud.

Honors: I become accustomed to taking pleasure in honors. Today is the *séance solennelle* (solemn convocation) at the Académie of Sciences. Going down the stairs between the raised sabers of the republican guards seems natural.

Malliavin is wearing his ceremonial outfit. Befittingly, the Académie's paleontologist has a sword with a dinosaur pommel. Malliavin is happy to be among his peers. He knows them all. He pushes a chair forward for Pierre Lelong, who is nearly ninety. He listens politely to Denisse, he teases Dercourt, he says a kind word to me. He jokes: "Here, we we all love researchers studying longevity and who search for a happiness pill to give the elderly"; and, all the while, he is predicting the election of Beaulieu [who specializes in geriatrics] as the new president of the Académie des Sciences.

Maneuvers: Malliavin has a plan for X: he sends stacks of mail, phones, counts his cards. He scrutinizes the weaknesses of the opposition's plans for Y. If the maneuver fails, it's a triumph for Z, and Y is chosen. Malliavin gives me sibylline advice. I interpret it as follows: as soon as someone is chosen for our section, we should all forget whatever bad things we once thought about him.

Should we do likewise with the dead? It might be wise, since we will have to spend eternity by their sides.

Haar Measures and Malliavin Measures: Is there some sort of Haar measure for loop groups? Which are the "natural" groups that admit unitary representations? Locally compact groups, thanks to Haar measure, but the unitary group also has a unitary representation. We may also construct ergodic measures for some natural groups, such as the infinite permutation group. These are questions that interest Paul Malliavin and Marie-Paule.

I naïvely believe that mathematical ideas have no genitors and come forth out of cauliflowers. But, no, Haar measure did not exist before Haar, Malliavin calculus did not exist before Malliavin any more than Itô's integral existed before Itô. Malliavin has a more human opinion: he is almost in tears when he learns that Itô received the Gauss Prize. Itô dies shortly thereafter, and the world without Itô seems less beautiful to Malliavin.

In the same way, for me the world without Malliavin is not quite the same. I miss him.