How a Medieval Troubadour Became a Mathematical Figure

Michael P. Saclolo

Lyric poetry of the Middle Ages may seem far removed from subgroups of the symmetric group or primitive roots of finite fields. However, one piece of medieval poetry has led to work in these mathematical disciplines, namely a sestina written in the Romance language of Old Occitan by a troubadour named Arnaut Daniel:

Lo ferm voler q'el cor m'intra
no'm pot ies becs escoissendre ni ongla
de lausengier, qui pert per mal dir s'arma
e car non l'aus batr'ab ram ni ab verga
si vals a frau lai o non aurai
oncle
jauzirai joi, en vergier o dinz cambra

Qan mi soven de la cambra
on a mon dan sai que nuills hom non intra
anz me son tuich plus que fraire ni oncle
non ai membre no'm fremisca, neis
l'ongla
aissi cum fai l'enfas denant la verga
tal paor ai no'l sia trop de l'arma

Del core li fos non de l'arma
e cossentis m'a celat deniz sa cambra
que plus mi nafra'l cor que colps de verga
car lo sieus sers lai on ill es non intra
totz temps serai ab lieis cum carns et ongla
e non creirai chastic d'amic ni d'oncle

Anc la seror de mon oncle
non amei plus ni tant per aqest'arma
c'aitant vezis cum es lo detz de l'ongla
s'a liei plagues volgr'esser de sa cambra
de mi pot far l'amors q'inz el cor m'intra
mieills a son vol c'om fortz de frevol verga

Pois flori la seca verga
Ni d'en Adam mogron nebot ni oncle
tant fin'amors cum cella q'el cor m'intra
non cuig fos anc en cors ni eis en arma
on q'ill estei fors on plaz' o dins
cambra
mos cors no' is part de lieis tant cum ten l'ongla

The firm desire that enters
my heart no beak can tear out, no nail
of the slanderer, who speaks his dirt and loses his soul.
And since I dare not beat him with branch or rod,
then in some secret place, at least, where I'll have no
uncle
I'll have my joy of joy, in a garden or a chamber.

When I am reminded of the chamber
where I know, and this hurts me, no man enters--
no, they're all more on guard than brother or uncle--
there's no part of my body that does not tremble, even
my nail,
as the child shakes before the rod,
I am that afraid I won't be hers enough, with all my soul.

Let me be hers with my body, not my soul,
let her hide me in her chamber,
for it wounds my heart more than blows from a rod
that where she dwells her servant never enters;
I will always be as close to her as flesh and nail,
and never believe the reproaches of brother and uncle.

Not even the sister of my uncle
did I love more, or as much, by my soul,
for as familiar as finger with nail
I would, if it pleased her, be with her chamber.
It can do more as it wills with me, this love that enters
my heart, than a strong man with a tender rod.

Since the flower was brought forth on the dry rod,
and from En Adam descended nephews and uncles,
a love so pure as that which enters
my heart never dwelt in body, nor yet in soul.
Wherever she stands, outside in the town or inside her
chamber,
my heart is not further away than the length of a nail.

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C'aissi s'enpren e s'enongla
mos cors e lei cum l'escorss'en la verga
q'ill m'es de joi tors e palaitz e cambra
e non am tant fraire paren ni oncle
q'en paradis n'aura doble joi m'arma
si ja nuills hom per ben amar lai intra

Arnautz tramet sa chanson d'ongl'e d'oncle
a grat de lieis que de sa verg'a l'arma
son Desirat cuit pretz en cambra intra

For my heart takes root in her and grips with its nail,
hold on like bark on the rod,
to me she is joy's tower and palace chamber,
and I do not love brother as much, or father, or uncle;
and there'll be double joy in Paradise for my soul,
if a man is blessed for loving well there and enters.

Arnaut sends his song of the nail and the uncle,
to please her who rules his soul with her rod,
to his Desired, whose glory in every chamber enters.

Note the rearrangement of the final word from each line of the stanzas to the next. The English translation by Frederick Goldin ([6]) to the right of the original illustrates the same phenomenon.

Arnaut's sestina is an example of courtly love poetry. The last three-line stanza unequivocally declares the poet's intention to please his noble lady. What interests us here, however, is how the six six-line stanzas, or sextets, use the same six ending words in a permutation given by the cycle (124536). The arrangement of the ending words in each sextet corresponds to a power of this cycle, and so if we were to compose a seventh sextet, we would arrive at one with the original arrangement of the six ending words. Poets and, lately, mathematicians have been fascinated by this poetic form, and the desire to generalize it has led to the investigation of the mathematical structures involved.

Arnaut Daniel and the Troubadours

I begin at the source, the poet. Information about Arnaut's life comes primarily from his *Vida*. The *Vidas*, very short biographical sketches of poet-troubadours in surviving manuscripts of their works (see, for example, [2]), were written primarily to promote interest in the troubadours, and therefore the information contained therein may not be entirely factual. What is commonly accepted, however, is that Arnaut lived in the latter part of the twelfth century. The *Vida* tells that he hailed from a castle named Ribérac in the bishopric of Périgueux in the Dordogne region of France. He was of noble birth and presumably had early contact with the nobles he would later entertain as a troubadour. The *Vida* mentions Arnaut as courting a married noble lady from Gascony, although it quickly points out that this forbidden love was believed to have remained unattained. For an extended account of Arnaut's life, refer to the introduction to the book edited by James Wilhelm [21].

As a troubadour, Arnaut was part of a group of musical poets, who entertained courtly audiences of southern France. Their language of Old Occitan counts modern Provençal as a descendant. Catalan, the language of Catalunya in northern Spain and in the Pyrenees region of France, is also closely related. The troubadours' northern French-speaking counterparts are called the *trouvères*. These northern poets were influenced by the troubadours and adopted the latter group's style and themes.

The troubadours and the trouvères celebrated *courtly love*. In this highly ritualistic literary tradition, full of rules and codes, the object of often forbidden desire is an idealized lady of the Court. The poetry composed and performed by the troubadours expresses her unattainability.

Arnaut himself was revered as a poet. One of his earliest and most prominent admirers was Dante Alighieri, the Italian poet and celebrated author of the *Divine Comedy*, who lived about a century after Arnaut. Dante mentions Arnaut several times in *De vulgari eloquentia*. He even depicts Arnaut as doing penance in *Purgatorio* in the *Divine Comedy*, imploring Dante in Occitan not to forget him.

Today, our most direct link to courtly love is its long and rich literary tradition. It is an understatement to say that the poetry of the troubadours and the trouvères influenced many poetic forms we enjoy today. Wilhelm discusses sources and influences in the introduction to [21].

The Sestina

The poetic form presented earlier is called the sestina, and Arnaut is credited with its invention ([4]). Both Dante and Petrarch composed in the form. Rudyard Kipling, Ezra Pound, Algernon Charles Swinburne, and Sir Philip Sidney are among poets who composed sestinas in English. Sidney composed an example of a double sestina.

As its base, the sestina has the structure of six sextets followed by a tercet called a *tornada* or an *envoi*; the former is an Occitan term meaning "turned" or "twisted", while the latter is French and means "send-off". The *envoi* traditionally incorporates all six ending words of the sextets. As previously mentioned, the ending words of each line of a sextet are rearranged in the subsequent sextet according to the cycle (124536). Of course, any permutation of order six can be used to compose a poem that shows a different rearrangement of the six ending words, while producing the same number of stanzas and lines. But the particular permutation used by Arnaut inspired poets and mathematicians alike. Poets sought either to compose in the same style or to develop variations. Mathematicians, instead, posed the question: What permutations have the "same" properties as the one used in the sestina? This is the story we are telling here.

Generalizing the Sestina: the Quenine

Interest in the sestina and the structure presented by the permutation began among literary figures in the twentieth century among literary figures. According to A. Tavera ([17]), it was Eugène Vinaver, Tavera's mentor and professor of French at the University of Manchester, who made him aware of the spiral shape of the permutation. To see this, number the six ending words of the sextet, top to bottom, 1 to 6. Now draw an inward "spiral" starting from the integer 6 heading towards 1 then 5, 2, 4, and finally 3 as in Figure 1. The resulting order of the six integers, following the trajectory of the spiral, is precisely the order of the six ending words in the subsequent strophe.

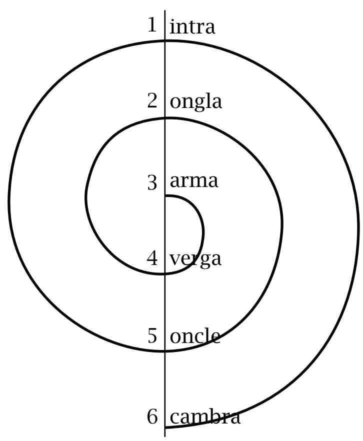

Figure 1. Spiral representation of a sestina. The six ending words of the strophes of Arnaut's sestina are listed in its original order.

The mathematical question arising from the particular permutation and its spiral representation was taken up by the poet Raymond Queneau and his colleague, the poet, writer, and mathematician Jacques Roubaud. Queneau was a member of the French mathematical society, the Société Mathématique de France. Moreover, he and Roubaud were central figures in the development of OULIPO ("OUvroir de LIttérature POtentielle"), a literary movement and association that Queneau and François Le Lionnais founded in 1960. Composed of mostly European members, the goal of OULIPO is to explore and compose what the group calls "potential" literature by using constraints drawn from mathematics and other technical disciplines. An example is George Pérec's novel, *La Disparition* (translated into English as *A Void*), a lipogrammatic novel, redacted without the letter *e* [9]. Queneau himself played with combinatorial possibilities in his collection of sonnets *Cent Mille Milliard de Poèmes* (*One Hundred Thousand Billion Poems*) [10]. For more information on the work of OULIPO, see, for example, the text translated and edited by Warren Motte [7].

Queneau noticed that the spiral representation can be extended to permutations of a set of any size n. Then he asked which positive integers n give an order n "spiral" permutation. He determined [11], apparently by rote calculation, a list of thirty-one positive integers less than 100 that have this property. Queneau called the generalized sestina for such n the *n-ine*. Thus the sestina is the same as a 6-ine, and any one-line poem is degenerately a 1-ine. The smallest nonadmissible number is $n = 4$, for the associated permutation that results, (124), fixes the 3. Later the n-ine became the *quenine*, coined by Roubaud in honor of Queneau. The French word for the sestina is *sextine*. By analogy one might call the generalization *n-ina* or *quenina* in English, but as yet there is no established tradition of this convention either in mathematical writing or poetry written in English.

The Search for Quenines

In this section I shall highlight the various developments to find positive integers n that give a generalized sestina, culminating with the definitive classification. First I shall set up some basic notations and definitions.

For x a positive integer, σ_n is defined by:

$$(1)\qquad \sigma_n(x) = \begin{cases} x/2 & \text{if } x \text{ is even,} \\ n - (x-1)/2 & \text{if } x \text{ is odd.} \end{cases}$$

The following terminology related to $\sigma_n(x)$ was formalized by Monique Bringer in [2]. The function $\sigma_n(x)$ is called a **spiral** permutation. Integers n such that $\sigma_n(x)$ is of order n are called **admissible**. Roubaud also referred to such n as **Queneau numbers**. A spiral permutation with admissible n is also a **quenine** or **n-ine**. In honor of Queneau and Daniel, Bringer called the cyclic subgroup of the symmetric group generated by $\sigma_n(x)$ the **Queneau-Daniel** group.

Thus the search for quenines became the search for admissible n. Moreover, it was the study of the inverse permutation to σ_n that eventually produced the desired results. Let δ_n be the inverse permutation to σ_n defined by:

$$(2)\qquad \delta_n(x) = \begin{cases} 2x & \text{if } 2x \le n, \\ 2n + 1 - 2x & \text{otherwise.} \end{cases}$$

As σ_n is of order n if and only if δ_n is also of order n, finding admissible n by way of δ_n gives the admissible n for σ_n.

Queneau

Queneau apparently did not realize this connection, at least not in published works (see [11], [12], [13]). He recognized early on that not every n is admissible, citing some basic counterexamples. In the case of $n = 4$, 7, or 10, for instance, σ_n fixes at least one number. For $n = 8$, σ_8 is of order 4. As for some initial characterization of nonadmissible n, he mentioned the cases in which $n = 2xy + x + y$ for x, y integers greater than or equal to 1 or if $n = 2^k$ for $k > 1$ as examples of such n, since the resulting permutation subgroups are of order less than n. (Consider the orbit of 2 under δ_n for example.) Alas, these conditions do not exhaust the cases in which n is not admissible.

After Queneau

Roubaud and his student Bringer continued the search for admissible n. In [14] Roubaud was interested in the various permutations employed by the troubadours. In considering the sestina, he attempted to connect the particular permutation used by Daniel to the assonant pairing of ending words in the stanzas, reiterating Queneau's problem of generalizing the sestina. But it was Bringer (in [2]) who took on Queneau's problem directly and was among the first to characterize such admissible and nonadmissible integers n. Bringer found that a necessary condition for n to be admissible is for $2n + 1$ to be prime. She also found that if 2 is a primitive root of the finite field $\mathbb{Z}/(2n+1)\mathbb{Z}$ (i.e., where $2n+1$ is prime), then n is admissible. (Note: While the mathematics that follows in the rest of the article is generally accessible, those who would like to read background and reference materials may look at [5] about finite fields and at classics such as [6], for a short treatment on symmetric groups, and [4], for a treatment of primitive roots.) The following is a summary of Bringer's results, listing conditions for both admissible and nonadmissible n.

Theorem 1 (Bringer)**.** (1) *If n is admissible, then $2n + 1$ is prime.*
(2) *If n and $2n + 1$ are prime, then n is admissible.*
(3) *If $n = 2p$ where p and $4p + 1 = 2n + 1$ are prime, then n is admissible.*
(4) *$n = 2^k - 1$ is not admissible.*
(5) *$n = 4k$ is not admissible.*

To show that $2n + 1$ is prime for n to be admissible, suppose the contrary. Then there exists a positive integer $q > 1$ that divides $2n+1$. Hence, if m is in the orbit of q under δ_n, then $m \equiv \pm 2^k q \pmod{2n+1}$. This implies that q divides m since it divides both $2n + 1$ and $\pm 2^k q$. Therefore the orbit of q only contains divisors of q. But, then, $2, \dots, q - 1$ do not divide q and thus cannot be in the orbit of q, a contradiction. Hence $2n + 1$ must be prime.

Bringer also points out that the necessary condition that $2n + 1$ is *prime* is equivalent to Queneau's prior remark that n is not admissible when $n = 2xy + x + y$, where x and y are integers greater than or equal to 1. For if $2n + 1$ were prime, suppose that $n = 2xy + x + y$ have noninteger solutions different from $x = -1$ and $y = -(n + 1)$ (which are always solutions), then $2n + 1 = 4xy + 2x + 2y + 1 = (2x + 1)(2y + 1)$. $2x + 1$ and $2y + 1$ are not equal to 1 (unless $n = 1$) and divide $2n + 1$, which is not possible. Now if $n = 2xy + x + y$ does not have integer solutions, suppose that $2n + 1$ is not prime. Then $2n + 1 = pq$ for p and $q \neq 1$ and both odd, i.e., $p = 2p' + 1$ and $q = 2q' + 1$. But it follows then that $n = 2p'q' + p' + q'$, which is false, and so $2n + 1$ must be prime for n to be admissible. Bringer also remarks that $2n + 1$ being prime is not sufficient, citing a counterexample, $n = 20$.

Recall that, earlier, Queneau made the claim that if $n = 2^k$, $k \neq 1$, then n is not admissible. This particular situation falls under the last item above of Bringer's list of results, but in [2] she includes a separate proof of this case. To see this, note that $\sigma_n(2) = 1$, $\sigma_n^2(2) = n = 2^k$. It follows that $\sigma_n^{k+1}(2) = 2$. The permutation cycle generated by 2 is of order $k + 1$, which is strictly less than 2^k if $k \neq 1$.

In 1993 Roubaud summarized what was known until that time about the search for quenines or admissible n. In [15] he even proposes extensions to the quenine. He also attempts to complete Bringer's result, stating a seemingly complete characterization of admissible n. He claims that a necessary and sufficient condition for n to be admissible is for 2 to be of order n or $2n$ in the multiplicative group of integers modulo $2n + 1$. Jean-Guillaume Dumas, in [3], points out that this is not true when n is even, citing the counterexample $n = 8$. Also 2 is of order 8 (mod 17). But, as we have seen, the "octine" is not possible as $n = 8$ gives a spiral permutation of order 4.

From the Middle Ages to the Twenty-First Century

Roubaud was almost correct, and Dumas provided the solution. In [3] Dumas gives a complete characterization of admissible numbers, and therefore quenines. He even adds to the list started by Queneau, naming all quenines under one thousand. (For example, the highest admissible n under 1000 is $n = 998$. Thus one can compose a poem with 998 strophe-ending words permuting "spirally" for 998 stanzas.) Extending Bringer's work, he investigates the properties of the finite field $\mathbb{F}_p$, for such n where $p = 2n + 1$ is prime. His characterization of Queneau numbers n is the following:

Theorem 2 (Dumas). *Let n be an integer such that $p = 2n + 1$ is prime. Then n is admissible if and only if:*

(1) *either 2 is of order $2n$ (i.e., 2 is a primitive root) of $\mathbb{F}_p$*

(2) *or n is odd and 2 is of order n in $\mathbb{Z}_p$.*

Recall that the first characterization above of admissible n was previously elucidated by Bringer. The following, however, is the entire proof by Dumas.

Proof. Beginning with the necessary condition, consider an admissible n. As the order of the 2 in $\mathbb{F}_p$ divides $2n$, the number of invertible elements of $\mathbb{F}_p$, then 2 can only be of order $2n$, n, or strictly less than n. Now suppose 2 is of order $j < n$. Then $\delta_n^j(2) \equiv \pm 2^j 2 \equiv \pm 2 \pmod{2n+1}$. So there are two cases to consider. First, if $\delta_n^j(2) = 2$, then the orbit of 2 only contains $j < n$ elements, which cannot be true since it is assumed that n is admissible. Thus, in the other case $\delta_n^j(2) = -2$. But by definition $1 \leq \delta_n^j(x)$, so $1 \leq 2n + 1 - 2 \leq 2$, or $n \leq 1$. But 2 is of order 2 $(\bmod\ 2(1) + 1 = 3)$. Therefore, if n is admissible, then 2 is of order n or $2n$ (mod $2n + 1$). But it remains to exclude the instance in which $n = 2k$ is even and 2 is of order n (mod $2n + 1$). But in this case $2^k \equiv -1 \pmod{2n+1}$. From here it follows that $\delta_n^k(2) = \pm 2^k 2 \equiv \pm 2 \pmod{2n+1}$. As $k = \frac{n}{2} < n$, then as before neither situation can exist if n is admissible.

Next, we prove the sufficient condition. Let us call ω the cardinality of the smallest orbit of the elements of the set $N = \{1, 2, 3, \ldots, n\}$ by δ_n. Now suppose that $u \in N$ is of order ω. Now there exists k such that $\delta_n^\omega(u) \equiv u \equiv (-1)^k 2^\omega u \pmod{2n+1}$. And, since u is invertible, $(-1)^k 2^\omega \equiv 1 \pmod{2n+1}$. This means that $2^\omega \equiv \pm 1 \pmod{2n+1}$. If $2^\omega \equiv 1$, then ω is greater than the order of 2. Therefore $\omega \geq n$ and so $\omega = n$ for the permutation is at most of order n. In the case that $2^\omega \equiv -1$, then 2 is of order $2n$ or an odd n. If 2 is of order $2n$, then $2^n \equiv -1 \pmod{2n+1}$. So, $2^{n+\omega} = 1$, and therefore $\omega = n$. If the order of 2 is n with n odd, then $(2^\omega)^2 = (-1)^2 = 1$, which means that n divides 2ω. As n is odd, Gauss's lemma implies that n divides ω. So once more, $\omega = n$. □

Another characterization of the quenine provided by Dumas is a corollary to the theorem above. According to Dumas it restates a conjecture by Joerg Arndt.

Corollary 3. *Let $\mathbb{F}_p$ be as above. Then n is admissible if and only if:*

(1) *either 2 is of order $2n$ in $\mathbb{F}_p$, and $n \equiv 1$ or $2 \pmod 4$,*

(2) *or 2 is of order n in $\mathbb{F}_p$, and $n \equiv 3 \pmod 4$.*

The characterizations of admissible n obviously provide a test of whether a quenine of a certain order n exists. At this point there is no way of *generating* such n.

Extensions and Connections

There are some very natural ways to extend mathematically the results inspired by the quenine. One generalization to δ_n, suggested by Roubaud, is to replace multiplication by 2 by any k. Call this permutation $\delta_{n,k}(x)$, the *k-quenines*. For example, the 3-*quenine* has formula:

$$(3) \qquad \delta_{n,3}(x) = \begin{cases} 3x & \text{if } 3x \leq n, \\ 2n + 1 - 3x & \text{if } n < 3x \leq 2n, \\ 3x - (2n+1) & \text{otherwise.} \end{cases}$$

In this case, when $n = 8$, the permutation $\delta_{8,3}$ is of order 8. Roubaud and Pérec developed an extension to σ_n, inspired by the absence of a quenine of order 10 ([15]). The permutation, π_n, is called the **pérecquine** after Pérec and is defined by the formula:

$$(4) \qquad \pi_n(x) = \begin{cases} 2x & \text{if } 2x \leq n, \\ 2x - (n+1) & \text{otherwise.} \end{cases}$$

As he did with the quenine, Dumas formalizes these extensions in [3] and provides analogous spiral-like representations for the permutations involved. With the pérecquine he also characterizes all admissible n, finding that π_n is of order n if and only if 2 is of order n (mod $n + 1$).

Finally, Dumas extends Theorem 2 to all admissible n to $\delta_{n,k}$ and all primitive roots.

Theorem 4 (Dumas). *Let n be an integer such that $p = 2n + 1$ is prime, and let $k \leq n$. Let $\mathbb{F}_p$ be the finite field containing $p = 2n+1$ elements; then $\delta_{n,k}$ is of order n if and only if:*

(1) *Either k is of order $2n$ (i.e., k is a primitive root) in $\mathbb{F}_p$, or*

(2) *n is odd and k is of order n in $\mathbb{F}_p$.*

Once more, $\delta_{n,k}$ is the inverse permutation to $\sigma_{n,k}$. The proof is similar to that of Theorem 2 except for the condition $k \leq n$.

Corollary 5. *Let n be a positive integer such that $2n+1$ is prime. Then there exists k, $1 \leq k \leq n$, such that $\delta_{k,n}$ is of order n.*

A Connection to Artin's Conjecture on Primitive Roots

Artin's conjecture states that a positive integer k that is not a perfect square is a primitive root of p, for infinitely many primes p. Moreover, the set of prime numbers p such that k is a primitive root has a positive asymptotic density in the set of primes. For example, when $k = 2$ (and, in general, if $k = ab^2$ and a is not congruent to 1 (mod 4)), then the density is Artin's constant

$$C := \prod_p \left(1 - \frac{1}{p(p-1)}\right) \approx 0.373955, \tag{5}$$

where the product is taken over all primes including 2. The conjecture then implies that there are infinitely many admissible or Queneau numbers n, where $p = 2n + 1$. Indeed, the sequence of such n corresponding to that of primes p for which 2 is a primitive root (sequence A001122 of N. J. A. Sloane of the On-Line Encyclopedia of Integer Sequences) appears to be a (proper) subset of the set of all Queneau numbers. There is a brief treatment of Artin's conjecture in the preface to his collected works [1], while the expository articles by Murty [8] and Stevenhagen [16] trace related history and summarize some developments in proving the conjecture.

Conclusion

The mathematical ideas inspired by Arnaut's sestina may not contribute anything to one's understanding of courtly love, to the culture of the troubadours, or even to the meaning of Arnaut's poem itself. However, to those who understand the mathematics involved, these results inform the reading of the poetry by providing another dimension to the appreciation of its composition. Whether or not Arnaut was aware of the spiral structure of the permutation employed in the sestina, this structure provided the natural link and spark to mathematical activity. Perhaps we shall never know what inspired Daniel to choose $\sigma_{n,6}$ to compose his sestina. His choice, however, inspired modern mathematicians to create and contribute to their own discipline. Now it is the poets' turn.

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